International Journal of Industrial Engineering & Production Research

September 2013, Volume 24, Number 3
pp. 207-216

http://IJIEPR.iust.ac.ir/

ISSN: 2008-4889

Deriving Fuzzy Inequalities Using Discrete Approximation of Fuzzy Numbers

N. Khademi*, M. Rajabi Bahaabadi, A. Shariat Mohaymany & J. Shahi

Navid Khademi, *Department of Highway and Transportation Engineering, School of Civil Engineering, College of Engineering, University of Tehran, Tehran, Iran,*
Mojtaba Rajabi Bahaabadi, *Department of Transportation Planning, Faculty of Civil Engineering, IUST, mojtaba_rajabi@civileng.iust.ac.ir*
Afshin Shariat Mohaymany, *Department of Transportation Planning, Faculty of Civil Engineering, IUST, shariat@iust.ac.ir*
Jalil Shahi, *Department of Transportation Planning, Faculty of Civil Engineering, IUST, jalil@iust.ac.ir*

KEYWORDS

Fuzzy number comparison;
Fuzzy inequality;
Fuzzy number ranking

ABSTRACT

Relative comparison of fuzzy numbers plays an important role in the domain of the fuzzy multi-criteria and multi-attribute decision making analysis. For making a comparison between two fuzzy numbers, beyond the determination of their order, the concept of inequality is no longer crisp. That is, an inequality becomes fuzzy in the sense of representing partial belonging or degree of membership when one makes a cardinal comparison between two or more hypothetical fuzzy numbers. It means that we need to calculate m ($m \in [0,1]$) in the fuzzy inequalities $\lesssim_m$ and $\gtrsim_m$ among two normal fuzzy numbers. In this paper we propose a method for capturing the membership degree of fuzzy inequalities through discretizing the membership axis (μ) into equidistant intervals. In this method, the two discretized fuzzy numbers are compared point by point and at each point the degree of preferences is identified. To show its validity, this method is examined against the essential properties of fuzzy number ordering methods in [Wang, X. and E.E. Kerre, Reasonable properties for the ordering of fuzzy quantities (I). Fuzzy Sets and Systems, 2001. 118(3): p. 375-385]. Furthermore it is compared numerically with some of the celebrated fuzzy number ranking methods. The results, which provides promising outcomes, may come in useful in the domain of fuzzy multi-criteria or multi-attribute decision making analysis, and more importantly, for fuzzy mathematical programming with fuzzy inequality constraints.

1. Introduction

The purpose of this paper is to introduce a simple comparative method for fuzzy numbers through which it could be possible not only to rank fuzzy numbers by deriving their absolute preference, but also to derive the membership function corresponding to the degrees of truth of the statements "less than" or "greater than".
The fuzzy number ranking problem has been studied by many researchers (see e.g. [1-29] for an overview and comparison of various methods). Some papers have also tried to provide a review of some methods for ranking fuzzy numbers or fuzzy subsets [e.g. see 1]. Moreover, some researchers have provided the criteria for evaluating ranking methods [e.g. see 9-10, 11]. According to the fundamental researches performed by

* **Corresponding author: Navid Khademi**
Email: *navid.khademi@ut.ac.ir*
Paper first received Feb, 18, 2012, and in accepted form Sep, 10, 2012.

Wang and Kerre [9-10], there are more than 35 methods proposed in the literature of fuzzy sets and systems aiming at ordering fuzzy quantities. Yet, there exist a lot of questions arisen by the researchers, none of which have been completely accepted.

Based on the research held by Wang and Kerre [30], the fuzzy number ranking and comparison methods were categorized to three approaches (with linguistic methods excluded). Through the first approach, a function (*F*) was employed to map the fuzzy numbers to the real line $\Re$.

These representative crisp quantities were then compared with each other to give the final ranking. The second approach employed a reference set or sets through which fuzzy numbers were ranked. In the last class, based on a defined fuzzy relation, pair-wise comparisons between fuzzy numbers were made and the final ranking orders were obtained.

More or less, most of the fuzzy number ranking methods have some degree of the well-known drawbacks such as inconsistency due to human judgments, indiscrimination, intricacy of interpretation, or counter intuition [1, 7-8].

Most of the research in the domain of fuzzy number comparisons serve the fuzzy number ordering purpose. However, fuzzy inequalities have received less attention, except for a few recent studies [e.g. see 7]. In other words, for making a comparison between two fuzzy numbers, first the order, and second, the magnitude of the order should be determined. In this way, the concept of inequality is no longer crisp; however it becomes fuzzy in the sense of representing partial belonging or degree of membership.

In fuzzy mathematics, an inequality is a statement regarding the relative size or magnitude of two fuzzy objects or about whether they are the same or not. Fuzzy inequality mostly appears in fuzzy mathematical programming and fuzzy inequality constraints [31]. However, each comparison made between two fuzzy numbers or a fuzzy number and a crisp one can be placed in the domain of fuzzy comparisons and fuzzy inequalities.

