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Information and Covariance Matrices for Multivariate Pareto (IV), Burr, and Related Distributions

GH. YARI and A. M. D JAFARI

Abstract: Main result of this paper is to derive the exact analytical expressions of information and covariance matrix for multivariate Pareto, Burr and related distributions. These distributions arise as tractable parametric models in reliability, actuarial science, economics, finance and telecommunications. We showed that all the calculations can be obtained from one main moment multidimensional integral whose expression is obtained through some particular change of variables. Indeed, we consider that this calculus technique for that improper integral has its own importance.

Keywords: gamma and beta functions, polygamma functions; Information matrix, covariance matrix; multivariate pareto models.

1. Introduction

In this paper the exact form of Fisher information matrix for multivariate Pareto (IV) and related distribution is determined. It is well-known that the information matrix is a valuable tool for derivation of covariance matrix in the asymptotic distribution of maximum likelihood estimations (MLE).

In the univariate case of the above distributions, the Fisher information matrix is found by Brazauskas [1]. As discussed in Serfling [2], section 4, under suitable regularity conditions, the determinant of the asymptotic covariance matrix of (MLE) reaches an optimal lower bound for the volume of the spread ellipsoid estimators.

In the univariate case of the above Pareto (IV), this optimality property of (MLE) is widely used in the robustness versus efficiency studies as a quantitative benchmark for efficiency considerations Brazauskas and Serfling [3,4], Brazauskas [5], Hampel et al [6], Abramowitz and Stegum [7], Huber [8], Klugman [9], Kimber [10,11] and Lehmann [12], chapter 5. These distributions are suitable for situations involving relatively high probability in the upper tails.

More specifically, such models have been formulated in the context of actuarial science, reliability, economics, finance and teletrafic.

These models arise whenever we need to infer the distributions of variables such as sizes of insurance claims, sizes of firms, income, and income in a population of people, stock price fluctuations and length of telephone calls.

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Dr. GH. YARI, *Iran University of Science and Technology, yari@iust.ac.ir*
A.M. DJAFARI, *Supelec, Plateau de Moulon,djafari@lss.supelec.fr*

For a broad discussion of pareto models and diverse applications see Amold [13], Johnson, Kotz and Balakrishnan [14], Chapter19.

Gomes, Selman and Crato [15] have recently discovered pareto (IV) tail behavior in the cost distributions of combinatorial search algorithms.This paper is organized as follows: Multivariate Pareto and Burr distributions are introduced and presented in section 2. Elements of the information and convariance matrix for multivariate pareto (IV) is derived in section 3. Elements of the informations matrices for multivariate Burr, Pareto (III), and Pareto (II) distributions are derived in section 4. Conclusion is presented in section 5. Derivation of first and second derivatives of the log density and the main moment integral calculations are given in Appendices A and B.

2. Multivariate Pareto and Burr distributions

As discussed in Arnold [13] Chapter 3, a hierarchy of Pareto distribution is established by starting with the classical Pareto(I) distribution and subsequently introducing additional parameters related to location, scale, shape and inequality (Gini index).

Such as approach leads to a very general family of distributions, called the Pareto (IV) family, with the cumulative distribution function

$$F_X(x) = 1 - \left(1 + \left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-\alpha}, \quad x > \mu, \tag{1}$$

where $-\infty < \mu < +\infty$ is the location parameter, $\theta > 0$ is the scale parameter, $\gamma > 0$ is the inequality parameter and $\alpha > 0$ is the shape parameter which characterizes the tail of the distribution. We denote this distribution by Pareto (IV) $(\mu, \theta, \gamma, \alpha)$. Parameter

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γ is called the inequality parameter because of its interpretation in the economics context.
That is, if we choose $\alpha = 1$ and $\mu = 0$ in expression (1), the parameter $(\gamma \le 1)$ is precisely the Gini index of inequality.
For the Pareto (IV) $(\mu, \theta, \gamma, \alpha)$ distribution, we have the density function

$$f_X(x) = \frac{\alpha\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\theta\gamma\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}}, x > \mu. \quad (2)$$

The density of the n-dimensional Pareto (IV) distribution is

$$f_n(\mathbf{x}) = \left(1+\sum_{j=1}^{n}\left(\frac{x_j-\mu_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)} \times \prod_{i=1}^{n}\frac{\alpha+i-1}{\theta_i\gamma_i}\left(\frac{x_i-\mu_i}{\theta_i}\right)^{\frac{1}{\gamma_i}-1}, \; x_i > \mu_i, \quad (3)$$

where $\mathrm{x} = [x_1,...,x_n]$, $x_i > \mu_i$, $\alpha > 0$, $\gamma_i > 0$, *and* $\theta_i > 0$ *for* $i = 1,...,n$.

