Acceleration of Lagrangian Method for the Vehicle Routing Problem with Time Windows

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ABSTRACT

The analytic center cutting plane method (ACCPM) is one of successful methods to solve nondifferentiable optimization problems. In this paper, ACCPM is used to accelerate Lagrangian relaxation procedure for solving a vehicle routing problem with time windows (VRPTW). First, a basic cutting plane algorithm and its relationship with a column generation technique is clarified. Then, the proposed method based on ACCPM is explained as a stabilization technique for Lagrangian relaxation. Both approaches are tested on a benchmark instance to demonstrate the advantages of the proposed method in terms of computational time and quality of lower bounds.

1. Introduction

The growing need for efficient supply chain management and logistics in recent years has caused the vehicle routing problems (VRP's) to be center of attention again, after more than half a century of their first introduction. Among them the vehicle routing problem with time windows (VRPTW) is one of the most basic and useful models introduced so far. In this problem a network of customer nodes and a depot node is given, with an amount of demand associated with each customer. A fleet of vehicles with limited capacity should service these customers through separate paths according to a time interval considered for each customer called a time window. The objective is finding a set of paths with minimum total distance traveled by vehicles.

The literature in VRPTW is quite abundant and not all of them will be covered in this paper. The readers can refer to [1], [2] and [3] about heuristic and metaheuristic studies and [3], [4] and [5] about exact methods. Most of developed exact methods for this problem are based on set partitioning formulation (column generation) or lagrangian relaxation. The introductory work in this area is [6] where a column generation method is used in a decomposition framework. This work was improved by [7] and [8]. Lagrangian relaxation and decomposition is studied primarily by [7], [9] and [10]. Since Lagrangian duality and Dantzig-Wolf (D-W) decomposition are closely related, considering both approaches in solution procedures has lead to a significant improvement in quality and efficiency of solution (see for example [11]).

Two best exact algorithms introduced so far in the literature are a lagrangian based branch-price-and-cut method [12] and a branch-price-and-cut algorithm based on set partitioning formulation [13].

The traditional approach in solving lagrangian relaxation is subgradient method which has been criticized in recent years [14]. The cutting plane method is another approach which has received more attention. Dual concept of cutting plane method is column generation and this method has many similarities with D-W decomposition and set partitioning based methods.

The main drawback of this method is its slow convergence which is treated by accelerating techniques. In this paper a method is proposed to stabilize and accelerate the lagrangian method for

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VRPTW for the first time. This method is called the analytic center cutting plane method (ACCPM), introduced basically for nondifferentiable optimization problems ([15], [16]). We will show that this method is well applicable in lagrangian method for VRPTW and results in more efficient convergence compared to classical cutting plane method.

In section two the mathematical model is described; section three is about lagrangian relaxation of the VRPTW and section four clarifies the relationship with column generation and set partitioning formulation. In fifth section our algorithm based on ACCPM is proposed and some computational results are presented in section six.

Finally in section seven concluding remarks and future research directions are addressed.

2. Mathematical Formulation

The VRPTW is defined on a directed graph $G = (N, A)$ where node set N consists of a subset $C = \{1,...,n\}$ of customers and nodes 0 and $n+1$ correspond to depot. Each customer is associated with a demand d_i and a time interval $[a_i, b_i]$, which is expected to be serviced within this time called a time window.

An interval $[a_0, b_0]$ is considered for depot nodes indicating the total time horizon for routing process. Each arc $(i, j) \in A$ is associated with a travel distance c_{ij} and a travel time t_{ij} that usually includes a service time for node i (it is assumed that the triangular inequality is satisfied for both). There are no arcs ending at depot 0 or originating from depot $n+1$. A fleet of U vehicles each with a capacity Q is due to service customers through a path starting from depot 0 and ending at depot $n+1$.

The VRPTW can be described as the following model:

$$\min \sum_{u \in U} \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}^u \tag{1}$$

$$st. \sum_{u \in U} \sum_{j \in N} x_{ij}^u = 1 \qquad \forall i \in C \tag{2}$$

$$\sum_{i \in C} \sum_{j \in N} d_i x_{ij}^u \leq Q \qquad \forall u \in U \tag{3}$$

$$\sum_{j \in N} x_{0j}^u = 1 \qquad \forall u \in U \tag{4}$$

$$\sum_{j \in N} x_{ij}^u - \sum_{j \in N} x_{ji}^u = 0 \qquad \forall i \in C, \forall u \in U \tag{5}$$

$$\sum_{j \in N} x_{i,n+1}^u = 1 \qquad \forall u \in U \tag{6}$$

$$s_i^u + t_{ij} - s_j^u + M x_{ij}^u \leq M \quad \forall i, j \in N, \forall u \in U \tag{7}$$

