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An Improved WAGNER-WHITIN Algorithm

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ABSTRACT

We present an improved implementation of the Wagner-Whitin algorithm for economic lot-sizing problems based on the planning-horizon theorem and the Economic- Part-Period concept. The proposed method of this paper reduces the burden of the computations significantly in two different cases. We first assume there is no backlogging and inventory holding and set-up costs are fixed. The second model of this paper considers WWA when backlogging, inventory holding and set-up costs cannot be fixed. The preliminary results also indicate that the execution time for the proposed method is approximately linear in the number of periods in the planning-horizon.

1. Introduction

An optimal solution to the single-level economic lot-sizing (ELS) problem has been known since Wagner and Whitin [1] published their dynamic programming analysis. The major limitations of the Wagner-Whitin approach (WWA) are the amount of computer memory and the computation time required for large problems. There have been tremendous efforts to reduce these limitations. For example, Evans [2] demonstrates an efficient implementation of WWA formula applicable to the general (non-concave cost) case. Other researchers improve this implementation under concave cost assumptions. Jacobs and Khumawala [3] use a branch and bound technique to achieve faster solutions. Saydam and McKnew [4] demonstrate an even faster algorithm based on the WWA formulation and the planning-horizon theorem. Silver and Meal [5] develop a heuristic method to select lot-size the case of a deterministic time-varying demand rate and discrete opportunities for replenishment. Groff [6] presents a decision rule for time phased component demand. Gaither [7] develops a near-optimal lot-sizing model for material requirement planning systems. Blackbum and Millen [8] analyze the single-stage lot-sizing using different numerical samples. Friend et al. [9] analyze the performance of seventeen lot-sizing methods and the effects they have on inventory for an aircraft application. They conclude, among other things, that the WWA ranks number one if the total cost of the system is the criteria used to assess the performance of the methods. The rest of the lot-sizing rules, only approximate the WWA, which is taken as benchmark problems.

Aryanezhad [10] presents a sufficient optimal condition for lot-sizing problem. Aryanezhad et. al. [11] extend the traditional lot-sizing for dynamic backlogging. There have been some attempts to find the near optimal solution for lot-sizing [12]. Vargas [13] studies the stochastic version of the WWA dynamic lot-size model. Richter et. al. [14] use the WWA for the natural resource stock control model. The basic assumption of most classical dynamic lot-sizing model is that the time varying demand is known in advance. Let N denote the length of the time planning and d_i be the demand where $d_i \in \{1,\cdots,T\}$. In other word, d_i represents the aggregate demand, placed by all customers, which is satisfied in period $i \in \{1,\cdots,T\}$. If backlogging is not allowed then d_i cannot be delivered earlier or later than $i \in \{1,\cdots,T\}$.

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If backlogging is allowed, d_i cannot be delivered earlier than *i*, but it can be delivered later at the expense of backordering costs.

In this paper we develop an improved implementation of the WWA for economic lot-sizing problems based on the planning-horizon theorem and the Economic Part-Period concept. We first assume there is no backlogging and inventory holding and set-up costs are fixed. The proposed method of this paper reduces the computations significantly. The second model of this paper considers WWA when backlogging, inventory holding and set-up costs cannot be fixed. For both cases, the proposed methods of this paper are examined using some rigorous test problems and the results are discussed in details.

2. The Lot-Sizing Problem

The lot-sizing problem described here considers a *T*-periods planning horizon problem with known demands $d_1,\cdots,d_T$. The problem is to find an ordering policy which leads to an optimum total cost of combined acquisition, purchasing, receiving and inspection, costs, set-up costs, and inventory carrying costs over the *T*-periods planning horizon. Before we go further we need to introduce the following assumptions,

I_t^+ on-hand inventory at the end of period *t*

A_t fixed cost of ordering/procurement, setup cost, in period *t*

h_t unit holding cost in period *t*

x_t production in period *t*

Q_t^* optimal quantity ordered or produced at the beginning of period *t* in terms of units

The solution of this problem is to determine the values $Q_i^*, i=1,\cdots,T$ such that all demands for *T* periods are met at a minimum total cost. There is also a case where the backorders are not permitted and the optimal lot-sizing decisions are the same as the solution given by WWA.

2.1. The Proposed Model

The objective of the proposed WWA is to determine the optimal state for each period, such that the total cost over the planning horizon is minimized.

