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# Fuzzy Network Analysis for Projects with High Level of Risks – Uncertainty in Time and Structure

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## KEYWORDS

Project network
Fuzzy PERT
Fuzzy structure
Non-convex fuzzy numbers
High-risk projects

## ABSTRACT

*In order to have better insight of project characteristics, different kinds of fuzzy analysis for project networks have been recently proposed, most of which consider activities duration as the main and only source of imprecision and vagueness, but as it is usually experienced in real projects, the structure of the network is also subject to changes. In this paper we consider three types of imprecision namely activity duration, activity existence and precedence relation existence which make our general fuzzy project network. Subsequently, a corrected forward recursion is proposed for analysis of this network. Since the convexity and normalization of traditional fuzzy numbers are not satisfied, some corrected algebraic operations are also presented. Employing the proposed method for a real project reveals that our method results in more applicable and realistic times for activities and project makespan in comparison to Classic fuzzy PERT.*



## 1. Introduction

A project network is basically defined as a set of activities with some specific characteristics like durations and precedence relations. Considering competitive advantages, project managers and other stakeholders of projects usually want to know more information about other characteristics of project, e.g. project completion time, early and late start (finish) times of activities, float times, and critical or risky activities. Therefore some methods have been developed to satisfy these needs. The well-known Critical Path Method (CPM) is one of the first methods which is widely applied for analyzing deterministic networks [1]. Although CPM is the most favorable method for project managers due to its simplicity, in real world it barely results in reliable outputs especially in the case of projects with high levels of risks and uncertainty.

To deal with project networks having uncertain and imprecise data, firstly the Program Evaluation and Review Technique (PERT) has been proposed [2]. The PERT, using probability theory, considers uncertainty in activities duration. Because of its simplification assumptions many researchers criticize the PERT method. Although many extending papers eliminated some deficiencies of the PERT, e.g. GERT type networks has been proposed to consider more uncertainty in projects [3], or Theory of Constraints has been recommended to apply for high-risk projects [4], there is still a fundamental and restricting assumption of using probability as a tool for expressing imprecision and uncertainty, which is one of the most important critics of the PERT [5].

The following deficiencies for using probability in project network analysis can be derived:

1) Probability axioms in decision making processes are barely satisfied, e.g. all possible events are not usually identified from the beginning of the project as it is known by progressive elaboration characteristic of project [6], and of course sum of the probabilities of all identified events does not equal one.

2) Uniqueness is the other major characteristic of projects, i.e. a project consists of non-repetitive activities and events. Therefore, project analysts usually do not have enough historical information to use probabilistic distributions.

Considering disadvantages of using probability in project network analysis, at the end of 70s and the first

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of 80s fuzzy project network analysis has been proposed [7]. There are two major concerns about fuzzy project network analysis. Some researchers have concentrated on determining possible values for project completion time, project critical paths and their degrees of criticality [8]; [9]; [10]; [11], and most others have worked on determining possible values for early and late activity times and activity floats [12]; [13]; [14]; [15]. The method used in the most of these researches is called fuzzy PERT, which considers activities duration as fuzzy numbers and tries to find the imprecise project characteristics. The imprecision in the calculated characteristics by fuzzy PERT is arisen only from fuzzy activity duration. However, the fuzzy structure of the project networks can also affects the output imprecision, which is barely considered in literature. Although Gavareshki [16], Mares [10], and Sharafi [17] have developed procedures to consider this kind of imprecision, there is no straight forward method yet which can calculate the start and finish time of each activity and project.

The aim of this paper is to propose a general fuzzy project network with imprecision in activities duration and network structure which concludes reliable and practical early start and early finish of each activity and project makespan by introducing corrected forward recursion and corrected fuzzy operators.

In section 2 some general concepts and definitions of fuzzy sets and their extensions are introduced. The proposed model for general fuzzy project network with imprecision in time and structure is described in section 3, and the method for considering these kinds of imprecision and analyzing the network with corrected forward recursion are presented in section 4. An example of a real project in section 5 shows the application of proposed method, and finally section 6 gives the results and conclusion.

## 2. General Concepts

This paper employs a combination of fuzzy theory [18] and theory of graph. Since there is no complication in application of graph theory in this text, only some definitions related to the fuzzy sets and fuzzy numbers are given. Most of the following definitions can be found in the fuzzy theory textbooks (e.g. [19]).

A fuzzy set $\tilde{A}$ in $X$ (discourse universe) is a mapping from $x \in X$ to a membership function (a real number in the interval [0,1]), presenting with $\tilde{A} = \{(x, \mu_{\tilde{A}}(x) | x \in X\}$.

A fuzzy set can be converted to a crisp set or multiple crisp sets through notions of support and $\alpha$-cut. The crisp set of those elements of the universe that have a nonzero degree of membership in a fuzzy set $\tilde{A}$ is called *support* of $\tilde{A}$ and represented by $S(\tilde{A})$:

$$S(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\} \tag{1}$$

A more general concept of support is $\alpha$-cut. The crisp set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha$ is called the $\alpha$-cut of $\tilde{A}$:

$$\tilde{A}_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2}$$

If all $\alpha$-cuts of $\tilde{A}$ be convex then we call $\tilde{A}$ a convex fuzzy set, in other words $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$, where $x_1, x_2 \in X$ and $\lambda \in [0,1]$, otherwise $\tilde{A}$ is a non-convex fuzzy number [19].

A very useful kind of fuzzy sets is fuzzy numbers. A fuzzy Number $\widetilde{M}$ is a convex normalized ($\sup \mu_{\tilde{A}}(x) = 1$) fuzzy set of the real line $\mathbb{R}$ that is piecewise continuous.