One of the main properties of this distribution is that, the joint density of any subset of the components of a Pareto random vector is again of the form (3) [13].
The n-dimensional Burr distribution has the density

$$f_n(\mathrm{x}) = \left(1+\sum_{j=1}^{n}\left(\frac{x_j-\mu_j}{\theta_j}\right)^{c_j}\right)^{-(\alpha+n)} \times \prod_{i=1}^{n}\frac{(\alpha+i-1)c_i}{\theta_i}\left(\frac{x_i-\mu_i}{\theta_i}\right)^{c_i-1}, \quad x_i > \mu_i, \quad (4)$$

Where
$x_i > \mu_i$, $\alpha > 0$, $c_i > 0$, $\theta_i > 0$ *for* $i = 1,...,n$.
We note that multivariate Burr distribution equivalent to the multivariate Pareto distribution with $\frac{1}{\gamma_i} = c_i$.

3. Information Matrix for Multivariate Pareto (IV)

Suppose X is a random vector with the probability density functions $f_\Theta(.)$ where $\Theta = (\theta_1, \theta_2, ..., \theta_K)$.
The information matrix $I(\Theta)$ is the $K \times K$ matrix with elements

$$I_{ij}(\Theta) = -E_\Theta\left[\frac{\partial^2 \ln f_\Theta(X)}{\partial\theta_i\partial\theta_j}\right], i, j = 1,..K \quad (5)$$

For the multivariate Pareto (IV), we have $\Theta = (\mu_1,...,\mu_n,\theta_1,...,\theta_n,\gamma_1,...,\gamma_n,\alpha)$.
In order to make the Multivariate Pareto (IV) distribution a regular family (in terms of maximum likelihood estimation), we assume that μ is known and, without loss of generality, equal to 0.
In this case information matrix is $(2n+1)\times(2n+1)$. Thus, further treatment is based on the following multivariate density function

$$f_n(\mathbf{x}) = \left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)} \prod_{i=1}^{n}\frac{\alpha+i-1}{\theta_i\gamma_i}\left(\frac{x_i}{\theta_i}\right)^{\frac{1}{\gamma_i}}, x_i > 0. \quad (6)$$

The log-density is:

$$\ln f_n(\mathrm{x}) = \sum_{i=1}^{n}\left[\ln(\alpha+i-1) - \ln\theta_i + \left(\frac{1}{\gamma_i}-1\right)\ln\left(\frac{x_i}{\theta_i}\right) - \ln\gamma_i\right] - (\alpha+n)\ln\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right). \quad (7)$$

Since the information matrix $I(\Theta)$ is symmetric it is enough find elements $I_{ij}(\Theta)$, where $1 \le i \le j \le 2n+1$.
The required first and second partial derivatives of the above expression are given in the Appendix A.
Looking at these expressions, we see that to determine the expression of the information matrix and score functions, we need to find the expressions of:

$$E\left[\ln\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)\right], \quad E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{\eta}{\gamma_l}}\right],$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{l}{\gamma_l}}\right] \; E\left[\ln\left(\frac{X_l}{\theta_l}\right)\right], E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{\eta}{\gamma_l}}\ln\left(\frac{X_l}{\theta_l}\right)\right],$$

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$$E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right],$$

and the general terms

$$E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{n_1}{\gamma_l}}\left(\frac{X_k}{\theta_k}\right)^{\frac{n_2}{\gamma_k}}\ln^{n_4}\left(\frac{X_k}{\theta_k}\right)\ln^{n_3}\left(\frac{X_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{n_5}}\right],$$

$$(n_1, n_2 > -1) \in R, n_3, n_4 \in N^+ and \quad n_5 \in R^+.$$

3.1. Main Strategy to Obtain Expression of the Expectations

Derivation of these expressions are based on the following strategy: first, we derive an analytical expression for the following integral

$$E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}}\right]=\int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{i=1}^{n}\left(\frac{x_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}} f_n(\mathrm{x})d\mathbf{x}, \quad (8)$$

and then, we show that all the other expressions can be found easily from it.