$$a_i \leq s_i^u \leq b_i \qquad \forall i \in N, \forall u \in U \tag{8}$$

$$x_{ij}^u \in \{0,1\} \qquad \forall i, j \in N, \forall u \in U \tag{9}$$

In this model x_{ij}^u and s_i^u are decision variables. x_{ij}^u is a binary variable indicating whether arc (i, j) is traversed by vehicle u or not. s_i^u is the time denoting when vehicle u starts to service customer i. M is a big quantity that can be replaced by $\max_{(i,j) \in A} \{b_i + t_{ij} - a_j\}$. Constraints(2) enforce that each customer should be visited once.

Constraints (3) concern vehicles capacity limit and constraints (4)-(6) define paths structure traversed by vehicles. Constraints (7) and (8) concern time windows restrictions.

3. Lagrangian Relaxation

The VRPTW is *NP*-hard since it is reducible to classic VRP with little modifications. So it is worthful to propose a good lower bound for the problem to be then used in more general frameworks like branch-bound to get optimal solutions. The lagrangian relaxation is among the methods that offers high quality lower bounds for most of problems. For computing a lagrangian lower bound for VRPTW we start with relaxing of constraints (2) and adding them to the objective function (1) as follows:

$$L(\lambda) = \min \sum_{u \in U} \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}^u - \sum_{i \in C} \lambda_i (\sum_{u \in U} \sum_{j \in U} x_{ij}^u - 1) \tag{10}$$
$$s.t. \ (3)-(9)$$

where coefficients $\lambda_i \geq 0$ are called Lagrangian multipliers. This lagrangian relaxation gives us a lower bound on the optimal solution of main problem (1)-(9). Let $\Re_\lambda$ be the set of all feasible solution for (3)-(10) This set can be split into U separate identical subsets $\Re$ such that $\Re_\lambda = \Re^U$. Therefore $L(\lambda)$ splits into U simpler subproblems for each vehicle u as:

$$L_u(\lambda) = \min \sum_{u \in U} \sum_{i \in N} \tilde{C}_{ij} x_{ij} \tag{11}$$

$$s.t. \sum_{i \in C} \sum_{j \in N} d_i x_{ij} \leq Q \tag{12}$$

$$\sum_{j\in N} x_{0j} = 1 \tag{13}$$

$$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = 0 \qquad \forall i \in C \tag{14}$$

$$\sum_{j\in N} x_{i,n+1} = 1 \tag{15}$$

$$s_i + t_{ij} - s_j + M x_{ij} \leq M \quad \forall i,j \in N \tag{16}$$

$$a_i \leq s_i \leq b_i \qquad\qquad \forall i \in N \tag{17}$$

$$x_{ij} \in \{0,1\} \qquad\qquad \forall i,j \in N \tag{18}$$

Where

$\tilde{c}_{ij} = c_{ij} - \lambda_i \ \forall i \in C, j \in N$, otherwise $\tilde{c}_{ij} = c_{ij}$. Each feasible solution according to (12)-(18) denotes a path $r \in \Re$ which starts at depot 0 and ends at depot $n+1$ regarding the capacity and time windows restrictions. This subproblem itself is a kind of elementary shortest path problem which has been attacked with several algorithms in literature.

Our aim is finding the best lagrangian lower bound. So the lagrangian dual problem can be described as follows:

$$LD(\lambda) = \max \sum_{i\in C} \lambda_i + U\left(\min_{r\in\Re} \sum_{i\in N}\sum_{j\in N} \tilde{c}_{ij} x_{ijr}\right) \tag{19}$$

where decision variable x_{ijr} corresponds to each feasible path $r \in \Re$. This problem is maximization over a finite number of linear functions and therefore a piecewise linear an concave function.

Two major solution procedures have been proposed for tackling lagrangian dual problem. At first subgradient method introduced as a simple way to updating lagrangian multipliers and improving lower bounds. Afterwards this method criticized in literature and attentions turned to other approach, the cutting planes methods. In cutting plane method the lagrangian dual problem is replaced with an equivalent linear problem as follows:

$$LD(\lambda) = \max \sum_{i\in C} \lambda_i + U\mu \tag{20}$$

$$\mu \leq \sum_{i\in N}\sum_{j\in N} \tilde{C}_{ij} x_{ijr} \qquad \forall r \in \Re \tag{21}$$

$$\lambda_i \geq 0 \tag{22}$$

Especially we start with a limited number of feasible paths, a subset $\Re'$ instead of $\Re$, generating limited number of constraints (21). Then we generate iteratively new constraints to narrow the space and reach to optimal point. These constraints are generated via lagrangian subproblem (11)-(18).