In this paper, a comparative method is proposed for fuzzy numbers which gives the membership function corresponding to the fuzzy number comparisons and fuzzy inequalities. It means that the concept of fuzziness is extended to the inequality measures. Also, the proposed method is checked against the reasonable properties of fuzzy ranking methods in [9-10] in order to prove the rationality of the procedure. Finally, through a numerical example, the proposed method is compared with one of the recent fuzzy number comparison methods.

2.Basic Definitions

In this paper, a fuzzy number is denoted by ζ_I where *I* is the index of the fuzzy number which can be *R* for the *reference fuzzy number* to which other fuzzy numbers are compared, or it could be *K* for the items which are compared with respect to the reference fuzzy number.

Generally, the fuzzy number ζ_I is a fuzzy set with membership function μ over the real number domain $\Re$. A *real fuzzy number* also is defined as a fuzzy set on $\Re$ if its membership function is continuous and has the following characteristics [32]:

$$\begin{array}{ll} \mu_{\zeta_I}(x_1) \geq \mu_{\zeta_I}(x_2), \text{ for } \forall x_1 > x_2 \in \Re, & \text{on } [i_1, i_2], \\ 1 & \text{on } [i_2, i_3], \\ \mu_{\zeta_I}(x_1) \leq \mu_{\zeta_I}(x_2), \text{ for } \forall x_1 > x_2 \in \Re, & \text{on } [i_3, i_4], \\ 0 & \text{on } (-\infty, i_1] \cap [i_4, +\infty). \end{array} \quad (1)$$

where $i_1 \leq i_2 \leq i_3 \leq i_4$ are real amounts.

The defined real fuzzy number ζ_I is consequently convex meaning that, for each $\lambda \in [0,1]$ and $x_1, x_2 \in X$, it gives $\mu_{\zeta_I}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\zeta_I}(x_1), \mu_{\zeta_I}(x_2)\}$ [33]. It is obviously normal because $\max(\mu_{\zeta_I}) = 1$. Also, $\mu_{\zeta_I}(x) \in [0,1]$ and

$$\begin{cases} \wedge_x \mu_{\zeta_I}(x) = 1 \\ \mu_{\zeta_I}(x_2) \geq \mu_{\zeta_I}(x_1) \vee \mu_{\zeta_I}(x_3), \forall x_1 \geq x_2 \geq x_3 \in R_I \end{cases} \quad (2)$$

where $\wedge$ is the max. operator and $\vee$ is the min. operator.

In this paper, the proposed methodology was extended to *triangular fuzzy numbers* (TFNs). ζ_I is a TFN if $\zeta_I = \{\underline{i}, i_M, \overline{i}\}$, where ($i_M$,1) is the main value (peak, kernel, or modal value). According to [34], the TFN ζ_I is defined on $\Re$ by:

$$\zeta_I \triangleq \mu_{\zeta_I}(x) = \begin{cases} \dfrac{x - \underline{i}}{i_M - \underline{i}} & \text{for } \underline{i} \leq x \leq i_M, \\ \dfrac{x - \overline{i}}{i_M - \overline{i}} & \text{for } i_M \leq x \leq \overline{i}, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where [$\underline{i}$, $\overline{i}$] is the supporting interval and is shown by Supp(ζ_I) and the symbol ≜ means "is defined by".

2-1. Fuzzy Inequalities

Suppose a fuzzy comparison should be made between two real fuzzy numbers (ζ_K, ζ_R) which are defined over the supports R_K and R_R where $R_K, R_R \subset \Re$ and specified through membership functions $(\mu_{\zeta_K}(x), x \in R_K)$ and $(\mu_{\zeta_R}(x), x \in R_R)$ respectively.

Considering the support of these *2* fuzzy numbers, the union of the supports $\tilde{R} = R_K \cup R_R$ is the domain in which the comparison is made.

The proposed method takes two fuzzy numbers and returns a degree of membership function. The membership function maps fuzzy numbers of a given universal set *X*, which is always a crisp set, into real numbers in [0, 1] as:

$$\mu_{\Diamond}: X \times X \rightarrow \Re^+$$

where $\Diamond$ is an arithmetic comparison relation such as fuzzy less than or fuzzy greater than ($\gtrsim$ and $\lesssim$).

At the outset, the first definition of the membership function made by Zadeh [35] needs to be recalled: a membership function characterized a function (μ) by which each crisp object *x* (from a space of crisp object *X* on the real domain $\Re$) is associated with a real number in the interval [0,1] according to the "grade of membership" of *x* to a specified property or characteristic ζ_I (i.e. $\mu_{\zeta_I}: X \rightarrow [0,1]$). This property can, for example, point to the numbers in the real domain whether they are much greater than 1 or not [35]. Seemingly, it could also be generalized to the domain of fuzzy numbers which are greater than a specific fuzzy number, named in this paper the *reference fuzzy number* (ζ_R). In other words, the degree to which a fuzzy number belongs to the set of fuzzy numbers which are greater (or less) than the reference fuzzy number can be defined.

Therefore, with a slight change to the first definition of the membership function, which inquires the degrees of truth of a property with respect to a crisp object (item), the membership function is defined which maps a set of fuzzy numbers to the real line $\Re$ with respect to a fuzzy object which is the reference fuzzy number (ζ_R).