We consider this derivation as one of the main contributions of this work.

This derivation is given in the Appendix B. The result is the following:

$$E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}}\right]$$

$$=\int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{i=1}^{n}\left(\frac{x_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}} f_n(\mathrm{x})\, d\mathbf{x}$$

$$=\frac{\Gamma(\alpha-\sum_{i=1}^{n} r_i)\prod_{i=1}^{n}\Gamma(r_i+1)}{\Gamma(\alpha)} \quad (9)$$

$$,\sum_{i=1}^{n} r_i < \alpha, r_i > -1, r_i \in R,$$

where Γ is the usual Gamma function,

$$\Gamma_{r_l r_k}\left(\alpha-\sum_{i=1}^{n} r_i\right)=\frac{\partial^2\Gamma(\alpha-\sum_{i=1}^{n} r_i)}{\partial r_k \partial r_l}$$

$$,1 \le l,k \le n,$$

$$\Psi^{(n)}(z)=\frac{d^n}{dz^n}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right), \quad z>0$$

$$,\frac{\partial^{(m+n)}}{\partial r_l^m \partial r_k^n}\left(\frac{\Gamma_{r_l r_k}(z)}{\Gamma(z)}\right)=\Psi^{(m+n)}(z), \quad z>0$$

and integers $n, m \ge 0$. Specifically, we use digamma $\Psi(z)=\Psi^{(.)}(z)$, trigamma $\Psi'(z)$ and $\Psi_{r_l r_k}(z)$ functions. To confirm the regularity of $\ln f_n(\mathbf{x})$ and evaluation the expected Fisher information matrix, we take expectations of the first and second order partial derivatives of (7). All the other expressions can be derived from this main result. Taking of derivative with respect to α, from the both sides of the relation

$$1=\int_0^{+\infty} f_n(\mathbf{x})d\mathbf{x}, \quad (10)$$

leads to

$$E\left[\ln\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{r_j}}\right)\right]=\sum_{i=1}^{n}\frac{1}{\alpha+i-1}. \quad (11)$$

From relation (9), for a pair of (*l*,k) we have

$$\varphi(r_l, r_k)=E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{r_l}{\gamma_l}}\left(\frac{X_k}{\theta_k}\right)^{\frac{r_k}{\gamma_k}}\right]= \quad (12)$$

$$\frac{\Gamma(\alpha-r_l-r_k)\Gamma(r_l+1)\Gamma(r_k+1)}{\Gamma(\alpha)},$$

and

$$\frac{\partial^{(n_3+n_4)}}{\partial r_l^{n_3}\partial r_k^{n_4}}\varphi(r_l=n_1, r_k=n_2)= \quad (13)$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{n_1}{\gamma_l}}\left(\frac{X_k}{\theta_k}\right)^{\frac{n_2}{\gamma_k}}\ln^{n4}\left(\frac{X_k}{\theta_k}\right)\ln^{n_3}\left(\frac{X_l}{\theta_l}\right)\right],$$

From relation(12), at $r_k=0$ we obtain

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{r_l}{\gamma_l}}\right]=\frac{\Gamma(\alpha-r_l)\Gamma(r_l+1)}{\Gamma(\alpha)}, \quad (14)$$

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and evaluating this expectation at $r_l = 1$ we obtain

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}\right] = \frac{1}{\alpha - 1}. \tag{15}$$

Writing the expression of the expectation

$$E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right],$$

as E_α to emphasis the role of the parameter α in (6), it can easily by shown that

$$E_\alpha\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right]$$

$$= \frac{\alpha}{\alpha+n} E_{\alpha+1}\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}\right]. \tag{16}$$

Using (15) with α replaced by $\alpha+1$, we now obtain an expression for the last expectation as

$$E_\alpha\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right] = \frac{1}{\alpha+n}.$$

Differentiating (14) with respect to r_l, and replacing for $r_l = 0$ and $r_l = 1$, we obtain the following relations:

$$E\left[\ln\left(\frac{X_l}{\theta_l}\right)\right] = \gamma_l[\Gamma'(1) - \Psi(\alpha)], \tag{17}$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}} \ln\left(\frac{X_l}{\theta_l}\right)\right] = \gamma_1\left[\frac{\Gamma'(2) - \Psi(\alpha-1)}{\alpha-1}\right], \tag{18}$$

and

$$E_\alpha\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}} \ln\left(\frac{X_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right]$$

$$= \frac{\alpha}{\alpha+n} E_{(\alpha+1)}\left[\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}} \ln\left(\frac{X_l}{\theta_l}\right)\right]. \tag{19}$$