The ways of generating new constraints leads to different cutting plane methods. In basic cutting plane method known as Kelley's method (introduced in [17] and [18]) a restricted lagrangian dual problem is solved to optimality and related optimal lagrangian multipliers are sent to lagrangian subproblem to generate new constraints.

If subproblem cannot generate new constraints the at hand best lower bound is the maximum lower bound we seek for.

4. Column Generation

Column generation method for VRPTW is closely related to stated cutting plane method. Investigating the similarities between these methods helps us to better understand the cutting plane methods and develop techniques to improve them. Column generation method for VRPTW is based on the set partitioning formulation for the problem.

Let $\Re$ be the set of paths satisfying capacity and time window constraints of (3)-(8) and let be the cost of each path $r \in \Re$. Binary coefficient a_{ir} takes value 1 if it is included in a path and takes 0 otherwise. Decision variable is z_r indicating whether a path is used in optimal solution or not. The set partitioning formulation of VRPTW is described as follows:

$$\min \sum_{r\in\Re} c_r z_r \tag{23}$$

$$s.t. \sum_{r\in\Re} a_{ir} z_r = 1 \qquad \forall i \in C \tag{24}$$

$$\sum_{r\in\Re} z_r = U \tag{25}$$

$$Z_r \in \{0,1\} \tag{26}$$

Constraints (24) ensure that every node will be visited exactly once time in optimal solution and constraint (25) denotes the number of required vehicles. This problem gives us the same optimal solution of main problem (1)-(9). Sometimes the equality constraints

(24) are changed to inequality by replacing "=" with "≥".

This replacement does not change the optimal solution (due to the triangular inequality) but makes the solution process easier with diminishing the dual space. It is worth mentioning that this formulation is also equivalent with Dantzig-Wolfe decomposition of VRPTW when any solution of the problem is expressed as a convex non-negative combination of paths admitting time window and capacity constraints (3)-(9) [3].

It is desirable to compute a lower bound from this model by relaxing it into a linear problem. The main difficulty in solving the linear relaxation is the huge number of columns in the model. To treat with this difficulty column generation method has been proposed. In this method we start the linear model with a limited number of paths or variables (a subset $\Re' \subset \Re$) and iteratively generate required columns (variables). We generate columns with minimum reduced cost via a subproblem called pricing subproblem.

This pricing subproblem is the same as lagrangian subproblem (11)-(18) with a slight difference in objective function by subtracting a single dual variable associating with constraint (25) (refer to [19] for more details).

To constructing this pricing subproblem we need the optimal dual variables of restricted linear problem. To this end these dual variables could be found by solving restricted linear model or the equivalent restricted dual. This restricted dual can be formulated with following model:

$$\max \sum_{i \in C} \lambda_r + U\lambda \qquad (27)$$

$$s.t. \sum_{i \in C} a_{ir}\lambda_r + \mu \le c_r \qquad r \in \Re \qquad (28)$$

$$\lambda_i \ge 0 \qquad (29)$$

The interesting result is that this dual problem is the same as restricted lagrangian dual problem (20)-(22). Especially if we apply these replacements:

$$c_r = \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ijr}$$

$$a_{ir} = \sum_{j \in N} x_{ijr}$$

The procedure of Kelley's cutting planes in lagrangian relaxation is equivalent to the column generation method on set partitioning formulation and both methods result in a same lower bound. Due to these similarities, Kelley's method is often called simply a column generation procedure.

5. Analytic Center Cutting Plane Method

The column generation approach to deal with lagrangian relaxation has received its own critiques in literature (see for example [20]). Therefore several ways have been developed to improve lagrangian relaxation method. These methods are usually called stabilization techniques. The main approach in these methods is diverting the dual variables from being optimal (extreme) points which slowed down the overall convergence rate. Interested reader can refer to [21] where several stabilization techniques have been reviewed.

The analytic center cutting plane method (ACCPM) is a successful method that has been developed for nondifferentiable optimization problems. Since the lagrangian dual problem (19) is a concave nondifferentiable problem, ACCPM can be considered a stabilization technique for VRPTW which have never been used for this problem before.

The key point in this method is using the analytic centers as candidate duals (lagrangian multipliers) to be sent to subproblem. We start with a bounded set pertaining to lagrangian dual space which contains optimal point(s) (this is called a localization set). Then the analytic center of this localization set is found and sent to lagrangian subproblem. The returning constraint from subproblem is appended to localization set and to decrease it. This sequence is continued until the localization set is small enough that contains only optimal point(s).