A fuzzy comparison between fuzzy numbers ζ_K and the reference fuzzy number ζ_R is denoted by $\zeta_K C \zeta_R$. This comparison between these two fuzzy numbers embodies the following fuzzy statements in this paper:

- $\zeta_K \hat{C} \zeta_R$: compares ζ_K with ζ_R to find out how much ζ_K is smaller than the reference fuzzy number (ζ_R).
- $\zeta_K \breve{C} \zeta_R$: compares ζ_K with ζ_R to find out how much ζ_K is greater than the reference fuzzy number (ζ_R).

The fuzzy set $\hat{C}_{\zeta_R}$ points to the grade or degree to which any fuzzy number (ζ_K) is less than the reference fuzzy number ζ_R as:

$$\hat{C}_{\zeta_R} = \left\{ \left(\zeta_K, \mu_{\hat{C}\zeta_R}(\zeta_K) \right) \middle| \mu_{\hat{C}\zeta_R} > 0 \right\} \quad (4)$$

where $\mu_{\hat{C}\zeta_R}(\zeta_K)$ specifies the degree to which any fuzzy number ζ_K belongs to the set of elements which are less than ζ_R. The fuzzy sets of $\breve{C}_{\zeta_R}$ is defined seemingly. Therefore, the nearer the value of $\mu_{\hat{C}\zeta_R}$ ($\mu_{\breve{C}\zeta_R}$) to unity, the higher the degree of being less (greater) than the reference fuzzy number.

Through these definitions, the aforementioned fuzzy inequalities is represented as $\lesssim_{\boxed{?}}$ and $\gtrsim_{\boxed{?}}$ where the real numbers between 0 and 1 are replaced in $\boxed{?}$ as the degree of membership of the inequality. For example, for two given TFNs ζ_K and ζ_R, $\left(\underline{k}, k_M, \bar{k}\right) \lesssim_{0.78} \left(\underline{r}, r_M, \bar{r}\right)$ can be presented meaning that ζ_K is smaller than ζ_R with the degree of 0.78.

2-2.Discretized Fuzzy Numbers

Discretizing the continuous membership function of a fuzzy number, which is held by discrete fuzzy set, embodies i) x_i-axis equidistant intervals, and ii) μ-axis equidistant intervals [36]. Through the former approach, for the fuzzy number ζ_K, the *x*-axis is subdivided into constant intervals $\Delta x = x_{i+1} - x_i, \quad \forall x_i \in [\underline{k}, \bar{k}]$, where $[\underline{k}, \bar{k}]$ is the supporting interval of ζ_K. In the latter approach, the membership axis is subdivided into constant discrete μ_i with the fixed interval $\Delta\mu = \mu_{j+1} - \mu_j, \quad \forall \mu_j \in [0,1]$.

In this paper, the discretization of the μ-axis is followed based on its inherent advantages over the discretization of the *x*-axis [36]. That is, the μ-axis is subdivided to *m* fixed intervals:

$$\Delta\mu = \frac{1}{m} \quad (5)$$

where *m* is named the *discretization value* [37]. Then, the fuzzy number ζ_K is figured approximately by the discrete fuzzy set:

$$\zeta_K = \left\{ (\underline{k}_0, \mu_0), (\underline{k}_1, \mu_1), (\underline{k}_2, \mu_2), \ldots, (k_M, \mu_M), \ldots, (\bar{k}_2, \mu_2), (\bar{k}_1, \mu_1), (\bar{k}_0, \mu_0) \right\} \quad (6)$$

where k_M is the modal value of the reference fuzzy number and $\underline{k}_i$ and $\bar{k}_i$ satisfy the following equations:

$$\mu_{\zeta_K}(\underline{k}_j) = \mu_j \textit{ and } \frac{d\mu_{\zeta_K}(x_j)}{dx_j}\bigg|_{x_j = \underline{k}_j} > 0,\ j = 1, 2, \ldots, m-1, \quad (7)$$

$$\mu_{\zeta_K}(\bar{k}_j) = \mu_j \textit{ and } \frac{d\mu_{\zeta_K}(x_j)}{dx_j}\bigg|_{x_j = \bar{k}_j} < 0,\ j = 1, 2, \ldots, m-1, \quad (8)$$

$$\underline{k}_0 = \underline{k}, \quad \bar{k}_0 = \bar{k} \quad \textit{with} \quad (\underline{k}, \bar{k}) = \mathrm{supp}(\zeta_K), \quad (9)$$

$$\underline{k}_M = \underline{k}_1 = \bar{k}_1 = \mathrm{core}(\zeta_K). \quad (10)$$

3. Proposed Fuzzy Numbers Comparison Method

In order to derive a general rule for the fuzzy comparisons in the next part, we start with a comparison between a crisp number (y) and a fuzzy reference number (ζ_R) where the comparison (C) encompasses $\hat{C}$ and $\breve{C}$.