3.2. Expectations of the Score Functions

The expectations of the first three derivations of the first order follow immediately from the corresponding results for their three corresponding parameters and we obtain:

$$E\left[\frac{\partial 1nf_n(\mathrm{X})}{\partial\alpha}\right]$$

$$= \sum_{i=1}^{n}\frac{1}{\alpha+i-1} - E\left[1n\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)\right] = 0, \tag{20}$$

$$E\left[\frac{\partial 1nf_n(\mathrm{X})}{\partial\theta_l}\right]$$

$$= -\frac{1}{\theta_l\gamma_l} + \left(\frac{\alpha+n}{\theta_l\gamma_l}\right) E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_i}}\right)}\right] = 0, \tag{21}$$

$$E\left[\frac{\partial 1nf_n(\mathbf{X})}{\partial\gamma_l}\right] = -\frac{1}{\gamma_l} - \frac{1}{\gamma_l^2} E\left[1n\left(\frac{X_l}{\theta_l}\right)\right]$$

$$+ \left(\frac{\alpha+n}{\gamma_l^2}\right) E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{\frac{1}{\gamma_l}} 1n\left(\frac{X_l}{\theta_l}\right)}{1+\sum_{j=1}^{n}\left(\frac{X_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}}\right] = 0. \tag{22}$$

3.3. The Expected Fisher Information Matrix

Main strategy is again based on the integral (9) which is presented in the Appendix B. However, derivation of the following expressions can be obtained

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mechanically but after some tedious algebraic simplifications:

$$I_{\mathbf{x}}(\alpha)=\sum_{i=1}^{n}\frac{1}{(\alpha+i-1)^2}, \tag{23}$$

$$I_{\mathbf{x}}(\theta_l,\alpha)=-\frac{1}{\theta_l\gamma_l(\alpha+n)}, \tag{24}$$

$$I_{\mathbf{x}}(\gamma_l,\alpha)=-\frac{1}{\gamma_l(\alpha+n)}[\Gamma'(2)-\Psi(\alpha)] \tag{25}$$

$, l=1,...,\ n,$

$$I_{\mathbf{x}}(\theta_l)=\frac{\alpha+n-1}{\theta_l^2\gamma_l^2(\alpha+n+1)},\quad l=1,...,n, \tag{26}$$

$$I_{\mathbf{x}}(\gamma_l)=\frac{\alpha+n-1}{\gamma_l^2(\alpha+n+1)}\left[\frac{\Gamma''(\alpha)}{\Gamma(\alpha)}+\Gamma''(1)+1\right]$$
$$+\frac{2(\alpha+n-2)}{\gamma_l^2(\alpha+n+1)}[\Gamma'(1)-\Psi(\alpha)]$$
$$-\frac{2(\alpha+n-1)}{\gamma_l^2(\alpha+n+1)}[\Gamma'(1)\Psi(\alpha)] \tag{27}$$

$, l=1,...,\ n,$

$$I_{\mathbf{x}}(\theta_l,\theta_k)=-\frac{1}{\theta_l\gamma_l\gamma_k\theta_k(\alpha+n+1)},\qquad k\neq l, \tag{28}$$

$$I_{\mathbf{x}}(\gamma_l,\gamma_k)=-\frac{1}{\gamma_l\gamma_k(\alpha+n+1)} \tag{29}$$
$$[(\Gamma'(2))^2-\Gamma'(2)(\Psi_{r_l}(\alpha)+\Psi_{r_k}(\alpha))+\Psi_{r_lr_k}(\alpha))]\ ,k\neq l,$$

$$I_{\mathbf{x}}(\theta_l,\gamma_k)=-\frac{1}{\theta_l\gamma_l\gamma_k(\alpha+n+1)} \tag{30}$$
$$[(\Gamma'(2)-(\Psi_{r_k}(\alpha)],\ k\neq l,$$

$$I_{\mathbf{x}}(\theta_l,\gamma_l)=\frac{\alpha+n-1}{\theta_l\gamma_l^2(\alpha+n+1)}[(\Gamma'(2)-(\Psi(\alpha)] \tag{31}$$
$$-\left[\frac{1}{\theta_l\gamma_l^2(\alpha+n+1)}\right],l=1,...,n.$$

