International Journal of Industrial Engineering & Production Research

December 2013, Volume 24, Number 4
pp. 285-295

http://IJIEPR.iust.ac.ir/

ISSN: 2008-4889

Stochastic Approach to Vehicle Routing Problem: Development and Theories

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KEYWORDS

Vehicle Routing Problem,
Chance Constrained Programming,
Linear approximation,
Optimization

ABSTRACT

This article proposes a stochastic vehicle routing problem within the frame-wok of chance constrained programming where one or more parameters are presumed to be random variables with known distribution function. The reality is that once we convert some special form of probabilistic constraint into their equivalent deterministic form then a nonlinear constraint generates. Knowing that reliable computer software for large scaled complex nonlinear programming problem with 0-1 type decision variables for stochastic vehicle routing problem is not easily available merely then the value of an approximation technique becomes imperative. In this article, theorems which build a foundation for moving toward the development of an approximate methodology for solving the stochastic vehicle routing problem are stated and proved. Using these theorems one can easily convert a nonlinear type vehicle routing problem of special type into an equivalently designed linear problem that can be solved fast and easy.

1. Introduction

Stochastic Vehicle Routing problem (SVRP) has attracted the attention of many researchers since its introduction into the literature of operations research in 1969 [28, 29]. A SVRP means to design a set of routes starting from and eventually returning to a central depot to deliver products to a fixed number of demand points such that capacity constraints, probabilistic demands, and the duration of the routes are satisfied. Vehicle routing problem (VRP) and SVRP are key issues in supply chain management and distribution systems. As management begins listening and paying attention to customers and wishes to fulfill their needs then the nature of this problem becomes more complex and demanding. To that end, management may consider time windows for on time delivery and suitable restrictions for answering customer's requirements. Vehicle routing with stochastic elements has received some attentions as well. Stewart, and Golden [26], Stewart [25], Golden and Yee [10], Dror and Trudeau [8], Laporte and Louveau [19] have employed stochastic optimization techniques to solve small size problems while Bertsimas et al. [2], Golden et al.[12, 13], Holmes and Parker [15], Magnanti [20], Nagy and Salhi [21], Salhi and Rand [23], Waters [31], Stacy [24], and Wassan and Osman [30] have looked into the variant approaches of this problem. This problem that fully satisfies the criteria of multiple objective programming has received very limited attention in the literature**.** It is believed that this is partially because this problem belongs to the class of the NP hard and it is of large scale problems.

Park [22], proposed a multiple objective analysis of the deterministic vehicle routing problem. In 2004, Bianchi

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Paper first received May 06, 2012, and in accepted form Dec. 03, 2012.

et al [3] studied vehicle routing problem with stochastic demand using meta-heuristic approaches. In 2006, Bianchi et al. [4] have considered five meta-heuristics of Simulated Annealing; Tabu Search; Iterated Local Search; Ant Colony Optimization and Evolutionary Algorithms to study VRP with stochastic demand. In 2005, Haugland, Ho and Laporte [14] studied the delivery districts for the VRP with stochastic demands. A two-stage hybrid algorithm for pickup and delivery vehicle routing problems with time windows and multiple vehicles is studies by Bent et al. [1].

In the first stage, a simple simulated annealing algorithm is used for decreasing the number of routes while the second stage uses large neighborhood search for decreasing total travel cost. Zheng and Liu [34] studied vehicle routing problem in which the travel times are assumed to be fuzzy variables. Jozefowiez et al. [35, 36, 37] have studied multi objective vehicle routing problem using set covering and meta-heuristic approaches. A good review of multiple objective programming is presented in the work of Zeleny [41], Deb et al. [38] and Zare Mehrjerdi [39, 40]. There are some methods available for generating the set of non-dominated solutions in multi objective programming [38-41].

Single and multiple objectives stochastic Vehicle Routing Problems can be classified into four categories as listed below:

1. Stochastic Customers: Jaillet t al. [16]; Jaillet et al. [17],
2. Stochastic Demands: Bianchi et al. [3]; Bianchi et al. [4]; Dror and Laporte [8]; Golden and Yee [10]; Golden and Stewart [12]; Haugland et al. [14]; Tan el al [27]; Tillman [28]; Yee and Golden [32];
3. Stochastic travel and unload times: Cook and Russell [7, 33];
4. Stochastic demands and stochastic travel and unload time: Zare Mehrjerdi [33].

The work presented here considers the SVRP of type 4 and studies the problem within the frame-wok of multiple objective goal programming problems [33]. The paper is organized as follows: a general view of SVRP is given in section 2; chance constrained programming model of the problem is discussed in section 3; assumptions and notations is the topic of section 4 while stochastic vehicle routing problem is discussed in section 5; distributions other than normal is the topic of section 6. Author's conclusion is given in section 7.