3-1. Fuzzy-Crisp Comparison ($yC\zeta_R$)

The reference fuzzy number ζ_R and the crisp amount y are considered as shown in Figure 1.a. The crisp number y is absolutely smaller than ζ_R if $y \le \underline{r}$ and is absolutely greater than ζ_R if $y \ge \overline{r}$. For $\underline{r} < y < \overline{r}$, y partially belongs to the set of numbers less than ζ_R or greater than ζ_R. The reference fuzzy number ζ_R could be divided to n discrete crisp numbers as an approximate representation of ζ_R (as shown in Figure 1.b) which is called the *possible discrete approximation of fuzzy number* ζ_R [38]. Consequently, the fuzzy-crisp comparison can be approximately performed through n crisp-crisp comparisons. The larger the n, the higher the precision.

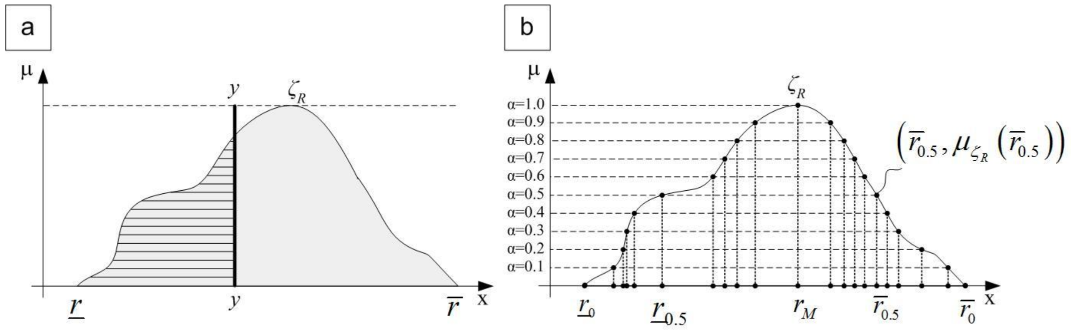

Fig. 1. Discrete approximation of the reference fuzzy number ζ_R.

In order to approximate a fuzzy number ζ_R by the possible discrete amounts, α-cuts of the fuzzy number are used. For the fuzzy number ζ_R, α-cuts $\left(\zeta_R\right)_\alpha = \left\{x \in \Re \middle| \mu_{\zeta_R}(x) \ge \alpha, \alpha \in [0,1]\right\}$ are convex subsets of $\Re$. The lower and upper limits for an α-cut are represented by:

$$\underline{r}_\alpha = \inf_{x \in \Re}\left\{x \middle| \mu_{\zeta_R}(x) \ge \alpha\right\} \tag{11}$$

$$\overline{r}_\alpha = \sup_{x \in \Re}\left\{x \middle| \mu_{\zeta_R}(x) \ge \alpha\right\} \tag{12}$$

For example, as shown in Figure 1, the comparison in terms of $y\breve{C}\zeta_R$, which derives the degree of membership for "y is greater than ζ_R", is held for the discrete approximation of fuzzy number ζ_R as:

$$\mu_{y\breve{C}\zeta_R} = \frac{\sum_\alpha I[y \succ r_\alpha] \times \mu_{\zeta_R}(r_\alpha)}{\sum_\alpha \mu_{\zeta_R}(r_\alpha)} \tag{13}$$

where $r_\alpha = \left\{\underline{r}_\alpha, \overline{r}_\alpha, r_M \middle| \alpha = \gamma n, \gamma \in [0,1/n], n \in Z^+\right\}$ (see Figure 1) and $I[\cdot]$ is an indicator function having the value of 1 for true and 0 for false statements.

As can be seen in Equation 13, the degree of membership of comparison inherits from the magnitude of the membership of the numbers coming into the comparison. As an example for a fuzzy-crisp comparison, a problem proposed by Bojadziev [34, pp.71-76] is recalled. In this example, a fuzzy Delphi method has been employed to enquire the technical realization of a brand new product through asking a group of experts. The average triangular number provided from experts' views is presented in Figure 2.a Bojadziev [34, p.76]. Suppose that the questions could be inquired from the experts in a different manner resulting in a simpler fuzzy number $\zeta_R = \left\{(2001, 0.5), (2009, 1)\right\}$ as shown in Figure 2.b.

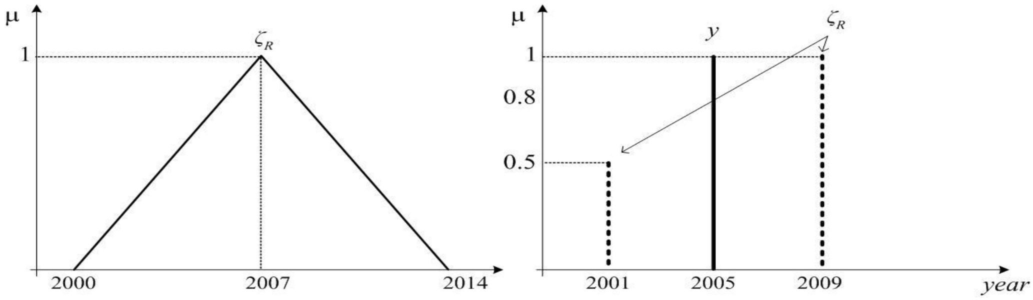

Fig. 2. Possible time of the technical realization of a new product.