Thus the information matrix, IMP (IV) (Θ), for the multivariate Pareto(IV) $(0,\theta,\gamma,\alpha)$ distribution is

$$I_{MP}(IV)(\Theta)=\begin{bmatrix}I(\theta_l,\theta_k) & I(\theta_l,\gamma_k) & I(\theta_l,\alpha)\\ I(\theta_l,\gamma_k) & I(\gamma_l,\gamma_k) & I(\gamma_l,\alpha)\\ I(\theta_l,\alpha) & I(\gamma_l,\alpha) & I(\alpha)\end{bmatrix}. \tag{32}$$

3.4. Covariance Matrix for Multivariate Pareto (IV)
Since the joint density of any subset of the components of a Pareto (IV) random vector is again a multivariate Pareto (IV), Arnold [13], we can calculate the expectation

$$E\left[\left(\frac{X_l-\mu_l}{\theta_l}\right)^{m_l}\left(\frac{X_k-\mu_k}{\theta_k}\right)^{m_k}\right]=$$

$$\int_0^\infty\int_0^\infty\left(\frac{x_l-\mu_l}{\theta_l}\right)^{m_l}\left(\frac{x_k-\mu_k}{\theta_k}\right)^{m_k}$$
$$f_{X_l,X_k}(x_l,x_k)dx_l\,dx_k=$$

$$\frac{\Gamma(\alpha-m_l\gamma_l-m_k\gamma_k)\Gamma(m_l\gamma_l+1)\Gamma(m_k\gamma_k+1)}{\Gamma(\alpha)},$$

$$m_l,m_k\in R, m_l\gamma_l,\ m_k\gamma_k>-1,$$
$$\alpha-m_l\gamma_l-m_k\gamma_k>0. \tag{33}$$

Evaluating this expectation at $(m_l=1, m_k=0)$, $(m_l=0, m_k=1)$ $and (m_l=1,\ m_k=1)$, we obtain

$$E[X_l]=\mu_l+\frac{\theta_l}{\Gamma(\alpha)}[\Gamma(\alpha-\gamma_l)\Gamma(\gamma_l+1)], \tag{34}$$
$$\gamma_l<\alpha,\gamma_l>-1,$$

$$E[X_k]=\mu_k+\frac{\theta_k}{\Gamma(\alpha)}[\Gamma(\alpha-\gamma_k)\Gamma(\gamma_k+1)], \tag{35}$$
$$\gamma_k<\alpha,\gamma_k>-1$$

$$E[X_lX_k]=\mu_kE[X_l]+\mu_lE[X_k]-\mu_l\mu_k$$

$$+\frac{\theta_l\theta_k}{\Gamma(\alpha)}[\Gamma(\alpha-\gamma_l-\gamma_k)\Gamma(\gamma_l+1)\Gamma(\gamma_k+1)], \tag{36}$$
$$\gamma_l+\gamma_k<\alpha,$$

$$E[X_l^m]=\frac{\theta_l^m}{\Gamma(\alpha)}[\Gamma(\alpha-m_l\gamma_k)\Gamma(m_l\gamma_k+1)], \tag{37}$$
$$\gamma_km_l<\alpha,\ -1<\gamma_km_l$$

$$\sigma^2_{X_l}=\frac{\theta_l^2}{\Gamma^2(\alpha)}[\Gamma(\alpha-2\gamma_l)\Gamma(2\gamma_l+1)\ \Gamma(\alpha)$$
$$-\Gamma^2(\gamma_l+1)\Gamma^2(\alpha-\gamma_l)],\qquad 2\gamma_l<\alpha, \tag{38}$$
$$\gamma_l>-1,\quad 2\gamma_l>-1,\quad \gamma_l<\alpha,$$

$$Cov[X_l,X_k]=\frac{\theta_l\theta_k\Gamma(\gamma_l+1)\Gamma(\gamma_k+1)}{\Gamma^2(\alpha)}$$
$$[\Gamma(\alpha-\gamma_l-\gamma_k)\Gamma(\alpha)-\Gamma(\alpha-\gamma_k)\Gamma(\alpha-\gamma_l)] \tag{39}$$
$$1\le l\le k\le n,\ \ k=2,...,n,\quad \gamma_l+\gamma_k<\alpha,\quad \gamma_l,\gamma_k<\alpha,$$
$$\gamma_l,\gamma_k>-1.$$

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4. Special Cases

4.1. Burr (θ,γ,α) distribution

The Burr family distributions are also sufficiently flexible and enjoy long popularity in the actuarial science literature (Daykin, Pentikainen, and Pesonen [16] and Klugman, Panjer, and Willmot [9]).