There are many types of fuzzy numbers. The most general and almost efficient fuzzy number is LR-type with the following membership function:

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\frac{\underline{m} - x}{\alpha}\right) & \text{for } x \geq \underline{m} \\ 1 & \text{for } \underline{m} \leq x \leq \overline{m} \\ R\left(\frac{x - \overline{m}}{\beta}\right) & \text{for } x \leq \overline{m} \end{cases} \tag{3}$$

where $\underline{m}$ and $\overline{m}$ are the upper and lower mean values (real number), and $\alpha$ and $\beta$ are the left and right spreads (greater than zero). The shape function $L$ (and $R$) maps from $\mathbb{R}^+$ to [0,1] with $L(0) = 1$, and $L(1) = 0$ or $L(+\infty) = 0$. LR-type fuzzy numbers are usually represented by $\widetilde{M} = (\underline{m}, \overline{m}, \alpha, \beta)_{LR}$.

If we denote $(\underline{m} = b, \overline{m} = c, \alpha = b - a, \beta = d - c)_{LR}$ with shape functions as $\max(0, 1 - x)$, then the famous trapezoidal fuzzy number would be obtained. Trapezoidal fuzzy numbers are also represented by $\widetilde{M} = (a, b, c, d)$. Other famous types of fuzzy numbers are triangular fuzzy number $\widetilde{M} = (a, b, d)$ and five pieces fuzzy number $\widetilde{M} = (\underline{m}^\varepsilon, \underline{m}^\lambda, \underline{m}, \overline{m}, \overline{m}^\lambda, \overline{m}^\varepsilon)$.

Fuzzy arithmetic can be deduced from the extension principle. If $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_r$ are fuzzy sets in $X_1, X_2, \dots, X_r$ and $y = f(x_1, x_2, \dots, x_r)$ then the fuzzy set $\tilde{B} = \{(y, \mu_{\tilde{B}}(y))\}$ has the membership function:

$$\mu_{\tilde{B}}(y) = \sup_{(x_1, \dots, x_r) \in f^{-1}(y)} \min_i \{\mu_{\tilde{A}_i}(x_i)\} \tag{4}$$

ome simple fuzzy operations for fuzzy numbers called extended operations can be derived from extension principle. For example extended operations for trapezoidal fuzzy numbers $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2)$ are represented as follows:

$$\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) \tag{5}$$

$$\widetilde{\max}(\tilde{A}, \tilde{B}) = \{\max(a_1, a_2), \max(b_1, b_2), \max(c_1, c_2), \max(d_1, d_2)\} \tag{6}$$

It is worthy to note that equation (6) is an approximate operation for maximum to yield a trapezoidal fuzzy number out of two trapezoidal fuzzy numbers. In fact, the actual output of maximum operator may be not a trapezoidal fuzzy number [20].

Another useful concept in fuzzy theory is fuzzy relation. A fuzzy relation $\tilde{R}$ in $X \times Y$ is a mapping from $X \times Y$ to [0,1] with following equation:

$$\tilde{R} = \{((x,y), \mu_{\tilde{R}}(x,y)) | (x,y) \in X \times Y\} \tag{7}$$

Fuzzy relation can also be defined in such a way that maps from (two) fuzzy sets in the universe $(X \times Y)$ to [0,1]. This definition of fuzzy relation can also be interpreted as fuzzy graph.

## 3. General Fuzzy Project Network Definition

A project network is denoted by a directed acyclic graph $G = (V, E; D)$ which is a triple of activities, relations and times, where $V = \{A_1, A_2, \dots, A_n\}$ representing the activities of network, $n \times n$ square matrix $E$ shows precedence relationships between project activities, and $D = \{d_1, d_2, \dots, d_n\}$ is the set of activity durations. It is also assumed, without loss of generality, that the activities are numbered such that a precedence relation always leads from a smaller activity number to a higher one. This kind of network is usually called activity-on-node (AON) network.

A project network may have some sources of imprecision; the most famous vagueness of the project is the estimation process of activities durations. Determining precedence relations can be a difficult task which is also a different aspect of project imprecision, and finally existence of activities may be in question.

### 3.1 Fuzzy Durations

Activity duration estimation is generally done through expert judgment process in initialization stage of the project; due to lack of enough information and experts linguistic estimations, there are usually vague quantities of time for activities duration, such as between 3 to 5 hours, or not more than 6 days to complete a job. In this case it would be better to use fuzzy quantities instead of crisp numbers or the well known probabilistic three point estimation.

Most of the papers in the field of fuzzy scheduling have considered this type of imprecision, so different types of fuzzy sets are proposed for activities duration. Activity duration has been introduced as a discrete fuzzy set (through $\alpha$-cut or independently) [7], or continuous fuzzy set (possibility distribution) [21], or different kind of fuzzy numbers like triangular fuzzy numbers [22], trapezoidal fuzzy numbers [23], five pieces fuzzy numbers [12]; [24], and L-R fuzzy numbers [9].

In this paper each activity duration $d_i$ is introduced by an arbitrary general fuzzy variable $\tilde{d}_i$ with a membership function $\mu_{\tilde{d}_i}(x)$ for $i = 1,2,\dots,n$ and $x \geq 0$, i.e. not even all types of traditional fuzzy numbers can be used but also non-convex fuzzy numbers can be assigned to the activities duration.

It also should be noted that almost in any AON networks there are two dummy activities which define the start and finish of the projects. Durations of these activities are zero with membership function:

$$\mu_{d_0}(x) = \mu_{d_{n+1}}(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \tag{8}$$

### 3.2 Fuzzy Precedence

Another type of imprecision and vagueness which emerges in project networks is due to the estimation of interrelations between activities of the network. Some precedence relations are clearly established because of inherent characteristics of the work, but there are always many cases in projects which an activity is not confidently dependent on another one.