2.Vehicle Routing Problem

Since 1959 when Dantzig and Ramser [9] first introduced the VRP and proposed a linear programming based heuristic for its solution the heuristic method has been widely researched. Christofieds and Eilon [6] indicated that the largest VRP of any complexity solved to date by exact methods and reported in the open literature contains only 31 demand points. Before considering model development for this research, a formulation of the VRP problem as a 0-1 integer program is given below. Problem that is known as "pure delivery" can be formulated as [9]:

P3: Minimize $$\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\sum_{k=1}^{NV} d_{ij}X_{ijk} \quad (1)$$

S.t.

S= { $$\sum_{i=1}^{N}\sum_{k=1}^{NV} X_{ijk} = 0 \qquad j=2,3,\ldots,N \quad (2)$$

$$\sum_{i=1}^{N} x_{ipk}\sum_{\substack{j=1\\ i\neq j}}^{N} X_{pjk} = 0 \quad k=1,2,\ldots,NV, p=1,2,\ldots,N \quad (3)$$

$$\sum_{j=2}^{N} X_{ijk} \le 1 \qquad \text{k=1,2,…,NV} \quad (4)$$

$$Z_i - Z_j + N\sum_{k=1}^{NV} X_{ijk} \le N-1 \qquad i \neq j = 1,2,\ldots,N \quad (5)$$

}

$$\sum_{i=1}^{N} d_i\left(\sum_{\substack{j=1\\ i\neq j}}^{N} X_{ijk}\right) \le Q_k \quad k=1,2,\ldots,NV \quad (6)$$

$$\sum_{i=1}^{N} t_{ijk}\sum_{\substack{j=1\\ i\neq j}}^{N} X_{ijk} + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} t_{ijk}X_{ijk} \le T_k \quad k=1,2,\ldots NV \quad (7)$$

$$X_{ijk} = \begin{cases} 0 \\ 1 \end{cases} \quad \text{i, } j, k \text{ and } i \neq j \quad (8)$$

Where;

N= Number of nodes

NV= Number of vehicles

Q_k = Capacity of vehicle k

T_k = Maximum time allowed for vehicle k on a route

d_i = Demand at node i (assuming that $d_1=0$)

t_{ik} = Time required for vehicle k to deliver or collect at node $i(t_{1k}=0)$

t_{ijk} = Travel time for vehicle k from node i to node j

d_{ij} = Distance from node i to j

$X_{ijk} =$ { 1 if arc (i, j) is traversed by vehicle k; 0 Otherwise

$Z_i =$ Arbitrary real numbers, i=1, 2,…,N.

The objective function (1) represents minimization of total distance traveled by NV vehicles. Alternatively, costs could be minimized by replacing d_{ij} with C_{ij}, depending on the vehicle type. Equation (2) ensures that each demand node is served by exactly one vehicle; equation (3) ensures that if a vehicle enters a demand node it must exit from that node; and (4) guarantees that vehicle availability is not exceeded; equation (5) prohibits sub-tours generation, equation (6) is vehicle capacity constraint, and finally (7) is the total elapsed route time constraint. Decision variables X_{ijk} that is of 0 and 1 type is represented by equation 8. The set of constraints (2), (3), (4), and (5) generate a space that we will refer to that as S in the remainder of this article. All feasible decision variables of problem belong to this space. For having any decision variable from set S as an acceptable solution of the problem it must satisfy constraints (6), (7), and (8) as well.

3. Chance Constrained Programming

The event of constrained violation must be regarded as a risk taking issue. Having identified (1-α) as the level of the constraint persistency then the risk level for that constraint is α. The presence of risk in the linear programming model adds another dimension into the managerial decision making problem. The input factors play a significant role in deteriorating systems reliability by violating one or more constraints. One well-defined methodology for treating problems with probabilistic constraints is known as chance constrained programming (CCP). Abraham Charnes and William Cooper [5] have proposed CCP models such as the E-model (expectation optimization model), the V-model (variance minimization model) and the P-model (probability maximization model).

Tillman [28] proposed a modification of the Clarke and Wright approach for multi-depot delivery and collection problems having probabilistic Poisson distributed demands. The objective function of the delivery problem for a given number of stop points on a proposed route is

Minimize

$$E(\cos t) = \underset{R}{Min}\{\int_0^R C_1(D)h(D)dD + \int_R^\infty C_2(D)h(D)dD\} \quad (9)$$

Where the first expression from the right indicates the cost of not hauling enough commodities to satisfy all customer demands on a route and the second expression from the right represents the cost of hauling excess commodity on the route that is not needed.