The localization set is a convex polyhedron related to lagrangian dual problem (19). Suppose that we are at k–th iteration of the solution procedure. A localization polyhedron can be described as:

$$F_k(\lambda, \mu) =$$

$$\left\{ \begin{array}{ll} (\lambda, \mu): & \sum_{i \in C} a_{ir}\lambda_i - \mu \le c_r \qquad \forall r \in \Re^k \\ & \sum_{i \in C} \lambda_i - U\mu \ge L(\hat{\lambda}) \\ & \lambda_i \ge 0 \end{array} \right\} \qquad (30)$$

Where $\Re^k$ is related subset of paths for k–th iteration and $L(\hat{\lambda})$ is the best lagrangian bound found so far. The analytic center of this polyhedron is used as lagrangian multipliers for subproblem in contrast to column generation algorithm where we use optimal

values of restricted dual problem (27)-(29) as lagrangian multipliers.

Perhaps the most challenging step in this algorithm is computing the analytic center of polyhedron $F(\lambda, \mu)$.

The rich literature of interior point methods provides us enough knowledge to deal with this step. For ease of exposition suppose that we have $\max\{b^T\lambda : A^T\lambda \leq c\}$ as a restricted dual problem. The aim is to find the analytic center of this localization set:

$$F(\lambda) = \begin{cases} \lambda : A^T\lambda \leq c \\ b^T\lambda \geq L(\hat{\lambda}) \end{cases} \qquad (31)$$

To find the weighted analytic center of this polyhedron we need to maximize this potential logarithmic function:

$$\max \varphi(s) = w \log s_0 + \sum_{i=1}^{m} \log s_i \qquad (32)$$

$$s.t. \quad s = c - A^T\lambda > 0 \qquad (33)$$

$$s_0 = b^T\lambda - L(\lambda) \rangle 0 \qquad (34)$$

Where w is the weight indicating the importance of lower bound constraint in localization set. Let z be the primal variable of stated dual problem. The necessary and sufficient conditions for optimality of this maximization problem are:

$$Sz = e \qquad (35)$$

$$S_0 z_0 = w \qquad (36)$$

$$z_0 b - Az = 0 \qquad z, z_0 > 0 \qquad (37)$$

$$s + A^T\lambda = c \qquad s > 0 \qquad (38)$$

$$s_0 - b^T\lambda = L(\lambda) \qquad s_0 > 0 \qquad (39)$$

Interior point methods literature suggests several techniques to solve this system of equations. If we have a primal feasible solution $(z > 0)$ then we can use a primal Newton's method. If only a dual feasible solution $(s > 0)$ is available then a dual Newton's method can be used.

And if we have both primal and dual feasible points $(z > 0, s > 0)$ at hand we can use a primal-dual Newton's method.

The overall algorithm of improved lagrangian method for VRPTW can be described now. Fig. 1 depicts the main steps of proposed method.

Initialization: Start with a subset $\mathfrak{R}' \subset \mathfrak{R}$ of paths containing at least one feasible solution for restricted dual problem (27)-(29).

Repeat: (While no new constraint could be found)
1. Construct restricted dual problem for k-th iteration and its related localization set $F_k(\lambda, \mu)$.
2. Find the analytic center (λ^k, μ^k) of $F_k(\lambda, \mu)$.
3. Send (λ^k, μ^k) to Lagrangian subproblem (11)-(18).
4. Find $L(\hat{\lambda}, \hat{\mu})$ and new constraint of type (28).
5. $k \to k+1$ and go to 1.

End (while).

Output: current analytic center (λ^*, μ^*) is the optimal point of $LD(\lambda)$ and $\sum_{i \in C} \lambda_i^* + U\mu^*$ is the best lagrangian lower bound for VRPTW.

Fig. 1. ACCPM based lagrangian method for VRPTW

Primal subset has significant effect of column generation and should be chosen carefully. Another critical consideration is first localization set which should be bounded and contained optimal point. If we do not have enough information in first iterations we can execute some Kelley iterations (i.e. getting extreme points instead of analytic centers) to tune an appropriate localization set.

6. Computational Results

To illustrate the differences between proposed method and classical approach both algorithms have been implemented and CPU time and convergence rate of them have been investigated. The algorithms have tested on a benchmark problem of VRPTW. An instance of 25 vertices (24 customers) was chosen from R1-type of well-known Solomon problems [22]. Both algorithms implemented in *MATLAB* but subproblems modeled and solved via *MATLAB/GAMS* interface using solver *CPLEX* versoin12 within *GAMS*. All tests were done on a Pentium-IV 2.2 GHz with 2 GB of RAM under Microsoft Windows operating system.