There are too many examples that an activity is related to the other one just to gain outputs with better quality, and because of time delays in preceding activities, these relations are usually ignored through execution of the project. Besides, regular and legal constraints also impose some precedence relations between activities, while law is always subject to change.

Not too many papers have considered vagueness in structure of the network. Mares [10] proposed a fuzzy set $\tilde{B}_j$ for each activity $A_j$. $\tilde{B}_j$ is a fuzzy subset of $V = \{A_1, A_2, \dots, A_n\}$ indicating possible **predecessors** of activity $A_j$. Sharafi et al [17] showed the uncertainty of precedence relationship between two activities $i$ and $j$ with a triangular fuzzy number $\tilde{S}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, where $a_{ij} \geq 0$ and $c_{ij} \leq 1$.

We present the imprecision in precedence relations by fuzzy graph concept. A fuzzy graph originated from fuzzy relation is defined for a (crisp) set $V$ as follows [19]:

$$\tilde{E}(A_i, A_j) = \{((A_i, A_j), \mu_{\tilde{E}}(A_i, A_j)) \,|\, (A_i, A_j) \in V \times V\} \tag{9}$$

where $V = \{A_1, A_2, \dots, A_n\}$ is the set of vertices (nods or activities) of the graph and $\tilde{E}$ is the fuzzy set of edges (arcs or relations) of the graph. The value of membership function $\mu_{\tilde{E}}(A_i, A_j)$ can be interpreted as vagueness of flow from node $A_i$ to node $A_j$. In the project network we can explain the membership function $\mu_{\tilde{E}}(A_i, A_j)$ as the possibility of activity $A_j$ succeeding activity $A_i$.

### 3.3 Fuzzy Activities

Although it is not considered vastly in literature, the activity itself may face with uncertainty. It is usually experienced in the real projects that an activity or a group of activities is skipped or canceled. A well known example of imprecision in existence or occurrence of an activity could be reworks or repairs. Regular activities of the project also may be canceled via change orders from client.

As noted before, there are few papers considering this kind of imprecision. Mares [10] considered the set of activities really applied in a project as a larger set of theoretically possible activities. Gavareshki [16] proposed to use a membership degree (as an estimated number between 0 and 1) for activity existence. In this paper, the membership function for activity $A_j$ is defined by $\mu_{\tilde{V}}(A_j)$ , where $\tilde{V}$ is the fuzzy set of project activities. The membership function $\mu_{\tilde{V}}(A_j)$ can be interpreted as the possibility of existence or occurrence of the activity $A_j$. We can also show the imprecision of activities existence in the mentioned fuzzy graph for precedence relations, i.e. $\mu_{\tilde{E}}(A_i, A_i) = \mu_{\tilde{V}}(A_i)$. It should be noted that most of fuzzy theory textbooks assume that the possibility of flow between two nodes of a graph cannot be higher than possibility of two side nodes, in the other words the inequality $\mu_{\tilde{E}}(A_i, A_j) \leq \min\{\mu_{\tilde{E}}(A_i, A_i), \mu_{\tilde{E}}(A_j, A_j)\}$ must holds. But this assumption is usually violated for fuzzy graph of project networks, because unless, the project network would be so complicated. Let's consider a simple network of 3 series activities in fig. 1, if we keep the above assumption then the possibility of A3 to succeed A2 and the possibility of A2 to succeed A1 should be less than possibility of A2, which also implies that A3 succeeds A1 with maximum possibility of A2. This inference may not be true, because if A2 is deleted from network (suppose there is no possibility for occurring A2), then the relation between A1 and A3 usually remains valid with possibility equals 1. If A1 is directly related to A3, this assumption can be kept, but in a project network with $n$ activities this will generate up to $\frac{n(n-1)}{2}$ precedence relations which makes the project network too complicated.

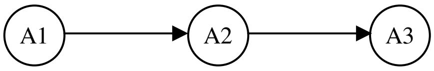



**Fig. 1. A simple project network with $\mu_{\tilde{V}}(A_2) = 0.3$**

Fuzzy activities can be extended to fuzzy projects or fuzzy subprojects. In the other words it is more practical to consider different subprojects or groups of activities, and estimate a membership function for their existence, because usually some activities are interconnected closely with each other and canceling an activity will result in canceling the other activities.

## 4. Corrected Fuzzy Project Network Analysis

From the previous section, a general fuzzy project network can be expressed by an acyclic fuzzy graph $\tilde{G} = (\tilde{V}, \tilde{E}; D')$, where $\tilde{V}$ is the fuzzy set of activities, $\tilde{E}$ is a fuzzy relation on support of $\tilde{V}$, and $D'$ is a crisp set of fuzzy activity durations based on support of $\tilde{V}$.

In this article, we only try to find early times of activities and makespan of the project through extension of forward recursion to deal with uncertainty in time and structure of general fuzzy project networks. The classic equations for calculating the earliest times of activities (forward recursion) are as follow:

$$TS_1 = 0 \tag{10}$$

$$TF_i = TS_i + d_i \qquad \text{for } i = 1, \dots, n \tag{11}$$

$$TS_j = \max_{i \in p_j} TF_i \qquad \text{for } j = 2, \dots, n \tag{12}$$

where $TS_i$ and $TF_i$ are respectively the earliest start and the earliest finish time of the $i$th activity, and $p_j$ is the crisp set of all activities that immediately precedes the $j$th activity.