If the year 2005 is defined as the deadline for the technical realization of the brand new product, the degree to which this deadline can be met is doubtful or, in other words, the degree to which the deadline (y) is greater than the possible time of the technical realization of the new product (ζ_R) is not clear. The reference fuzzy number ζ_R has a discrete support of $\underline{r} = 2001$ and $\bar{r} = 2009$ with the degrees of membership 0.5 and 1, respectively. In a comparison made between ζ_R and $y=2005$, $y \gtrsim_1 \underline{r}$ and $y \gtrsim_0 \bar{r}$. To sum up then, y is greater than ζ_R with the following degree of membership:

$$\left(1\times\mu_{\zeta_R}\left(\underline{r}\right)+0\times\mu_{\zeta_R}\left(\bar{r}\right)\right)\Big/\left(\mu_{\zeta_R}\left(\underline{r}\right)+\mu_{\zeta_R}\left(\bar{r}\right)\right)=$$
$$\left(1\times 0.5\right)\Big/\left(0.5+1\right)=0.33$$

which means that $\mu_{y\breve{C}\zeta_R} = 0.33$ or $y \gtrsim_{0.33} \zeta_R$. It means that $y \lesssim_{0.77} \zeta_R$.

When n (number of α-cuts)→∞, the degree to which the crisp number y is smaller than the reference fuzzy number ζ_R (i.e. the degree of membership of $y\breve{C}\zeta_R$ or, stated equivalently, $y \gtrsim_{y\breve{C}\zeta_R} \zeta_R$), is calculated from the following integral over ζ_R:

$$\mu_{y\breve{C}\zeta_R} = \frac{\int_{\underline{r}}^{y}\mu_{\zeta_R}\left(x\right)dx}{\int_{\underline{r}}^{\bar{r}}\mu_{\zeta_R}\left(x\right)dx} \tag{14}$$

Equation (14) may be perceived intuitively and can be represented graphically in Figure 1.a by the ratio of hatched area to the overall area underneath the fuzzy number ζ_R.

3.2 Fuzzy-Fuzzy Comparison

A comparison can be considered between the reference fuzzy number ζ_R and a given fuzzy number ζ_K which can approximately be divided to n discrete fuzzy amounts $\left\{\left(y_k,\mu_{\zeta_K}\left(y_k\right)\right)\right\}$. Then, the comparison is divided approximately to n comparisons and the membership function is calculated as:

$$\mu_{\zeta_K C\zeta_R} = \frac{\sum_{\zeta_K}\mu_{\zeta_K}(y_k)\times\left(y_k C\zeta_R\right)}{\sum_{\zeta_K}\mu_{\zeta_K}(y_k)} \tag{15}$$

where $\cdot C\cdot$ refers to any of the two comparisons $\cdot\breve{C}\cdot$ and $\cdot\hat{C}\cdot$; $\sum_{\zeta_K}$ point to the discretization of fuzzy number ζ_K, and $xC\zeta_R$ is the fuzzy-crisp comparisons derived from Equation 13.

Similar to Equations 13 and 14, when the number of α-cuts tends to infinity, Equation 15 becomes:

$$\mu_{\zeta_K\breve{C}\zeta_R} = \int_{\underline{k}}^{\bar{k}}\mu_{\zeta_K}(y)\times\left(\frac{\int_{\underline{r}}^{y}\mu_{\zeta_R}(x)dx}{\int_{\underline{r}}^{\bar{r}}\mu_{\zeta_R}(x)dx}\right)dy \Bigg/ \int_{\underline{k}}^{\bar{k}}\mu_{\zeta_K}(y)\,dy \tag{16}$$

$$\mu_{\zeta_K\hat{C}\zeta_R} = \int_{\underline{k}}^{\bar{k}}\mu_{\zeta_K}(y)\times\left(\frac{\int_{y}^{\bar{r}}\mu_{\zeta_R}(x)dx}{\int_{\underline{r}}^{\bar{r}}\mu_{\zeta_R}(x)dx}\right)dy \Bigg/ \int_{\underline{k}}^{\bar{k}}\mu_{\zeta_K}(y)\,dy \tag{17}$$

4. Wang and Kerre's Properties

Based on the properties introduced by Wang and Kerre [10], the proposed method was examined against these properties. Let S be a set of fuzzy quantities for which, method M can be used and suppose Z as a finite subset of S. The notations $\zeta_K \succ \zeta_M$, $\zeta_K \sim \zeta_M$ and $\zeta_K \succsim \zeta_M$ mean that ζ_K has a higher ranking than ζ_M, the same ranking as ζ_M, and at least the same ranking as ζ_M.

Property 1. For an arbitrary finite subset Z of S and $\zeta_I \in Z$, by the proposed method on Z we have $\zeta_I \succsim \zeta_I$.

Proof.

$$\mu_{\zeta_I\breve{C}\zeta_I} = \int_{\underline{i}}^{\bar{i}}\mu_{\zeta_I}(y)\times\left(\frac{\int_{\underline{i}}^{y}\mu_{\zeta_I}(x)dx}{\int_{\underline{i}}^{\bar{i}}\mu_{\zeta_I}(x)dx}\right)dy \Bigg/ \int_{\underline{i}}^{\bar{i}}\mu_{\zeta_I}(y)\,dy = 0.5 \succ 0$$

In the same way, $\mu_{\zeta_I\hat{C}\zeta_I} = 0.5$.