The value of R is determined for each route is the load assigned to the truck for that route. Notations are:

$C_1(D) =$ { Cost of hauling excess commodity on the route that is not needed or; Cost of completing scheduled route and having unfilled capacity

$C_2(D) =$ { Cost of not hauling enough commodity to satisfy all the demands on the route, or; Cost of filling truck prior to completing the scheduled route

$$D = d_1 + d_2 + ... + d_n \quad (10)$$

Where,

d_i = the probabilistic demand for the i^{th} stop

$f(d_i)$ =the probability density function of the random variables d_i

h(D) = the probability density function of D

Golden and Stewart [12] have extended Tillman's SVRP in a different way considering only a single depot problem. In this technique the locations on the route are n_1, n_2 ,.., n_k, and it is assumed that all vehicles have the same capacity Q and that the total demand for all locations is

$$X = dn_1 + dn_2 + ... + dn_k \quad (11)$$

Where, d_{ni} is the demand at location i which is described by the independent Poisson distribution with mean and variance Y_{ni}. Then

$$E(X) = Var(X) = Yn_1 + Yn_2 + ... + Yn_k \quad (12)$$

Using the central limit theorem and approximating with normal distributions, then

$$U = Yn_1 + Yn_2 + ... + Yn_k \quad (13)$$

and

$$\sigma = (U)^{1/2}. \quad (14)$$

The mathematics of the primary and secondary errors is based upon the following definitions:

3.1. Primary Error

A primary error occurs when a vehicle cannot satisfy the demands of the customers on the route to which it has been assigned.

3.2. Secondary Error

A secondary error occurs when a vehicle returns to the central depot after satisfying the demands on its route with more than 100(1-α) % of its original load where $0 \le \alpha \le 1.$

According to above definitions we can write the following formula for the primary and secondary errors.

$$P(X \ge Q) = P(\text{Pr}imaryError) = P(Z \ge \frac{(Q-\mu)}{\mu^{1/2}}) \ge (1-\alpha) \quad (15)$$

$$P(X \le \alpha Q) = P(SecondaryError) = P(Z \le \frac{(\alpha Q-\mu)}{\mu^{1/2}}) \ge \alpha \quad (16)$$

Where Q is the truck capacity and $0 \le \alpha \le 1.$

Assuming that μ is nearly the same for most of K routes, then an artificial capacity μ^{-}, as the vehicle capacity, can be used along with the Yni as demand points and the "Saving" approach of Clarke and Wright to obtain a fixed set of routes. Therefore, the following problem is the one that must be solved:

Minimize {Expected total cost}
S.t.
{
1. A fixed set of routes
2. Satisfaction of customers
3. Vehicle capacity is obeyed
4. P {total demand $\ge$ truck capacity } >= (1-α)
}

where $0 \le \alpha \le 1.$

Golden and Yee [11] proposed a dynamic formulation of the problem for determining the driver operating strategies when customer demands on a route are probabilistic. Specifically, after delivery of goods to a demand point on a fixed route, which has already been determined by the Clarke and Wright procedure, the driver is faced with the decision of whether to return to the depot to replenish the supply. However, the optimal decision is based on whether the remaining supply of goods in the vehicle is greater or less than some critical value which must take into account the following criteria:

1. The probabilistic demands on the remaining portion of the route, and the distribution between the remaining customers
2. The distances between the remaining customers

Cook and Russell [7] have successfully treated a large routing problem with timing constraints and stochastic travel times and demands. The authors approach to the problem is by generating a deterministic solution using the MTOUR algorithm and then testing these routes via simulation to demonstrate that they are effective; however, the stochastic nature of the problem is not explicitly considered in the route generation stage. The basic procedure for the generation of travel times and pick up times is based on the development of the multiple regression equations for the intra-city transit times is derived by employing the Euclidean distance and the average speed limit as the independent variables. The service time (pickup time) is considered to be a function of two independent variables: number of containers and the total capacity of the containers. Based on these assumptions, the second regression equation for pickup times is determined.

4. Assumptions and Notations

The assumptions and notations used in the SVRP model development are as listed below.

4.1. Assumptions

1. Customer demands are random variables with know distribution functions
2. Traveling time from one point to next is random variables with known distribution function
3. Unloading time at each customer demand point is random variable with known distribution function
4. There are NV vehicles available for shipping purposes
5. All vehicles have the same capacity
6. The commodity to be transported is homogeneous
7. The shortest distance between two stations is considered to be Euclidian
8. For each route, a total travel time is identified beforehand
9. For each route, a total unloading time is identified beforehand
10. $\alpha, \beta, \psi, \alpha_k, \beta_k, \psi_k$ are predetermined constant numbers.