However, this family can be treated as a special case of Pareto (IV). Burr (θ,γ,α) = Pareto (IV) $(0,\theta,\frac{1}{\gamma},\alpha)$ (Klugman, Panjer, and Willmot [9], p.574).

Since the Burr distribution is a reparametrization of Pareto (IV) $(0,\theta,\gamma,\alpha)$, it follows from Lehmann (8), Section 2.7, that its information matrix $I_B(\Theta)$ can be derived from $I_{P(IV)}(\Theta)$ *by* $JI_{P(IV)}(\Theta)J'$, where J is the Jacobian matrix of the transformation of variables.

Thus, the transformation matrix of multivariate Burr distribution, $I_{MB}(\Theta)$ is then given by $JI_{MP(IV)}(\Theta)J'$, where

$$J=\begin{bmatrix} I & 0 & 1 \\ 0 & I\gamma^2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{40}$$

which is obtained by noting that J is the Jacobian matrix of the transformation

$(\theta,\gamma,\alpha) \rightarrow (\theta,\frac{1}{\gamma},\alpha)$

4. 2. Pareto (III) $(0,\theta,\gamma)$ Distribution

This is a special case of Pareto (IV) with $\alpha=1.$

Therefor, last row and last column of $I_{MP(IV)}(\Theta)$ vanish (these represent information about parameter α) and we obtain

$$I_{MP(III)}(\Theta)=\begin{bmatrix} I(\theta_l,\theta_k) & I(\theta_l,\gamma_k) \\ I(\theta_l,\gamma_k) & I(\gamma_l,\gamma_k) \end{bmatrix}, \tag{41}$$

where we have to substitute $\alpha=1$ in all the remaining expressions.

4. 3. Pareto (II) $(0,\theta,\alpha)$ Distribution

This is a special case of Pareto (IV) with $\gamma=1$. There for $I(\theta_l,\gamma_k),\ I(\gamma_l,\gamma_k)$ and $I(\gamma_l,\alpha)$ in $I_{MP(IV)}(\Theta)$ vanish and we obtain

$$I_{MP(II)}(\Theta)=\begin{bmatrix} I(\theta_l,\theta_k) & I(\theta_l,\alpha) \\ I(\theta_l,\alpha) & I(\alpha) \end{bmatrix}, \tag{42}$$

where we have to substitute $\gamma=1$ in all the remaining expressions.

5. Conclusion

In this paper we obtained the exact form of Fisher information and covariance matrix for multivariate Pareto (IV) distribution.

We showed that all the calculations can be obtained from one main moment multi dimensional integral which has considered and whose expression is obtained through some particular change of variables.

A short method of obtaining some of the expectations as a function of α is used.

To confirm the regularity of the $1n\ f_n(\mathbf{x})$, we showed that the expressions of the score functions are equal to 0. Information matrices of multivariate Burr, Pareto (III) and Pareto (II) distributions are derived as special cases of multivariate Pareto (IV) distribution.

Appendix A

Expression of the derivatives in this Appendix, we give detailed expressions of the first and second derivatives of $1n\ f_n(x)$ which are needed for obtaining expression of the information matrix:

$$\frac{\partial 1nf_n(\mathrm{x})}{\partial\alpha}=\sum_{i=1}^{n}\frac{1}{\alpha+i-1}-1n\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right), \tag{1}$$

$$\frac{\partial 1nf_n(\mathrm{x})}{\partial\theta_l}=-\frac{1}{\theta_l\gamma_l}+\left(\frac{\alpha+n}{\theta_l\gamma_l}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}, \tag{2}$$

$l=1,\ldots,n,$

$$\frac{\partial 1nf_n(\mathbf{x})}{\partial\gamma_l}=-\frac{1}{\gamma_l}-\frac{1}{\gamma_l^2}1n\left(\frac{x_l}{\theta_l}\right)+\left(\frac{\alpha+n}{\gamma_l^2}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}1n\left(\frac{x_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)} \tag{3}$$