It means that a given fuzzy number is less (or greater) than itself with the degree of membership 0.5. □

Property 2. For an arbitrary finite subset Z of S and $\zeta_K,\zeta_M \in Z$, $\zeta_K \succsim \zeta_M$ and $\zeta_M \succsim \zeta_K$ by the proposed method on Z, the proposed method gives $\zeta_K \sim \zeta_M$ on Z.

Proof. According to Equations 16 and 17, $\zeta_K \succsim \zeta_M$ means $\mu_{\zeta_K\breve{C}\zeta_M} \geq 0.5$. On the contrary, $\zeta_M \succsim \zeta_K$ means $\mu_{\zeta_M\breve{C}\zeta_K} \geq 0.5$. Hence, it can be concluded that $\zeta_K \sim \zeta_M$. □

Property 3. For an arbitrary finite subset Z of S and $\zeta_K,\zeta_M,\zeta_N \in Z$, $\zeta_K \succsim \zeta_M$ and $\zeta_M \succsim \zeta_N$, then $\zeta_K \succsim \zeta_N$ is supposed by the proposed method on Z.

Proof. According to Equations 16 and 17, $\zeta_K \succsim \zeta_M$ means that $\mu_{\zeta_K\breve{C}\zeta_M} \geq 0.5$. In addition, $\zeta_M \succsim \zeta_N$ gives $\mu_{\zeta_M\breve{C}\zeta_N} \geq 0.5$. From the two inequalities $\mu_{\zeta_K\breve{C}\zeta_M} \geq 0.5$ and $\mu_{\zeta_M\breve{C}\zeta_N} \geq 0.5$, we can conclude that $\mu_{\zeta_K\breve{C}\zeta_N} \geq 0.5$ which indicates $\zeta_K \succsim \zeta_N$. □

Property 4. For an arbitrary finite subset Z of S and $\zeta_K,\zeta_M \in Z$, if $\inf Supp\left(\zeta_K\right) > \sup Supp\left(\zeta_M\right)$, by the proposed method on Z, $\zeta_K \succsim \zeta_M$ on Z is supposed.

Proof. Straightforward. □

Property 5. Let S and S' be two arbitrary finite sets of fuzzy quantities in which the proposed method can be applied and ζ_M and ζ_K which are in $S \cap S'$. The ranking order $\zeta_K \succ \zeta_M$ can be obtained by the proposed method on S' iff $\zeta_K \succ \zeta_M$ by the proposed method on S.

Proof. See [10, proposition 4.14, p. 383]. □

Property 6. Let $\zeta_K, \zeta_M, \zeta_K + \zeta_N$ and $\zeta_M + \zeta_N$ be elements of S. If $\zeta_K \succsim \zeta_M$ by the proposed method on $\{\zeta_K, \zeta_M\}$, then $\zeta_K + \zeta_N \succsim \zeta_M + \zeta_N$ by the proposed method on $\{\zeta_K + \zeta_N, \zeta_M + \zeta_N\}$.

Proof.

Let $(\zeta_K)_\alpha = [\underline{k}_\alpha, \bar{k}_\alpha], (\zeta_M)_\alpha = [\underline{m}_\alpha, \bar{m}_\alpha], (\zeta_N)_\alpha = [\underline{n}_\alpha, \bar{n}_\alpha]$.

$(\zeta_K + \zeta_M)_\alpha = [\underline{k}_\alpha + \underline{m}_\alpha, \bar{k}_\alpha + \bar{m}_\alpha]$ and $(\zeta_K + \zeta_N)_\alpha = [\underline{k}_\alpha + \underline{n}_\alpha, \bar{k}_\alpha + \bar{n}_\alpha]$.

Because $\underline{n}_\alpha$ and $\bar{n}_\alpha \geq 0 \; \forall \alpha \in \Re$, if $\underline{k}_\alpha \geq \underline{m}_\alpha$ then $\underline{k}_\alpha + \underline{n}_\alpha \geq \underline{m}_\alpha + \underline{n}_\alpha$

and if $\bar{k}_\alpha \geq \bar{m}_\alpha$ then $\bar{k}_\alpha + \bar{n}_\alpha \geq \bar{m}_\alpha + \bar{n}_\alpha$.

when $(\zeta_K)_\alpha \geq (\zeta_M)_\alpha$ then it holds $(\zeta_K + \zeta_N)_\alpha \geq (\zeta_M + \zeta_N)_\alpha$

$\Rightarrow \mu_{\zeta_{K+N} \check{C} \zeta_{M+N}} \geq 0.5.$ □

Property 7. Let $\zeta_K, \zeta_M, \zeta_K\zeta_N$; and $\zeta_M\zeta_N$ be elements of S. If $\zeta_K \succsim \zeta_M$ by the proposed method on $\{\zeta_K, \zeta_M\}$ and $\zeta_N \succ 0$, then $\zeta_K\zeta_N \succsim \zeta_M\zeta_N$ by the proposed method on $\{\zeta_K\zeta_N, \zeta_M\zeta_N\}$.

Proof. Straightforward and similar to the proof of property 6.