The basic question of the lot-sizing problem is to determine the lot-sizing in each period, such that customer demand is satisfied in each period and total costs are minimized.

In this paper, we assume a discrete model horizon $t=1,\cdots,T$ and d_t is known for each period. This is, for example, the case when incoming orders are known upfront or when the accurate demand forecasts are available for the first *T* periods. Furthermore, the following costs are considered in the model. If an order is placed in period *t*, then a fixed setup cost *A* is incurred and the unit holding cost h_t is incurred for carrying inventory from period *t* to period *t + 1* or unit backordering cost π is incurred for stock out from period *t* to period *t - 1*. Therefore, the problem can be formulated as follows:

$$\text{Min} \quad z=\sum_{t=1}^{T}(A_t^* y_t + h_t^* I^+) \tag{1}$$

subject to

$$I_t^+ = I_{t-1}^+ + x_t - d_t \qquad t=1,\cdots,T \tag{2}$$

$$x_T + I_{T-1}^+ = D_T \tag{3}$$

$$x_t \geq 0, \qquad I_t^+ \geq 0 \tag{4}$$

$$Y_t = \begin{cases} 1 & if \quad x_t > 0; j=1,\cdots,T \\ 0 & if \quad x_t = 0; j=1,\cdots,T \end{cases}$$

In this model, the objective function includes the sum of the set up costs and the inventory expenditure and the main idea is to reduce the amount of calculation.

2.2. The Wagner and Whitin Algorithm

Wagner and Whitin show that the following formula can be used for *t = 1* to *T* to yield the minimal cost policy.

Let $K(x_t, I_t)$ be the cost of production x_t units in period *t* and the inventory holding cost I_t units at the end of period *t* where,

$$K(x_t, I_t) = \begin{cases} hI_t & if \quad x_t = 0 \\ A + Cx_t + hI_t & if \quad x_t > 0 \end{cases} \tag{5}$$

$$f_t(I) = \min\left[k_t(x_t, I_t) + f_{t-1}(I_t)\right] \tag{6}$$

$$I_t = I_{t-1} + x_t - d_t \tag{7}$$

This is a forward-recursive procedure which builds to a solution of the overall *T*-period problem by first solving a one-period problem and sequentially solving sub-problems until the overall optimum is found. Wagner and Whitin prove several theorems to break the ELS problem into small sub-problems. First, they prove that if the optimal-ordering policy requires an order arrival at period i* the problem can be separated into two sub problems consisting of periods i < z* and i > z* and i* is a cutoff point. Then, they prove that if, for any given *t*, a minimum in the *C*, series occurs at

period j where $j \leq t$, one cutoff period is $z^* = j$. The last statement, called the planning-horizon theorem is a powerful concept that proves large problems can be broken up into smaller sub problems.

2.3. The Proposed Method

In this section, we reduce the calculation of WWA and remove the additional (excess) calculation. We first assume that the setup cost, the inventory holding cost and the cost of purchasing items are constants. Our procedure for building the solution is the same as the forward recursive WWA except that we avoid calculating some branches, as justified by the proof of the following proposition. The first assumption is that the inventory holding cost and the production cost are fixed. Therefore we have,

$$K(x_t, I_t) = \begin{cases} I_t \\ \dfrac{A}{h} + I_t \end{cases}$$

Proposition

When the ordering costs which include the possible setup are divided by the inventory holding costs per part per period, the ordering costs are expressed in the Derived Part Periods (DDP). Therefore we have,

DPP = A/H

When a new period j is added with ($j>i$) and the sum of the demand in period $(i+1)$ to j is less than DPP we have,

$$DPP_T^2 = DPP_{T-1} + D_T$$

and

$$DPP_0 = 0.$$

Proof

The proof is established by induction. Start by expanding Equation (1) for the cases $t=1$, $t=2$and $t=3$ and summarizing the costs associated with all alternatives in the last order positions as follows:

Tab. 1.The cost for each period

j / i	1	2	3	4
1	A	$A+H\times D_2$	$A+H\times(D_2+2D_3)$	$A+H\times(D_2+2D_3+3D_4)$
2		f_1+A	$f_1+A+H\times D_3$	$f_1+A+H\times(D_3+2D_4)$
3			f_2+A	$f_2+A+H\times D_4$
4				f_3+A

Let $x_t = 0$.