, $l=1,\ldots,n,$

$$\frac{\partial^2 \ln f_n(X)}{\partial\alpha^2}=-\sum_{i=1}^{n}\frac{1}{(\alpha+i-1)^2} \tag{4}$$

$$\frac{\partial^2 1nf_n(\mathbf{x})}{\partial\theta_k\partial\alpha}=\left(\frac{1}{\theta_k\gamma_k}\right)\frac{\left(\frac{x_k}{\theta_k}\right)^{\frac{1}{\gamma_k}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}, \tag{5}$$

$k=1,\ldots,n,$

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$$\frac{\partial^2 1nf_n(\mathbf{x})}{\partial\gamma_k\partial\alpha}=\left(\frac{1}{\gamma_k^2}\right)\frac{\left(\frac{x_k}{\theta_k}\right)^{\frac{1}{\gamma_k}}1n\left(\frac{x_k}{\theta_k}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}, \quad (6)$$

$k = 1,..., n,$

$$\frac{\partial^2 1nf_n(\mathbf{x})}{\partial\theta_l^2}=\frac{1}{\theta_l^2\gamma_l}-\left(\frac{\alpha+n}{\theta_l^2\gamma_l}\right)\left(1+\frac{1}{\gamma_l}\right)\times\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}+\left(\frac{\alpha+n}{\theta_l^2\gamma_l^2}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^2}, \quad (7)$$

$l = 1,..., n,$

$$\frac{\partial^2 1nf_n(x)}{\partial\theta_\gamma^2}=\frac{1}{\gamma_l^2}+\frac{2}{\gamma_3}1n\left(\frac{x_l}{\theta_1}\right)-2\left(\frac{\alpha+n}{\gamma_l^3}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}1n\left(\frac{x_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}$$

$$-\left(\frac{\alpha+n}{\gamma_l^4}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}1n^2\left(\frac{x_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}+\left(\frac{\alpha+n}{\gamma_l^4}\right)\left(\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}1n\left(\frac{x_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right)^2, \quad (8)$$

$l = 1,..., n,$

$$\frac{\partial^2 1nf_n(\mathrm{x})}{\partial\theta_k\partial\theta_l}=\left(\frac{\alpha+n}{\gamma_l\gamma_k\theta_l\theta_k}\right)\times\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}\left(\frac{x_k}{\theta_k}\right)^{\frac{1}{\gamma_k}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^2}, \quad (9)$$

$k \neq l,$

$$\frac{\partial^2 1nf_n(\mathrm{x})}{\partial\gamma_k\partial\gamma_l}=\left(\frac{\alpha+n}{\gamma_l^2\gamma_k^2}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}\left(\frac{x_k}{\theta_k}\right)^{\frac{1}{\gamma_k}}1n\left(\frac{x_l}{\theta_l}\right)1n\left(\frac{x_k}{\theta_k}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^2}, \quad (10)$$

$k \neq l,$

$$\frac{\partial^2 1nf_n(\mathbf{x})}{\partial\gamma_k\partial\theta_l}=\left(\frac{\alpha+n}{\gamma_l\theta_l\gamma_k^2}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}\left(\frac{x_k}{\theta_k}\right)^{\frac{1}{\gamma_k}}1n\left(\frac{x_k}{\theta_k}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^2}, \quad (11)$$

$k \neq l,$

$$\frac{\partial^2 1nf_n(\mathrm{x})}{\partial\gamma_l\partial\theta_l}=\frac{1}{\theta_l\gamma_l^2}-\left(\frac{\alpha+n}{\theta_l\gamma_l^2}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}-\left(\frac{\alpha+n}{\theta_l\gamma_l^3}\right)\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}1n\left(\frac{x_l}{\theta_l}\right)}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}$$

$$+\left(\frac{\alpha+n}{\theta_l\gamma_l^3}\right)\left(\frac{\left(\frac{x_l}{\theta_l}\right)^{\frac{1}{\gamma_l}}}{\left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)}\right)^2 1n\left(\frac{x_l}{\theta_l}\right), \quad (12)$$

$l = 1,..., n.$

Appendix B.
Expression of the Main Integral

This Appendix gives one of the main results of this paper which is the derivation of the expression of the following integral

$$E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}}\right]=\int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{i=1}^{n}\left(\frac{x_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}} \quad (1)$$