There are three kinds of imprecision in $\tilde{G}$ that should be noted for extending relations of forward recursion. First of all we try to consider the imprecision in activities duration. As noted before, this case interested more researchers, and many papers proposed different methods for it. Most of these papers used only a special kind of fuzzy numbers to denote fuzzy activities duration. The main reason for using simple fuzzy numbers (e.g. triangular) is easy arithmetic required by these fuzzy numbers with approximate fuzzy algebra. Although fuzzy operations on simple fuzzy numbers are easier than other fuzzy sets, in more complex networks like the network with uncertainty in its structure, fuzzy numbers with more complexity or exempting some of its assumptions like convexity or normalization should be used. So without loss of generality, equations (11) and (12) can be changed into equation (13) and (14), considering fuzzy variables instead of crisp variables.

$$\widetilde{TF}_i = \widetilde{TS}_i \oplus \tilde{d}_i \qquad \text{for } i = 1, \dots, n \tag{13}$$

$$\widetilde{TS}_j = \widetilde{\max_{i \in p_j}} \widetilde{TF}_i \qquad \text{for } j = 2, \dots, n \tag{14}$$

where $p_j$ is still the crisp set of all activities that immediately precedes the $j$th activity with some possibility greater than zero, but operators $\oplus$ and $\widetilde{\max}$ do not go with equations (5) and (6) anymore, since the fuzzy numbers $\tilde{d}_i$, $\widetilde{TS}_i$ and $\widetilde{TF}_i$ are not supposed to be convex fuzzy numbers. The corrected operators will be discussed in sections 4.1 and 4.2.

In addition to the activity duration imprecision, the uncertainty of activity existence should be noted. If we denote the possibility of existence of the $i$th activity as $\mu_{\tilde{V}}(A_j)$ then the fuzzy number $\tilde{d}_i$ representing the $i$th activity duration, would be converted to a new fuzzy quantity $\tilde{d}'_i$ with the following membership function:

$$\mu_{\tilde{d}'_i}(x) = \begin{cases} \max\left(\mu_{\tilde{d}_i}(x), 1 - \mu_{\tilde{V}}(A_j)\right) & \text{for } x = 0 \\ \min\left(\mu_{\tilde{d}_i}(x), \mu_{\tilde{V}}(A_j)\right) & \text{for } x \neq 0 \end{cases} \tag{15}$$

Fig. 2 shows an activity which its possibility to occur is 0.8, and its duration is a trapezoidal fuzzy number (2,4,7,8). Using equation (15) will result in a new fuzzy quantity, shown by thick lines, for activity duration.

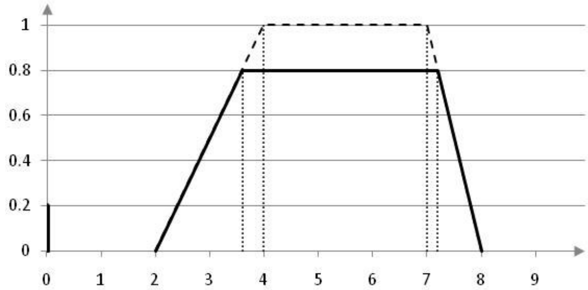



**Fig. 2. New fuzzy quantity for activity duration**

It can be easily seen that the new fuzzy quantity is no more fuzzy number, because it consists of two separated segments (one point at zero with 0.2 possibility and one interval from 2 to 8 with maximum 0.8 possibility), and it is neither convex nor normalized. In this paper, these kinds of fuzzy quantities that originate from fuzzy numbers are called non-convex fuzzy numbers. The non-convex fuzzy number of fig. 2 could be represented by $\big((0;0.2),(2,3.6,7.2,8;0.8)\big)$. The resulted non-convex fuzzy number can also be converted to a normalized non-convex fuzzy number. Although normalizing non-convex fuzzy numbers can lead to better judgments, it should be noted since the calculated possibility for each quantity of time (duration) is the basis for calculating other quantities, the normalization should be done just after finishing all calculations of the project network.

The imprecision in the precedence relations between activities also appears in the recognition of predecessors of activities. So, calculating equation (14) firstly needs to distinguish the arguments of the $\widetilde{\max}$ operator ($\widetilde{TF}_i$'s). If we denote the possibility of activity $i$ preceding activity $j$ by $\mu_{\tilde{E}}(A_i, A_j)$ then the equation (14) should use the corrected times for early finish of activity $i$ based on relation $i$-$j$, $\widetilde{TF}_i^{i-j}$, with the following membership function:

$$\mu_{\widetilde{TF}_i^{i-j}}(x) = \begin{cases} \max\left(\mu_{\widetilde{TF}_i}(x), 1-\mu_{\tilde{E}}(A_i, A_j)\right) & \text{for } x = 0 \\ \min\left(\mu_{\widetilde{TF}_i}(x), \mu_{\tilde{E}}(A_i, A_j)\right) & \text{for } x \neq 0 \end{cases} \qquad (16)$$

It should be noted that there probably exist different values for possibility of precedence relations that originates from activity $i$, say $\mu_{\tilde{E}}(A_i, A_j)$ for $j = i+1, \dots, n$, so the corrected early finish time of activity $i$ may be different for each of these relations, therefore the corrected early finish time should be considered only for next activity calculation through equation (14), and not for any other interpretation.