For $j=2$

If $f_1 + H \times D_2 > A$ then the demand for period 2 is not placed in period 1. In other word,

$$f_1 = A \qquad x_t > 0 \Rightarrow h \times D_2 > A \Rightarrow D_2 > A/h$$

For $j=3$

If $A + H \times D_2 + H \times 2 \times D_3 > f_1 + A + D_3 \times H$ the demand for the second period is not placed in the first period,

$$f_1 = A \Rightarrow h \times D_2 + h \times D_3 > A \Rightarrow D_2 + D_3 > A/h$$

Note that when we compare the period i with $i+1$, we assume all $i-1$ periods are ordered in the same period which is the worst possible state and this completes the proof. ■

2.4. Example:

Suppose we want to plan for 12 periods, the beginning and the ending inventory levels are equal to zero, and the backordering is not permitted. Table (2) shows the information of demand. Assume that the inventory holding costs per part per period is equal to 0.4 (unit), and the ordering cost and the set up costs or the ordering cost is equal to 54 (unit). Table (3) shows the optimal ordering policy $Q_i^*, i = 1, \cdots, 12$ using traditional method and Table (4) shows the optimal ordering policy using the proposed method of the paper.

Tab. 2. The input data

T	1	2	3	4	5	6
D	10	62	12	130	154	129
T	7	8	9	10	11	12
D	88	52	124	160	238	41

Tab.3. The optimal solutions using WWA

	1	2	3	4	5	6	7	8	9	10	11	12
1	54	78.8	88.4	244.4	490.8	748.8	960	1105.6	1502.4	2078.4	3030.4	3210.8
2		108	112.8	216.8	401.6	608	784	908.8	1256	1768	2624.8	2788.8
3			132.8	184.8	308	462.8	603.6	707.6	1005.2	1453.2	2214.8	2362.4
4				142.4	204	307.2	412.8	496	744	1128	1794.4	1925.6
5					196.4	248	318.4	380.8	579.2	899.2	1470.4	1585.2
6						250.4	285.6	327.2	476	732	1208	1306.4
7							302	322.8	422	614	994.8	1076.8
8								339.6	389.2	517.2	802.8	868.4
9									376.8	440.8	631.2	680.4
10										430.8	526	558.8
11											484.8	501.2
12												538.8

Tab. 4. The optimal solutions using the improved WWA

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	62	86									
2		135	147									
3			197	327								
4				221								
5					356	485						
6						491	579					
7							620	672				
8								714	838			
9									807			
10										942		
11											1077	1118
12												1212

Some the of the details of the computations of the proposed method are summarized as follows,

$$\frac{A}{H} = \frac{54}{0.4} = 135$$

$$D_2 \le 135 \Rightarrow Cell(1,2) = D_2 = 62$$

$$D_2 + D_3 = 74 < 135 \Rightarrow Cell(1,3) = 86$$

$$D_2 + D_3 + D_4 = 62 + 12 + 130 = 204 > 135$$

As we can observe, we do not perform any calculation and move to period 2. In period 3 we have, $D_3 \le 135 \Rightarrow \text{Cell}(2,3) = 147$ since

$D_3 + D_4 = 12 + 130 = 142 > 135$ and we do not perform further calculation and move to period 3. We continue this procedure till we have $D_{12} \le 135$ and Cell(11,12) = 1118. In the next section we present some experimental results to verify the performance of the proposed method.

2.5. Computational Results

In order to evaluate the performance of the proposed method we use randomly generated data in different sizes. The resulted problems are solved using both the traditional method as well as the proposed method of this paper. For both methods, demands are selected randomly distributed with N (μ=0,σ =9) and demand is calculated as $d_t = \mu + \sigma \times z_t$. Table (5) shows the details of the computations.

Tab. 5. Numerical experience for the proposed method

Size		40×40	
Holding cost	0.4	0.8	1.2
CPU Time for the Old Method	0.178	0.18	0.18
CPU Time for the New Method	0.085	0.071	0.0625
Size		60×60	
Holding cost	0.4	0.8	1.2
CPU Time for the Old Method	0.38	0.382	0.39
CPU Time for the New Method	0.15	0.14	0.12
Size		80×80	
Holding cost	0.4	0.8	1.2
CPU Time for the Old Method	0.68	0.68	0.7
CPU Time for the New Method	0.26	0.21	0.187

As we can observe from Table 5, the proposed method of the paper demonstrates better results in terms of CPU time for different sizes.