4.2. Notations

$d_i =$ the i^{th} customer demand, and it is a random variable (assuming that d_1=0)
NV= total number of trucks
N= Number of nodes
Q = truck capacity
S_{NV} = indicates the convex set of feasible region for NV trucks
$X_{ijk} = 1$ if vehicle k travels from node i to node j
= 0 otherwise
$UT_k =$ A predetermined maximum total travel time allowed for the k^{th} vehicle route
$TR_k =$ A predetermined maximum total travel time allowed for the k^{th} vehicle route

α_k = A predetermined level of constrained violation of the k th route considering the vehicle capacity

β_k = A predetermined level of constrained violation of the k th route considering total unloading time

φ_k = A predetermined level of constrained violation of the k th route considering the total travel time

t_{ik} = Time required for vehicle k to deliver or collect at node $i (t_{1k} = 0)$

t_{ijk} = Travel time for vehicle k from node i to node j

d_{ij} = Distance from node i to j

$$X_{ijk} = \begin{cases} 1 \text{ if arc (i, j) is traversed by vehicle k} \\ 0 \text{ Otherwise} \end{cases}$$

Z_i = Arbitrary real numbers, i=1, 2,…,N.

5. Stochastic Vehicle Routing Problem

The CCP formulation of the VRP within the framework of multiple objective goal programming is led to the development of the stochastic goal programming model of the vehicle routing problem (SVRP). This model allows decision makers (DMs) involvement in the solution process of the problem to obtain satisfactory vehicle routes. The problem formulation for the SVRP is divided into two groups according to the type of the criteria that is to be minimized. The objective functions are realized to be:

1. Total cost (or distance) minimization
2. Total time minimization

5.1. Problem Type I (Cost Minimization)

When the cost (or distance) is considered as a criterion, the objective function is linear in terms of decision variable $X_{ijk.}$ [9, 33]

P1: *Minimize* $$\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\sum_{k=1}^{NV} c_{ij}X_{ijk} \qquad (17)$$

S.t.

$$P\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} d_i X_{ijk} \le Q\} \ge (1-\alpha_k) \text{ for } k=1,2,...,N \qquad (18)$$

$$P\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} t_i X_{ijk} \le UT_k\} \ge (1-\beta_k) \text{ for } k=1,2,...,N \qquad (19)$$

$$P\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} t_{ij} X_{ijk} \le TR_k\} \ge (1-\psi_k) \text{ for } k=1,2,...,NV \qquad (20)$$

$$\sum_{i=1}^{N}\sum_{k=1}^{NV} X_{ijk} = 0 \quad j=2, 3,...,N \qquad (21)$$

$$\sum_{i=1}^{N} x_{ipk}\sum_{\substack{j=1\\ i\neq j}}^{N} X_{pjk} = 0 \quad k=1,2,...,NV, \quad p=1,2,...,N \qquad (22)$$

$$\sum_{j=2}^{N} X_{ijk} \le 1 \qquad \text{k=1,2,…,NV} \qquad (23)$$

$$Z_i - Z_j + N\sum_{k=1}^{NV} X_{ijk} \le N-1 \quad i \neq j = 1,2,...,N \qquad (24)$$

$$X_{ijk} = \begin{cases} 0 \\ 1 \end{cases} \quad i, j, k \text{ and } i \neq j \qquad (25)$$

The set of constraints (21), (22), (23), (24) and (25) produce a space that we will refer to that as S in the remainder of this article. All feasible decision variables of problem belong to this space. For having any decision variable from set S as an acceptable solution of the problem it must satisfy other constraints of the problem as well.

The EDF of problem P1 is as shown in P2 provided that demand, travel time and unload time are normally distributed random variables [6, 18]:

P2: *Minimize* $$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\ i\neq j}}^{NV} c_{ij}X_{ijk} \qquad (26)$$

S.t.

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N} \mu_{d_i} X_{ijk} + N^{-1}(1-\alpha_k)\{\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N} \sigma^2_{d_i} X^2_{ijk}\}^{1/2} \le Q \quad \text{for all } k=1,...,NV \qquad (27)$$

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N} \mu_{t_i} X_{ijk} + N^{-1}(1-\beta_k)\{\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N} \sigma^2_{t_i} X^2_{ijk}\}^{1/2} \le UT_k \quad \text{for } all\ k=1,...,NV \qquad (28)$$

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\mu_{t_{ij}}X_{ijk}+N^{-1}(1-\psi_k)\{\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sigma^2_{t_{ij}}X^2_{ijk}\}^{1/2}\leq TR_k \quad \text{for } all\ k=1,\ldots,NV \tag{29}$$

$$X=[X_{ijk}]\in S \tag{30}$$

5.2. Problem Type II (Time Minimization)

When total time is realized as criterion then the objective function to be minimized is nonlinear. This is shown by the following problem:

P3: Minimize $\sum_{k=1}^{NV}UT_K+TR_k$ (31)

S.t.