Regarding equations (15) and (16), the corrected equations of forward recursion based on equations (13) and (14) are:

$$\widetilde{TF}_i = \widetilde{TS}_i \oplus \tilde{d}'_i \qquad \text{for } i = 1, \dots, n \qquad (17)$$

$$\widetilde{TS}_j = \widetilde{\max_{i \in p_j}} \widetilde{TF}_i^{i-j} \qquad \text{for } j = 2, \dots, n \qquad (18)$$

where $\tilde{d}'_i$ is corrected $i$th activity duration and $\widetilde{TF}_i^{i-j}$ is corrected early finish of $i$th activity considering $j$th (succeeding) activity. But as noted previously, ordinary extended operations for fuzzy numbers are no more applicable to non-convex fuzzy numbers. To overcome this problem, the corrected operations on non-convex fuzzy numbers are presented in the following sections.

### 4.1 Corrected Fuzzy Addition

First, addition of non-normalized convex fuzzy numbers is dicussed. Since supreme of membership functions of non-normalized fuzzy numbers do not equal 1, and for two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, it can also be different, from extension principle (equation (4)) the supreme of membership function for the resulted number cannot be greater than $\min(\sup_x \mu_{\tilde{A}}(x), \sup_x \mu_{\tilde{B}}(x))$. If we denote $\beta = \min(\sup_x \mu_{\tilde{A}}(x), \sup_x \mu_{\tilde{B}}(x))$ then it makes no difference to add the two mentioned fuzzy numbers with their membership functions greater than $\beta$ to be fixed to $\beta$.

This is called flattening a non-normalized fuzzy number to $\beta$ [25]. The membership function of the flattened number is:

$$\mu_{\tilde{A}^{\beta}}(x) = \min\{\mu_{\tilde{A}}(x), \beta\} \qquad (19)$$

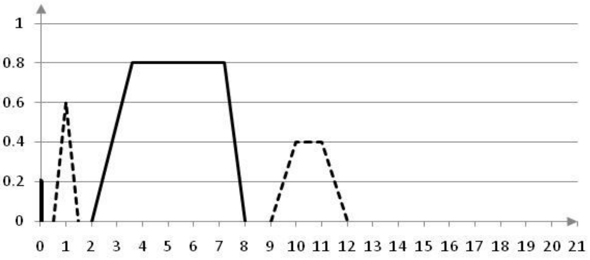
For example, in fig. 2 trapezoidal fuzzy number (2,4,7,8) is flattened to 0.8, resulting in the fuzzy non-normalized number (2,3.6,7.2,8;0.8). After reducing $\tilde{A}$ and $\tilde{B}$ to the flattened fuzzy numbers, the addition for convex fuzzy numbers would be the same as the ordinary extended addition of fuzzy numbers. But in this paper fuzzy numbers are usually non-convex and have several segments (e.g. two segments in fig. 2). There are two cases that can be considered:

1) All segments of two fuzzy non-convex numbers ($\tilde{A}$ and $\tilde{B}$) are in fact convex fuzzy numbers (not essentially normalized), e.g. the non-convex fuzzy number in fig. 2 consists of two convex non-normalized fuzzy numbers (a point at zero and an interval from 2 to 8).

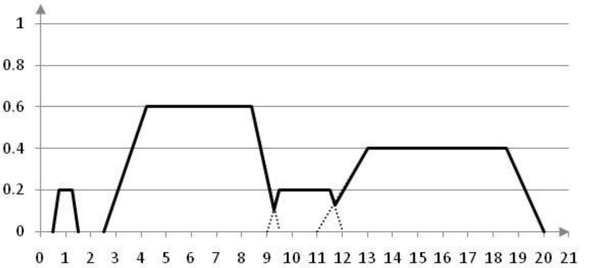
In this case, the proposed fuzzy addition for non-normalized fuzzy numbers can be used for adding one segment of $\tilde{A}$ with one segment of $\tilde{B}$. In the other words, if $\tilde{A}$ has $h$ segments and $\tilde{B}$ has $k$ segments then $h \times k$ additions should be done for calculating $\tilde{A} \oplus \tilde{B}$. Then by combining all the resulted segments, the fuzzy non-convex number for $\tilde{A} \oplus \tilde{B}$ is calculated. Also if the calculations for these different segments have intersection, then from the extension principle the supreme of the intersected area should be considered.

An Example of adding two non-convex fuzzy numbers, each having convex segments, is shown in fig. 3.

2) There may be some segments in the non-convex fuzzy numbers that are not convex themselves, fig. 3.b is a good example, which consists of a convex segment in the interval (0.5,1.5) and a non-convex segment in the interval (2.5,20). A proposed method by Dubois and Prade decompose the non-convex fuzzy set (number) into the union of convex, possibly non-normalized, fuzzy sets whose membership functions are either strictly increasing or decreasing or constant in the only interval where they are not zero. Subsequently, the operation is performed for every two flattened parts (one part for each fuzzy set) that belonging to the same kind of sets (nondecreasing or nonincreasing), and at last the final result is the union of these resulted fuzzy sets [25]. An example by Dubois and Prade is shown in fig. 4.



**a. Two non-convex fuzzy number $\tilde{A}$ and $\tilde{B}$**



**b. The resulted non-convex fuzzy number $\tilde{A} \oplus \tilde{B}$**

**Fig. 3. Corrected fuzzy addition for two non-convex fuzzy numbers**

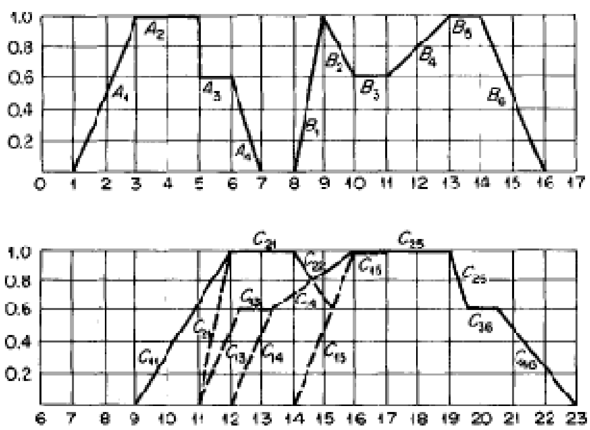


**Fig. 4. Method of Dubois and Prade for fuzzy addition of two non-convex fuzzy numbers [25]**

## 4.2 Corrected Fuzzy Maximum

For the corrected fuzzy maximum operator, the above method of Dubois and Prade also can be used [25], but since the non-convex fuzzy numbers of general fuzzy project network may be non-continuous (having several segments), this method fails to handle network analysis. Considering the extension principle (equation (4)), a general equation for $\widetilde{\max}$ operator can be derived as follows:

$$\mu_{\widetilde{\max}(\tilde{A},\tilde{B})}(x) = \max\left\{\min\left(\mu_{\tilde{A}}(x), \max_{y\le x}\mu_{\tilde{B}}(y)\right), \min\left(\mu_{\tilde{B}}(x), \max_{y\le x}\mu_{\tilde{A}}(y)\right)\right\} \quad (20)$$

where $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are membership functions of non-convex (non-normalized, non-continuous) fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

## 5. Application and Numerical Example

For having better illustration of potential applications of the proposed method a subproject of a real construction project is considered. Ilam Gas Refinery construction project was started at the beginning of 2003 in the west of Iran near Ilam city, and it was planned to complete in 2 years, but actually the refinery began to work at the end of 2007 with 3 years of delay. Besides too much delay in known project activities, there were plenty of activities which were not considered in the base plan of the project, which forced the client to even change the contract fundamentally after 2 years from the start of the project. We think that many omitted activities are due to their possible existence, that convinced the project team to delete them from base plan, and we claim that our proposed method can handle this problem to some extent. Generally, a construction project consists of some main subprojects e.g. designing, procurement, building, erection, commissioning and start-up. One of the subprojects that has significantly less activities (weight) compare to the others, is site mobilization. Although site mobilization has few activities, it is usually an important part of project which its progress is monitored continuously, and delay in it may cause delay in the whole project. Here, we consider the set of site mobilization activities as our project network. A typical list of activities for this project with their fuzzy durations is given in Table 1. Although trapezoidal fuzzy numbers are used as activity duration inputs, all kinds of fuzzy numbers can be substituted for activities duration.

**Tab. 1. The list of activities of site mobilization**

| Activity Code | Activity Name | Activity Duration |
|---|---|---|
| A0 | Start | (0,0,0,0) |
| A1 | Drawing local sketches | (6,9,10,12) |
| A2 | Site visiting | (1,1,2,2) |
| A3 | Selecting workshop location | (1,1.5,2,2.5) |
| A4 | Acquiring legal permissions | (7,10,12,15) |
| A5 | Uprooting trees | (2,2.5,3.5,4) |
| A6 | Leveling field | (1,1.5,2.5,3) |
| A7 | Installing prefabricated offices | (9,13,15,20) |
| A8 | Constructing storehouse | (12,16,19,22) |
| A9 | Putting up fences | (6,7,7,8) |
| A10 | Checking and verifying | (3,3.5,4,5) |
| A11 | Reworks | (5,8,13,16) |
| A12 | Finish | (0,0,0,0) |

The AON representation of the project is shown in fig. 5, which also presents precedence relations between activities. This network with information of table 1 can be easily analyzed with ordinary fuzzy PERT, but paying more attention it can be admitted that there are some uncertainties that are not shown in fig. 5 nor table 1.

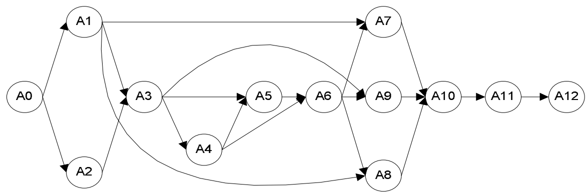



**Fig. 5. AON network for site mobilization project**

There are some activities that their existence are in question, for example workshop location may be selected in an area that does not require any legal permission, or there may be not any trees to cut, rework also may not be done because of work acceptance. So the fuzzy set of activities is:

$$\tilde{V} = \{(A_0,1),(A_1,1),(A_2,1),(A_3,1),(A_4,0.9),(A_5,0.4),(A_6,1),(A_7,1),(A_8,1),(A_9,1),(A_{10},1),(A_{11},0.6),(A_{12},1)\}$$

Considering precedence relations, a fence can be put up ($A_9$) right after specifying workshop location ($A_3$) before field leveling ($A_7$) due to security issues. There may be some other possible uncertainty in precedence relations, for example getting permissions may not depend on workshop location, or because of natural type of trees it may be not necessary to get permissions to uproot them, also a project manager may not wait until finishing of the sketches, and choose workshop location with only site visiting. Based on these uncertainties, the membership function of fuzzy graph including the membership function of activity existence can be derived:

$$\mu_{\tilde{E}}(A_i,A_j) = \begin{array}{cc} & \begin{array}{cccccccccccc} A0 & A1 & A2 & A3 & A4 & A5 & A6 & A7 & A8 & A9 & A10 & A11 \end{array} \\ \begin{array}{c} A0\\A1\\A2\\A3\\A4\\A5\\A6\\A7\\A8\\A9\\A10\\A11\\A12 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.8 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