$f_n(\mathrm{x})\ d\ \mathrm{x},$

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where $f_n(\mathbf{x})$ is the multivariate Pareto (IV) density function (3). This derivation is done in the following steps:
First consider the following one dimensional integral:

$$C_1 = \int_0^{+\infty} \frac{\alpha}{\theta_1 \gamma_1} \left(\frac{x_1}{\theta_1}\right)^{\frac{r_1}{\gamma_1}} \left(\frac{x_1}{\theta_1}\right)^{\frac{1}{\gamma_1}-1} \left(1+\sum_{j=1}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)} dx_1$$

$$= \int_0^{+\infty} \frac{\alpha}{\theta_1 \gamma_1} \left(\frac{x_1}{\theta_1}\right)^{\frac{r_1}{\gamma_1}} \left(\frac{x_1}{\theta_1}\right)^{\frac{1}{\gamma_1}-1}$$

$$\times \left(1 + \sum_{j=2}^{n} \left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)}$$

$$\times \left(1+\frac{\left(\frac{x_1}{\theta_1}\right)^{\frac{1}{\gamma_1}}}{1+\sum_{j=2}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}}\right) dx_1. \qquad (2)$$

Note that, goings from first line to second line is just a factorizing and rewriting the last term of the integrand. After many reflections on the links between Pareto and Burr families and Gamma and Beta functions, we found that the following change of variable

$$\left(1+\frac{\left(\frac{x_1}{\theta_1}\right)^{\frac{1}{\gamma_1}}}{1+\sum_{j=2}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}}\right) = \frac{1}{1-t}, \quad 0<t<1, \qquad (3)$$

simplifies this integral and guides us to the following result

$$C_1 = \frac{\alpha \Gamma(r_1+1)\Gamma(\alpha+n-r_1-1)}{\Gamma(\alpha+n)} \left(1 + \sum_{j=2}^{n} \left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)+r_1+1}. \qquad (4)$$

Then we consider the following similar expression:

$$C_2 = \int_0^{+\infty} \frac{\alpha(\alpha+1)}{\theta_2 \gamma_2} \frac{\Gamma(r_1+1)\Gamma(\alpha+n-r_1-1)}{\Gamma(\alpha+n)} \times \left(\frac{x_2}{\theta_2}\right)^{\frac{r_2}{\gamma_2}} \left(\frac{x_2}{\theta_2}\right)^{\frac{1}{\gamma_2}-1} \left(1+\sum_{j=2}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)+r_1+1} dx_2$$

$$= \int_0^{+\infty} \frac{\alpha(\alpha+1)}{\theta_2 \gamma_2} \frac{\Gamma(r_1+1)\Gamma(\alpha+n-r_1-1)}{\Gamma(\alpha+n)} \left(\frac{x_2}{\theta_2}\right)^{\frac{r_2}{\gamma_2}} \left(\frac{x_2}{\theta_2}\right)^{\frac{1}{\gamma_2}-1} \left(1+\sum_{j=3}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)+r_1+1}$$

$$\times \left(1+\frac{\left(\frac{x_2}{\theta_2}\right)^{\frac{1}{\gamma_2}}}{1+\sum_{j=3}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}}\right)^{-(\alpha+n)+r_1+1} dx_2,$$

and again using the following change of variable:

$$\left(1+\frac{\left(\frac{x_2}{\theta_2}\right)}{1+\sum_{j=3}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}}\right) = \frac{1}{1-t}, \qquad (5)$$

we obtain:

$$C_2 = \frac{\alpha(\alpha+1)\Gamma(r_1+1)\Gamma(r_2+1)}{\Gamma(\alpha+n)} \times \frac{\Gamma(\alpha+n-r_1-r_2-2)}{\Gamma(\alpha+n)} \times \left(1+\sum_{j=3}^{n}\left(\frac{x_j}{\theta_j}\right)^{\frac{1}{\gamma_j}}\right)^{-(\alpha+n)+r_1+r_2+2}. \qquad (6)$$

Continuing this method, finally, we obtain the general expression:

$$C_n = E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{\frac{r_i}{\gamma_i}}\right] = \Gamma(\alpha-\sum_{i=1}^{n} r_i) \times \frac{\prod_{i=1}^{n}\Gamma(r_i+1)}{\Gamma(\alpha)}, \qquad (7)$$

$$\sum_{i=1}^{n} r_i < \alpha, \quad r_{i.} > -1.$$

We may note that to simplify the lecture of the paper we did not give all the details of these calculations.

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