P $\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}d_iX_{ijk}\leq Q\}\geq(1-\alpha_k)$ for $k=1,2,\ldots,NV$ (32)

P $\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}t_iX_{ijk}\leq UT_k\}\geq(1-\beta_k)$ for k=1,2,…,NV (33)

P $\{\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}t_{ij}X_{ijk}\leq TR_k\}\geq(1-\psi_k)$ for k=1,2,…,NV (34)

$$X=[X_{ijk}]\in S \tag{35}$$

The EDF of problem P3 is shown in P4 [6, 18]:

P4: Minimize { $\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\ i\neq j}}^{NV}\mu_{t_i}X_{ijk}+N^{-1}(1-\beta_k)\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\ i\neq j}}^{NV}\sigma^2_{d_i}X^2_{ijk}\}^{1/2}$ + (36)

$\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\ i\neq j}}^{NV}\mu_{t_{ij}}X_{ijk}+N^{-1}(1-\psi_k)\{\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\substack{k=1\\ i\neq j}}^{NV}\sigma^2_{t_{ij}}X^2_{ijk}\}^{1/2}$ }

S.t.

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\mu_{d_i}X_{ijk}+N^{-1}(1-\alpha_k)\{\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sigma^2_{d_i}X^2_{ijk}\}^{1/2}\leq Q \tag{37}$$

$$X=[X_{ijk}]\in S \quad \text{for } k=1,2,\ldots,NV \tag{38}$$

6.Distributions Other than Normal

The deterministic constraints (27), (28), and (29) can be replaced with some other constraints in an easier form, if the time and demand distributions are of the same special forms. There are several distributions that satisfy the following condition:

$$\sigma_i^2=\gamma\mu_i \tag{39}$$

This means that the variance is some constant multiple of the mean of that distribution. Distributions such as Poisson and Chi-square satisfy the above condition. The values of γ for these distributions are as shown in table 1.

Tab. 1. Specific distribution with simple relationship between mean and variance

Distribution	Relationship	γ
Poisson	$\sigma_i^2=\gamma\mu_i$	1
Chi-square	$\mu_i=\tau$ $\sigma_i^2=\gamma\tau$	2

The following theorem shows the existence of a set of deterministic linear time and demand constraints which are equivalent to the nonlinear set of the time and demand constraints of the RIS problem.

Theorem 1

Under the following conditions:

1. The probability distributions of t_{ij} are independent and stable and $\sigma^2_{t_{ij}}=\gamma\mu_{t_{ij}}$
2. The probability distributions of t_i are independent and stable and $\sigma^2_{t_i}=\gamma\mu_{t_i}$
3. The probability distributions of d_i are independent and stable and $\sigma^2_{d_i}=\gamma\mu_{d_i}$

There exist values T_1^-, T_2^- and Q^- such that:

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}}\mu_{t_{ij}}X_{ij}=T_1^- \tag{40}$$

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}}\mu_{t_i}X_{ij}=T_2^- \tag{41}$$

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{d_i} X_{ij} = Q^- \quad (42)$$

are equivalent to the following deterministic constraint, respectively.

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij} + N^{-1}(1-\beta)\{\sum\sum \sigma^2_{t_{ij}} X^2_{ij}\}^{1/2} = T_1 \quad (43)$$

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_i} X_{ij} + N^{-1}(1-\psi)\{\sum\sum \sigma^2_{t_i} X^2_{ij}\}^{1/2} = T_2 \quad (44)$$

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{d_i} X_{ij} + N^{-1}(1-\alpha)\{\sum\sum \sigma^2_{d_i} X^2_{ij}\}^{1/2} = Q \quad (45)$$

Proof

The proof is developed only for (40). One can prove similarly for (41) and (42) as well. Since decision variable X_{ij} is either zero or 1 then

$$X_{ij} = X^2_{ij} \quad (46)$$

Therefore

$$\sigma^2_{t_{ij}} X_{ij} = \sigma^2_{t_{ij}} X^2_{ij} \quad (47)$$

Or

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \sigma^2_{t_{ij}} X_{ij} = \sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \sigma^2_{t_{ij}} X^2_{ij} \quad (48)$$

$$\{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \gamma\mu_{t_{ij}} X_{ij}\}^{1/2} = \{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \sigma^2_{t_{ij}} X^2_{ij}\}^{1/2} =$$

$$\{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \sigma^2_{t_{ij}} X_{ij}\}^{1/2} = \gamma^{1/2}\{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij}\}^{1/2} \quad (49)$$