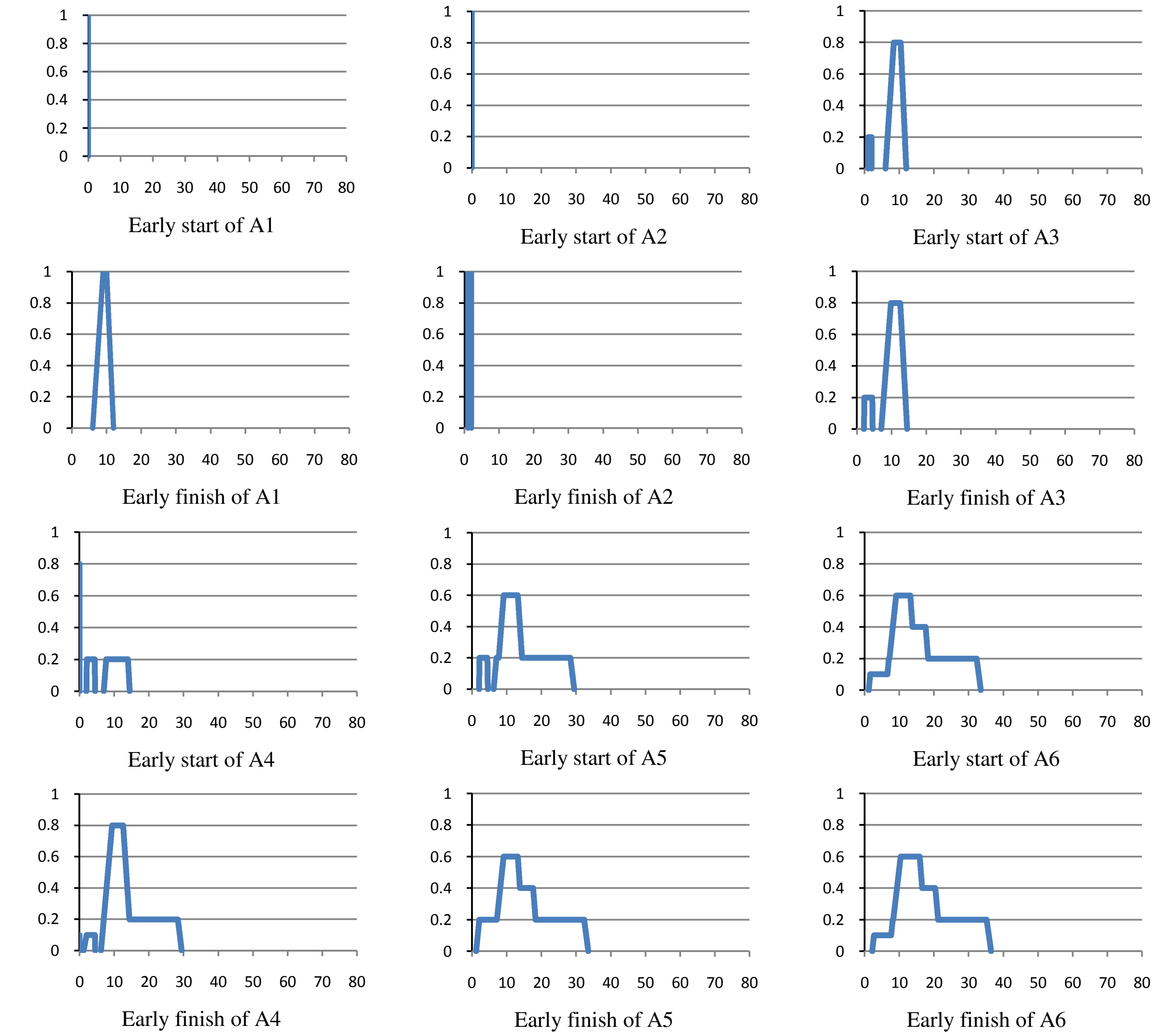



**Fig. 6. Early times of activities for site mobilization project**

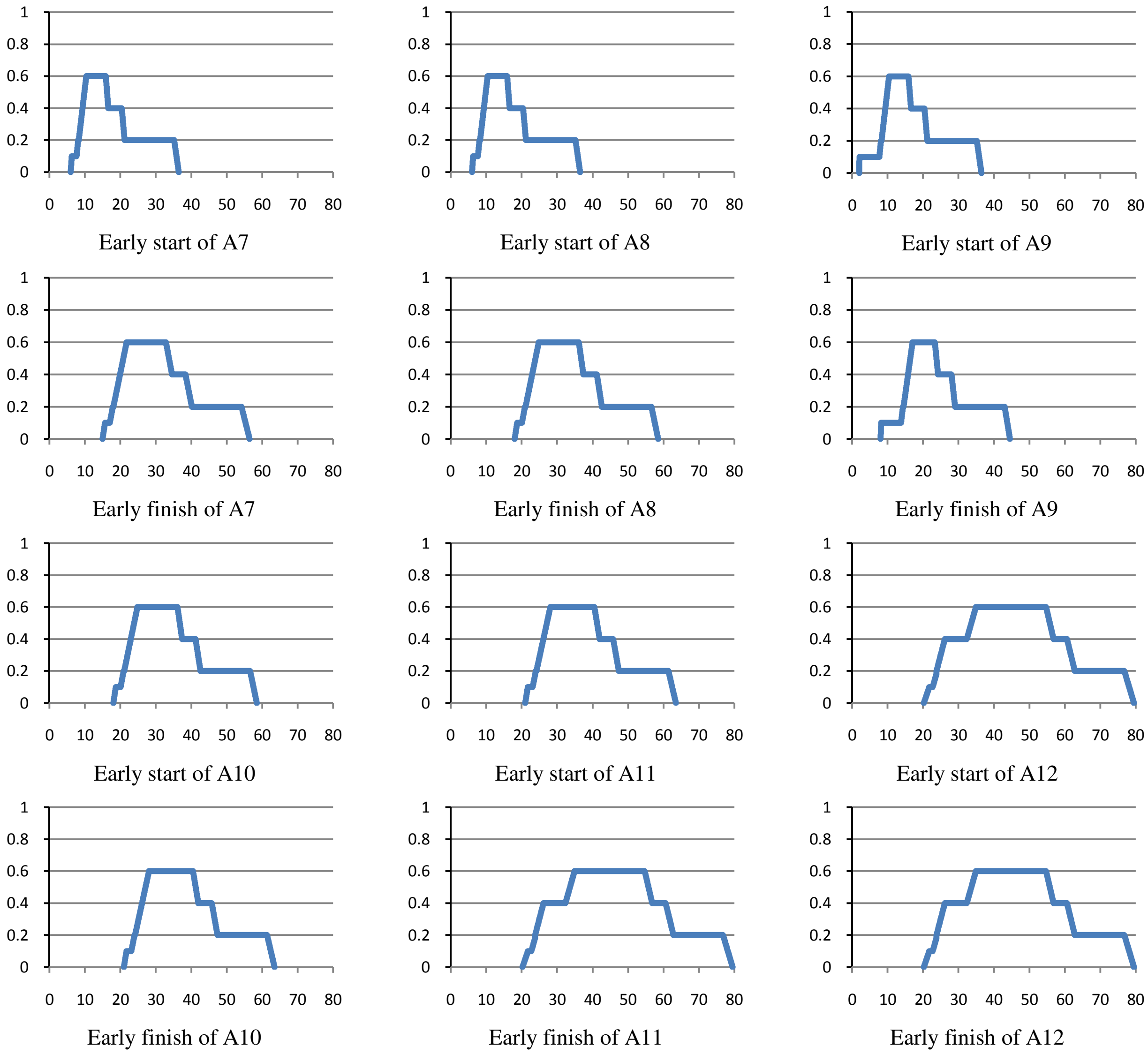



**Fig. 6. Cont.: Early times of activities for site mobilization project**

Using equations (17) and (18) for corrected forward recursion and applying corrected fuzzy operators presented in sections 4.1 and 4.2, the resulted fuzzy quantities for early start and early finish of each activity are shown in fig. 6. The makespan of the project has possibility to occur in interval (20.2 79.5) with maximum 0.6 possibility in interval (34.9 55.7). One can also normalize the fuzzy quantities of each early start and early finish of activities in fig. 6. If all possible activities and precedence relations, which their possibilities are less than one, are deleted from network of fig. 5, then the classic fuzzy PERT for deterministic structure conclude the project makespan as a trapezoidal fuzzy number (21,28.5,33,39). This trapezoidal fuzzy number denotes the best case that can happen during execution of the project (fig. 7). If project planning is done on the basis of this number then the project may face with lack of budget and resources, and it may also fail to reach its objectives. In the other hand, the worst case occurs when all possible activities and precedence relations exist necessarily; in this case, using the classic fuzzy PERT, the project makespan would be trapezoidal fuzzy number (37,52,66,79.5), (fig. 7).

Therefore, project may face with inflation in times of activities and project makespan, which leads project organization to allocate more budget and resources than what is really needed. In site mobilization of Ilam Gas Refinery construction project, the best case was supposed to occur, i.e. activities and precedence relations that had possibilities less than one was omitted from network, which cause an optimistic estimation in project activity times and project makespan.

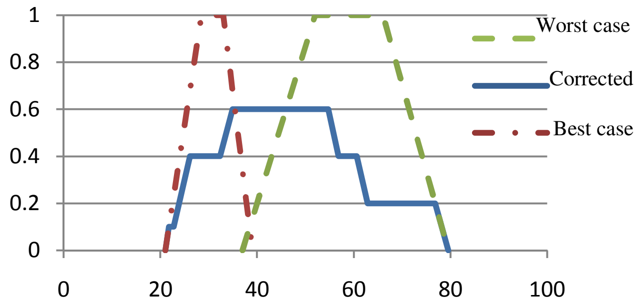