5. Numerical Example

Because of the principal role of TFNs in fuzzy set theory, this section provides the extended solution of Equations 16 and 17 for TFNs. Let us consider two TFNs: $\zeta_R = (\underline{r}, r_M, \bar{r})$ and $\zeta_K = (\underline{k}, k_M, \bar{k})$ such that $\underline{r} \leq \underline{k}$. According to Figure 3, it is obvious that, there are 10 permutations for sequencing the two fuzzy numbers ζ_R and ζ_K (considering $\underline{r} \leq \underline{k}$). For simplicity, Equation 16 is computed for each permutation and the consequent results are presented in Table 1.

As shown in Figure 4, a software is developed to compare two triangular fuzzy numbers for the proposed method. The computer codes were written in C# programming language and the software is downloadable from the following URL: http://webpages.iust.ac.ir/khademi/supplementary/. In order to assess the validity and the robustness of the proposed method, 15 sets of triangular fuzzy numbers are employed to compare the proposed method of this paper with the most validated and acclaimed fuzzy number ranking methodologies devised by Cheng [17], Chu and Tsao [18], Wang et al.[39], Wang and Lee [27], Deng et al. [19], Nasseri and Sohrabi [40], and Modarres and Sadi-Nezhad [7] as shown in Figure 5. The results of the comparison (see Table 2), reveals the fact that the results of the proposed method are almost the same as the results of the approved above-mentioned methods.

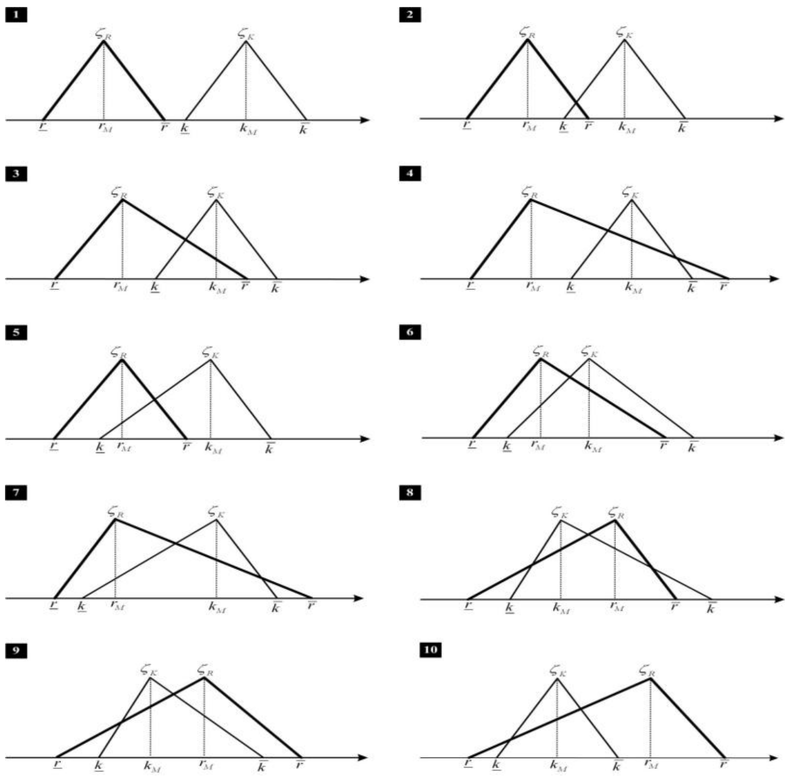

Fig. 3. Comparison between two triangular fuzzy numbers

Tab. 1. (continued from above)