Substituting (48) in equality (43), then

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij} + N^{-1}(1-\psi)\gamma^{1/2}\{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij}\}^{1/2} = T_1 \quad (50)$$

Let

$$M = \{\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij}\}^{1/2} \quad (51)$$

And

$$q = N^{-1}(1-\psi) \quad (52)$$

Therefore

$$T_1 = M^2 + q\gamma^{1/2}M \quad (53)$$

or

$$M^2 + q\gamma^{1/2}M - T_1 = 0 \quad (54)$$

After solving for M we get

$$M = \{-q\gamma^{1/2} + (q^2\gamma + 4T_1)^{1/2}\}/2 \quad (55)$$

However,

$$M^2 = \{(-q\gamma^{1/2} + (q^2\gamma + 4T_1)^{1/2})/2\}^2 = T_1^- \quad (56)$$

Hence,

$$M^2 = \{[\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij}]^{1/2}\}^2 = T_1^- \quad (57)$$

Or

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_{ij}} X_{ij} = T_1^- \quad (58)$$

A similar analysis yields

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{t_i} X_{ij} = T_2^- \quad (59)$$

$$\sum_{i\in I}\sum_{\substack{j\in J\\ i\neq j}} \mu_{d_i} X_{ij} = Q^- \quad \text{Q.E.D.} \quad (60)$$

Theorem 2

If α, the probability of route failure increases up to 0.50 then, the value of T_1^- will increase provided that T_1 is a fixed value.

Proof

Let us reconsider equation (56) where

$$q = Z = Z_{1-\alpha} = N^{-1}(1-\alpha) \quad (61)$$

and T_1 is fixed value.

$$T_1^- = (2Z^2\gamma + 4T_1 - 2Z\gamma^{1/2}(Z^2\gamma + 4T_1)^{1/2})/4 \quad (62)$$

$$\partial T_1^- / \partial\alpha = \partial T_1^- / \partial Z . \partial Z / \partial\alpha \quad (63)$$

$\partial z / \partial\alpha < 0$, and this is because when α increases then (1-α) and $z_{1-\alpha}$ decreases. However, after some calculations one gets the following equation:

$$\partial T_1^- / \partial z = (2\gamma(z^2\gamma + 4T_1)^{1/2} - 3/2z^2\gamma^{3/2} - 2T_1\gamma^{3/2} - 2T_1\gamma^{1/2}) / (z^2\gamma + 4T_1)^{1/2}) / 4 \quad (64)$$

Notice that when a and b are two positive numbers the following inequality exists:

$$(a+b)^{1/2} < a^{1/2} + b^{1/2} \quad (65)$$

Therefore

$$z\gamma(z^2\gamma + 4T_1)^{1/2} < z\gamma(z^2\gamma)^{1/2} + z\gamma(4T_1)^{1/2} = z^2\gamma^{3/2} + 2z\gamma T_1^{1/2} \quad (66)$$

Hence, due to inequality (48)

$$\partial T_1^- / \partial z < (z^2\gamma^{3/2} + 2z\gamma T_1^{1/2} - 3/2z^2\gamma^{3/2} - 2T_1\gamma^{1/2}) / (z^2\gamma + 4T_1)^{1/2} \quad (67)$$
$$= (-1/2\gamma^{3/2}z^2 + 2z\gamma T_1^{1/2} - 2T_1\gamma^{1/2}) / (z^2\gamma + 4T_1)^{1/2}$$

The numerator on the right hand side of (50) is

$$-[2^{-1/2}z\gamma^{3/4} - (2T_1\gamma^{1/2})^{1/2}]^2$$
$$\partial T_1^- / \partial z < -[2^{-1/2}z\gamma^{3/4} - \{(2T_1\gamma^{1/2})^2 / (z^2\gamma + 4T_1)^{1/2}\}^{1/2} < 0 \quad (68)$$

Hence,

$$\partial T_1^- / \partial \alpha = \partial T_1^- / \partial Z . \partial Z / \partial \alpha > 0 \quad (69)$$

Corollary 1 to Theorem 2
If α the probability of route failure increases up to 0.5 then the value of T_2^- will increase provided that T_2 is a fixed value.