**Fig. 7. Classic fuzzy PERT vs Corrected fuzzy PERT**

Using classic fuzzy PERT, project managers usually have an interval for project makespan, but actually this interval does not include all possible times for project completion. As illustrated in fig. 7, proposed method in this article provides all possible values for project makespan with better estimation of possibility for each specific time. Therefore, the proposed method can prevent project managers from mal-estimation of project finish time which have direct influence on costs of the project.

## 6. Conclusion

Real projects usually encounter with many sources of imprecision, but up to now only imprecision in activities duration is usually considered in literature and applications. We have proposed a general fuzzy project network, by using fuzzy graph concept, which can show imprecision in activities duration and project structure (activity existence and precedence relation existence). A new method has been also extended on the basis of forward recession for analyzing a network with these kinds of imprecision; in addition, since resulted fuzzy quantities are not convex nor normalized, corrected arithmetic for these quantities (numbers) are also introduced.

Solving a real project network for calculating early times of activities and project makespan showed that the proposed method for fuzzy network structure results in a wider range of possible times. Comparing the new proposed method with traditional fuzzy PERT, our results are more realistic, because the resulted times for fuzzy PERT are too much optimistic if we do not consider the activities and precedence relations with possibility less than a pre-specified level, and too much pessimistic if we consider all possible activities and precedence relations in a deterministic structure.

For future researches, a corrected backward recursion can be developed to estimate late times of activities in a project network with uncertainty in time and structure. Also, a project with high levels of risk may have cycles, i.e. some activities may repeat until a specified objective is gained, which can affect the fuzzy structure of the project.

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