3. An Extended Proposed Method

In this section, we extend the proposed method of this paper when the inventory holding and set up expenditures are not fixed.

Proposition

For any finite planning-horizon of length *T*, let *l* be any period in which an order is received. When there is an optimal solution such that:

$$\sum_{i=l}^{k} D_i > \frac{A_{l+1}}{h_l}, \qquad (8)$$

D_k cannot be considered for ordering in the Q_L order.

Proof

Start by expanding Equation (5) for the cases *t =1, t =2*and *t=3* and summarize the costs associated with all alternative last order positions as follows:

i \ j	1	2	3	
1	A_1	$A_1 + H_1 \times D_2$	$A_1 + H_1 \times (D_2 + D_3) + H_2 \times D_3$	….
2		$f_1 + A_1$	$f_1 + A_2 + H_2 \times D_3$	….
3			$f_2 + A_3$	….
⋮	⋮	⋮	⋮	⋮

We prove this proposition by examining the first two periods, period 1 and period 2.

For *j=2*

If $A_1 + H_1 \times D_2 > f_1 + A_2$ then we do not place order period 2 in period 1

$$f_1 = A_1 \Rightarrow H_1 \times D_2 > A_2 \Rightarrow D_2 > \frac{A_2}{H_1}$$

For *j=3*

If $A_1 + H_1 \times (D_2 + D_3) + H_2 \times D_3 > f_1 + A_2 + H_2 \times D_3$ then we do not place order period 2 in period 1

$$f_1 = A_1 \Rightarrow H_1 \times (D_2 + D_3) > A_2 \Rightarrow D_2 + D_3 > \frac{A_2}{H_1}$$

We perform similar steps for all the remaining *j*. The following proposition to determine the station is also useful:

i. Order period *i, i+1,…,k* in period *i-1* or
ii. Order period *i-1* in period *i-1* and order *i, i+1,…,k* in period *i*

The total cost for these stations are calculated as follows:

Station 1:

$$K_1 = A_{i-1} + H_{i-1} \times D_i + (H_{i-1} + H_i) \times D_{i+1} + \ldots + (H_{i-1} + H_i + \ldots + H_{k-1}) \times D_k + C_{i-1} \times (D_{i-1} + \ldots + D_K)$$

Station 2:

$$K_2 = A_{i-1} + A_i + H_i \times D_{i+1} + \ldots + (H_i + \ldots + H_{k-1})D_k + \ldots C_{i-1} \times D_{i-1} + C_i \times (D_i + D_{i+1} + \ldots + D_k)$$

If $k_1 < k_2$ then we order period *i,i+1,...,k* in period *i-1* otherwise order them in period *i*

$$k_1 < k_2 \Rightarrow A_{i-1} + H_{i-1} \times D_i + (H_{i-1} + H_i) \times D_{i+1} + \ldots + (H_{i-1} + H_i + \ldots + H_{k-1}) \times D_k < A_{i-1} + A_i + H_i \times D_{i+1} + \ldots + (H_i + \ldots + H_{k-1}) \times D \Rightarrow D_i + D_{i+1} + \ldots + D_k < \frac{A_i}{H_{i-1}} \Rightarrow \sum_{j=i}^{k} D_j < \frac{A_i}{H_{i-1}}$$

Note that if purchasing cost is not fixed then the Equation (8) is changed as follow:

$$\sum_{i=1}^{k} D_i > \frac{A_{l+1}}{H_l + C_l - C_{l+1}} \qquad if\ (H_l + C_l - C_{l+1}) \geq 0$$

$$\sum_{i=1}^{k} D_i < \frac{A_{l+1}}{H_l + C_l - C_{l+1}} \qquad if\ (H_l + C_l - C_{l+1}) < 0$$

and D_k is not placed in the Q_l order. Therefore, the following hold,

i. order period *i, i+1,...,k* in period *i-1* or
ii. order period *i-1* in period *i-1* and order *i, i+1,...,k* in period *i*

The total cost for these stations are also calculated as follows:

Station 1:

$$K_1 = A_{i-1} + H_{i-1} \times D_i + (H_{i-1} + H_i) \times D_{i+1} + \ldots + (H_{i-1} + H_i + \ldots H_{k-1}) \times D_k) + C_{i-1} \times (D_{i-1} + \ldots + D_k).$$

Station 2:

$$K_2 = A_{i-1} + A_i + H_i \times D_{i+1} + \ldots + (H_i + \ldots H_{k-1}) \times D_k + C_{i-1} \times D_{i-1} + C_i \times (D_i + D_{i+1} + \ldots + D_k).$$

If $K_1 < K_2$ then we order period i, i+1,…, k in period i-1 unless order them in period i

$$K_1 < K_2 \Rightarrow$$

$$\sum_{i=1}^{k} D_i > \frac{A_{l+1}}{H_l + C_l - C_{l+1}} \qquad if\ (H_l + C_l - C_{l+1}) \geq 0$$

$$\sum_{i=1}^{k} D_i < \frac{A_{l+1}}{H_l + C_l - C_{l+1}} \qquad if\ (H_l + C_l - C_{l+1}) < 0$$

and this completes the proof. ■

Using the proposed method of this paper we determine Q_i^*, i = 1, 2…12 such that all demands for the 12 periods are met at a minimum total cost.

Example:

Suppose there are 12 periods, the beginning and the ending inventories are equal to zero, and the back-order is not permitted. The other necessary information of demand, setup costs and inventory holding cost are shown in the Table 6.

Tab. 6. the input information of the example

I	1	2	3	4	5	6
A_i	85	102	102	101	98	114
H_i	1.1	1	1	1	1	1
D_i	69	29	36	61	61	26
I	7	8	9	10	11	12
A_i	105	86	119	110	98	114
H_i	1	1.1	1.2	1.2	1.2	1.2
D_i	34	67	45	67	79	56

Tab. 7. Numerical results of the optimal policy

	1	2	3	4	5	6	7	8	9	10	11	12
1	85	116.9	192.5	-								
2		187	223	345	-							
3			218.9	279.9	-							
4				293.5	354.5	406.5	-					
5					377.9	403.9	471.9	-				
6						468.5	502.5	638.5	-			
7							508.9	575.9	-			
8								557.9	607.4	-		
9									676.9	757.3	-	
10										717.4	812.2	-
11											815.4	882.6
12												926.2

$$K^* = 882.6$$

$$Q_1^* = D_1 + D_2 = 69 + 29 = 98$$

$$Q_3^* = D_3 + D_4 = 97$$

$$Q_5^* = D_5 + D_6 + D_7 = 121$$

$$Q_8^* = D_8 + D_9 = 112$$

$$Q_{11} = D_{11} + D_{12} = 79 + 56$$

Next section, the performance of the proposed method is examined using some randomly selected test problems.

3-1. Numerical Results

In order to study the performance of the proposed method of this paper we generate some input data. We have examined the performance of our proposed methods with the performance of traditional WWA. The demand, the inventory holding and the setup costs are selected using random numbers with normal distribution with N (30,9), N (2,0.8) , N(100,10), .

Tab. 8. Numerical experience for the proposed method versus the traditional one

Size	**40×40**	
Holding cost	N(2,0.8)	N(3,1)
CPU Time for the Old Method	0.45	0.5
CPU Time for the New Method	0.25	0.11
Size	**60×60**	
Holding cost	N(2,0.8)	N(3,1)
CPU Time for the Old Method	1.32	1.4
CPU Time for the New Method	0.30	0.27
Size	**80×80**	
Holding cost	N(2,0.8)	N(3,1)
CPU Time for the Old Method	2.80	2.91
CPU Time for the New Method	0.5	0.37

Table (8) summarizes the results of the performance of the proposed method versus the old one for three different test examples. For each test problem, we use different input data. The results indicate that the proposed method can reach the optimal solution much faster than the old technique.

4. Conclusions

In this paper we have presented a new modified WWA method. We have shown that the proposed method of this paper could reduce the burden of the computations, significantly. The implementation of the proposed method has been demonstrated using some numerical examples and using some randomly generated test problems, the performance of the proposed method with the traditional WWA have been compared. As a future research, one can consider the input data in an uncertain environment where neither traditional method nor the proposed method can be implemented directly and we leave it for future research.

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