Corollary 2 to Theorem 2
If α the probability of route failure increases up to 0.5 then the value of Q will increase provided that Q is a fixed value.

Theorem 3
Suppose that β_k and ψ_k the probability of route failures are set such that $\partial\psi_k / \partial\beta_k > 0$ then by increasing β_k and ψ_k up to 0.5 the total elapsed time of the k[th] route will decrease.

Proof
Let

$$z_1 = z(1-\beta_k) = N^{-1}(1-\beta_k) \quad (70)$$

and

$$z_2 = z(1-\psi_k) = N^{-1}(1-\psi_k) \quad (71)$$

The total elapsed time of the k[th] route is

$$T_k = \sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \mu_{t_{ij}} x_{ijk} + \sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \mu_{t_i} x_{ijk} + z_1\{\sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \sigma^2_{t_{ij}} x^2_{ijk}\}^{1/2} + z_2\{\sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \sigma^2_{t_i} x^2_{ijk}\}^{1/2} \quad (72)$$

Now, it is necessary to show that $\partial T_k / \partial \beta_k < 0$ where

$$\partial T_k / \partial \beta_k = \partial T_k / \partial z_1 . \partial z_1 / \partial \beta_k + \partial T_k / \partial Z_2 . \partial z_2 / \partial \psi_k . \partial \psi_k / \partial \beta_k \quad (73)$$

$\partial z_1 / \partial \beta_k$ and $\partial z_2 / \partial \psi_k$ are both less than zero because by increasing β_k and ψ_k, (1- β_k) and (1- ψ_k) decrease and consequently $z_1 = z(1-\beta_k) = N^{-1}(1-\beta_k)$ and $z_2 = z(1-\psi_k) = N^{-1}(1-\psi_k)$ decrease. However,

$$\partial T_k / \partial z_1 = \{\sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \sigma^2_{t_{ij}} x^2_{ijk}\}^{1/2} > 0 \quad (74)$$

$$\partial T_k / \partial z_2 = \{\sum_{i\in I}\sum_{\substack{j\in J \\ i\neq j}} \sigma^2_{t_i} x^2_{ijk}\}^{1/2} > 0 \quad (75)$$

$$\partial T_k / \partial \beta_k = (+)(-) + (+)(-)(+) = (-) + (-) < 0 \quad Q.E.D.$$

Theorem 4
If the condition of theorem 3 exists for all NV truck routes then the total elapsed time of the whole system decreases.

Proof
According to theorem 3 the total elapsed time of each vehicle route decreases and thus it can be concluded that the total elapsed time of the whole delivery system will decrease since NV remains unchanged.

Theorem 5
For a SVRP having only probabilistic customer demands, if α, the probability of route failure, increase up to 0.50 then the total travel distance of the whole delivery system will decrease.

Proof
To proof this theorem, the following, which is a mathematical model for a SVRP having only probabilistic customer demand and no time restrictions, is considered:

(P$_3$) Minimize $D = \sum_{i=0}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sum_{k=1}^{NV} d_{ij}X_{ijk}$ (76)

S.t

$$\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\mu_{d_i}X_{ijk} + N^{-1}(1-\alpha_k)(\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sigma_{d_i}^2 X_{ijk}^2)^{1/2} \le Q \quad (77)$$

$$X = [X_{ijk}] \in S \quad (78)$$

By setting

$$z = N^{-1}(1-\alpha_k) \quad (79)$$

$$Y = \{\sum_{i=1}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sigma_{d_i}^2 X_{ijk}^2\}^{1/2} > 0 \quad (80)$$

It will be noticed that $\partial z/\partial\alpha_1 < 0$ because by increasing α_k, $(1-\alpha_k)$ decreases and consequently $N^{-1}(1-\alpha_k)$ decreases. Hence, Z.Y decreases and consequently, $Q^{-1} = Q - Z.Y$ will increase. However,

$$\partial D/\partial\alpha = \partial D/\partial Q^{-}.\partial Q^{-}/\partial z.\partial z/\partial\alpha \quad (81)$$

It is obvious that $\partial Q^{-}/\partial z = -Y < 0$. Hence, it remains to show that $\partial D/\partial Q^{-} < 0$, this is because $dQ^{-}/dz.dz/d\alpha > 0$. Now, it is only necessary to prove that in the deterministic VRP where customer demands are equal to their demand's mean, by increasing the artificial capacity of truck the traveled distance will decrease. If the transportation cost depends linearly on the weight of goods delivered and the distance traveled, then the following equation can be used:

$$C_{ij} = u_{ij}W_{ij}d_{ij} \quad (82)$$

Where

u_{ij} = Cost per unit weight per unit distance from node i to node j

W_{ij} = Weight transported from node i to node j,

d_{ij} = The distance from node i to node j

r_j = Number of times that weight W_{ij} can be fitted in Q^{-}.