Table 1 indicates final results of implementation. In this table improvement of lower bound (***LB***) is compared for our ACCPM based lagrangian method (**ACCPM**) and classical column generation method (**Kelley**). For each iteration the CPU time (***time***) is included too. The best lagrangian lower bound we seek for is 616. As expected the proposed method has reached to this value in less iteration and less total CPU time.

Tab. 1.Improvement of Lower Bounds

Iteration	ACCPM		Kelley	
	LB	*time*	*LB*	*time*
1	-9627	0.35	-8144	0.5
2	-7159	0.29	-8112	0.47
3	-6782	0.27	-8032	0.6
4	-6782	0.39	-8000	0.57
5	-6782	0.3	-8000	0.66
6	-6782	0.36	-8000	0.56
7	-6782	0.37	-8000	0.59
8	-6782	0.3	-7968	0.52
9	-6782	0.35	-7872	0.56
10	-6782	0.35	-7872	0.53
11	-6782	0.32	-7856	0.57
12	-6139	0.34	-7856	0.62
13	-5415	0.36	-7856	0.59
14	-4302	0.3	-7856	0.61
15	-3856	0.36	-7856	0.59
16	-3856	0.32	-7856	0.54
17	-2288	0.3	-7856	0.57
18	-1100	0.37	-7856	0.55
19	-411	0.36	-7856	0.53
20	71	0.28	-7856	0.62
21	375	0.36	-7856	0.55
22	469	0.35	-7856	0.59
23	469	0.29	-7808	0.56
24	494	0.38	-7808	0.59
25	496	0.35	168	0.57
26	532	0.3	168	0.58
27	542	0.44	168	0.61
28	543	0.37	168	0.56
29	556	0.27	168	0.55
30	569	0.37	168	0.57
31	583	0.36	168	0.55
32	583	0.36	168	0.58
33	583	0.33	168	0.59
34	594	0.37	168	0.56
35	594	0.32	168	0.64
36	594	0.31	168	0.6
37	602	0.34	184	0.62
38	605	0.33	319	0.59
39	605	0.38	383	0.56
40	613	0.38	415	0.7
41	615	0.33	463	0.58
42	**616**	0.28	487	0.59
43			487	0.59
44			487	0.57
45			516	0.78
46			516	0.58
47			539	0.6
48			539	0.58
49			539	0.66
50			545	0.62
51			545	0.55
52			545	0.56
53			558	0.59
54			574	0.57
55			574	0.66
56			574	0.57
57			574	0.6
58			574	0.58
59			580	0.59
60			592	0.63
61			**616**	0.59
Time		*14.21*		*35.71*

Fig. 2 depicts the convergence rate of both methods. In this figure number of iterations is compared against the difference of lower bound and best value. It is obvious that convergence rate of proposed method is more stable than classic approach regardless of primal lower bounds which are smaller in our approach.
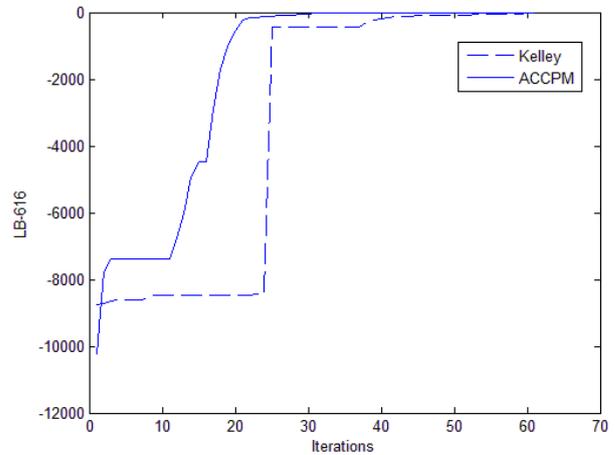
Fig. 2. Total convergence rate of two methods

7. Concluding Remarks

In this paper, the analytic center cutting plane method has been introduced into the vehicle routing problem for the first time to deal with lagrangian relaxation of the problem. At first we have described basic cutting plane methods and its relationship with column generation approach in VRPTW; then we have incorporated ACCPM into classic cutting plane method to improve its efficiency (as a stabilization technique) of finding lagrangian lower bound. The computational result on a benchmark instance demonstrates the superiority of proposed algorithm in rate of overall convergence and quality of lower bounds.

Our proposed method to compute lower bounds could be a good starting point to develop a more general framework like branch-and-price or some heuristic algorithms. This method can also be applied to other related problems to VRPTW.

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