However,

$$d_{ij} = C_{ij}/u_iW_{ij} \quad (83)$$

and

$$Q^{-} = r_jW_{ij} \quad (84)$$

Or

$$W_{ij} = Q^{-}/r_j \quad (85)$$

Since,

$X_i = \begin{cases} 1 \\ 0 \end{cases}$

Then,

$$D = \sum_{i=0}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sum_{k=1}^{NV} d_{ij}X_{ijk} = \sum_{i=0}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sum_{k=1}^{NV} d_{ij} \quad (86)$$

Or

$$D = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=1}^{NV} C_{ij}r_j/u_{ij} < 0 \quad (87)$$

Hence,

$$\partial D/\partial Q = \{-1/Q^2\}\sum_{i=0}^{N}\sum_{\substack{j=0\\ i\neq j}}^{N}\sum_{k=1}^{NV} C_{ij}r_j/u_{ij} < 0 \quad (88)$$

This result indicates that $\partial D/\partial Q^{-} < 0$, which proves this theorem. Q.E.D.

The following theorem concerns with the number of structured vehicle routes for the SVRP having probabilistic customer demands. It indicates that when number of constructed vehicle routes increases then the total demand to be served by the vehicle will increase. However, this situation will not happen when customer demands are deterministic.

Theorem 6

The larger number of routes is equivalent to the larger total demand to be served by all vehicles.

Proof

Suppose that μ_i and σ_i^2 are the mean and variance of demand point i, it is clear that

$$\sum_{i=1}^{N} u_i = u_1 + u_2 + \ldots + u_N \quad (89)$$

But,

$$\{\sum_{i=1}^{N}\sigma_i^2\}^{1/2} < \sigma_i^{1/2} + \ldots + \sigma_N^{1/2} \quad (90)$$

If only one vehicle can be used to deliver all customer demands, then the following inequality is needed:

$$\sum_{i=1}^{N} u_i + N^{-1}(1-\alpha)\{\sum_{i=1}^{N}\sigma_i^2\}^{1/2} \le Q \quad (91)$$

If two vehicles are used instead of one vehicle to deliver the customer demands, the following inequalities are needed:

$$\sum_{i=1}^{m<N} u_i + N^{-1}(1-\alpha)\{\sum_{i=1}^{m<N}\sigma_i^2\}^{1/2} \le Q \qquad (92)$$

and

$$\sum_{i=m+1}^{N} u_i + N^{-1}(1-\alpha)\{\sum_{i=m+1}^{N}\sigma_i^2\}^{1/2} \le Q \qquad (93)$$

Hence, after the addition of both sides of inequalities (92) and (93) the result is the following inequality.

$$\sum_{i=1}^{N} u_i + N^{-1}(1-\alpha)[\{\sum_{i=1}^{m<N}\sigma_i^2\}^{1/2} + \{\sum_{i=m+1}^{N}\sigma_i^2\}^{1/2}] \le 2Q \qquad (94)$$

The left hand side of inequality (94) represents the total generated demands to be served by two vehicles. Using the concept of inequality (90), inequality (94) can be written in the following form:

$$\sum_{i=1}^{N} u_i + N^{-1}(1-\alpha)\{\sum_{i=1}^{N}\sigma_i^2\}^{1/2} \le$$
$$\sum_{i=1}^{N} u_i + N^{-1}(1-\alpha)[\{\sum_{i=1}^{m<N}\sigma_i^2\}^{1/2} + \{\sum_{i=m+1}^{N}\sigma_i^2\}^{1/2}] \le 2Q \qquad (95)$$

The inequality (95) indicates that the total generated demand using two vehicles is larger than using one vehicle. However, one can extend the previous discussion for NV vehicles which are needed to satisfy all customer demands.
Q.E.D.

7. Conclusion

This article presented the development of the mathematical formulation of the route construction stage and route improvement stage of the SVRP problem. The equivalent deterministic forms of problem "P1" and "P3" were developed and presented by "P2" and "P4" type problems, respectively. The existence of a set of deterministic linear time constraints which are equivalent to the nonlinear set of time constraints of the problem for distributions such as Poisson and Chi-Square is proved through theorem1. The effects of route failure probabilities on the total elapsed time of the whole delivery system were proved through theorems 2 to 4. Theory 5 illustrates that the total traveled distance decreases when route failure probability increases. Additionally, theorem 6 is provided to demonstrate that the larger number of vehicle routes is equivalent to the larger customer demands.

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