Type	$\mu_{\zeta_R \breve{C} \zeta_K} =$
1	0
2	$\frac{1}{24(r_M-\bar{r})(\underline{k}-k_M)}(6\underline{r}\bar{r}\underline{k}k_M - 6\bar{r}r_M k_M\bar{k} - 6\underline{r}\bar{r}k_M\bar{k} + 6\bar{r}r_M\underline{k}k_M + 6\bar{r}^2k_M\bar{k} - 6\bar{r}^2\underline{k}k_M - 6\underline{r}r_M\underline{k}k_M + 6\underline{r}r_M k_M\bar{k} + 4\bar{r}^3\underline{k} - 6\underline{r}\bar{r}\underline{k}^2 - 6r_M\bar{r}\underline{k}^2 + 4\bar{r}\underline{k}^3 + 6\underline{r}r_M\underline{k}^2 - \bar{r}^4 - \underline{k}^4 + 6\underline{r}\bar{r}\underline{k}\bar{k} + 6r_M\bar{r}\underline{k}\bar{k} - 6\bar{r}^2\underline{k}\bar{k} - 6\underline{r}r_M\underline{k}\bar{k}$
3	$\frac{1}{24(r_M-\bar{r})(k_M-\bar{k})}(\bar{r}^4 + 6\bar{r}r_M\underline{k}\bar{k} - 6\bar{r}r_M\underline{k}k_M + 6\bar{r}r_M k_M\bar{k} + 6\underline{r}\bar{r}\underline{k}\bar{k} - 6\underline{r}\bar{r}\underline{k}k_M + 6\underline{r}\bar{r}k_M\bar{k} - 6\underline{r}r_M\underline{k}\bar{k} - 4\bar{r}\underline{k}^2\bar{k} + 4\bar{r}\underline{k}^2k_M - 4\bar{k}\bar{r}^3 + \bar{k}k_M^3 - \underline{k}k_M^3 + 4\underline{k}\bar{r}k_M^2 - 4\bar{k}\bar{r}k_M^2 - 6\bar{r}\bar{k}^2r_M - 6\underline{r}\bar{k}^2\bar{r} + 6\underline{r}\bar{k}^2r_M + \underline{k}k_M^2\bar{k} + 6\bar{k}^2\bar{r}^2 + \underline{k}^3\bar{k} - \underline{k}^3k_M - \underline{k}^2k_M^2 - 4\underline{k}k_M\bar{k}\bar{r})$
4	$\frac{1}{24(r_M-\bar{r})}(-\underline{k}k_M^2 + k_M^2\bar{k} + 4\bar{r}\underline{k}k_M - 4\bar{r}k_M\bar{k} - \underline{k}^2k_M + k_M\bar{k}^2 - 6\underline{r}\bar{r}\underline{k} - 6r_M\bar{r}\underline{k} + 6\underline{r}r_M\underline{k} - \underline{k}^3 + 4\bar{r}\underline{k}^2 + 6r_M\bar{r}\bar{k} - 6\underline{r}r_M\bar{k} + 6\underline{r}\bar{r}\bar{k} - 4\bar{r}\bar{k}^2 + \bar{k}^3$
5	$\frac{1}{24(\underline{r}-r_M)(\underline{k}-k_M)}(\underline{r}r_M^2 - 4\underline{r}\underline{k}^3 - r_M^3\bar{r} - r_M\bar{r}^3 + \underline{r}\bar{r}^3 - r_M^2\bar{r}^2 + 4r_M^2\bar{r}\underline{k} + \underline{r}r_M^2\bar{r} - 4\underline{r}r_M\bar{r}\underline{k} - 6\underline{r}r_M k_M\bar{k} - 6\underline{r}\bar{r}k_M\bar{k} - 6r_M\bar{r}\underline{k}\bar{k} + 6\underline{r}r_M\underline{k}\bar{k} - 6r_M\bar{r}\underline{k}k_M + 6\underline{r}r_M\underline{k}k_M + 6r_M\bar{r}k_M\bar{k} - 6\underline{r}^2\underline{k}\bar{k} - 6\underline{r}^2\underline{k}k_M + 6\underline{r}^2k_M\bar{k} + 6\underline{r}^2\underline{k}^2 + \underline{k}^4 + 6\underline{r}\bar{r}\underline{k}\bar{k} + 6\underline{r}\bar{r}\underline{k}k_M + \underline{r}r_M\bar{r}^2 + 4r_M\bar{r}^2\underline{k} - 4\underline{r}\bar{r}^2\underline{k} - 4\underline{r}r_M^2\underline{k}$
6	$\frac{1}{24(\underline{k}-k_M)(\underline{r}-r_M)(r_M-\bar{r})(k_M-\bar{k})}(-4r_M^3\bar{r}\underline{k}\bar{k} + 4r_M^3\bar{r}\underline{k}k_M - 4\underline{r}\bar{r}\underline{k}^3 + 4\underline{r}\bar{r}\underline{k}^3k_M + 4\underline{r}r_M\underline{k}^3\bar{k} - 4\underline{r}r_M\underline{k}^3k_M + 4\underline{r}r_M^3\underline{k}\bar{k} - 4\underline{r}r_M^3\underline{k}k_M + 6\underline{r}^2\bar{r}\underline{k}^2\bar{k} - 6\underline{r}^2\bar{r}\underline{k}^2k_M - 6\underline{r}^2r_M\underline{k}^2\bar{k} + 6\underline{r}^2r_M\underline{k}^2k_M - 4r_M\bar{r}k_M^3\bar{k} + 4\underline{r}\bar{r}k_M^3\bar{k} + 4r_M\bar{r}\underline{k}k_M^3 - 4\underline{r}\bar{r}\underline{k}k_M^3 + 6r_M^2\bar{r}k_M^2\bar{k} - 6\underline{r}^2\bar{r}k_M^2\bar{k} - 6\underline{r}r_M^2k_M^2\bar{k} + 6\underline{r}^2r_M k_M^2\bar{k} - 6r_M^2\bar{r}\underline{k}k_M^2 + 6\underline{r}^2\bar{r}\underline{k}k_M^2 + 6\underline{r}r_M^2\underline{k}k_M^2 - 6\underline{r}^2r_M\underline{k}k_M^2 + r_M^4\bar{r}\bar{k} - r_M^4\bar{r}k_M + \bar{r}\underline{k}^4\bar{k} - \bar{r}\underline{k}^4k_M - r_M\underline{k}^4\bar{k} + r_M\underline{k}^4k_M - \underline{r}r_M^4\bar{k} + \underline{r}r_M^4k_M + r_M k_M^4\bar{k} - \underline{r}k_M^4\bar{k} - r_M\underline{k}k_M^4 + \underline{r}\underline{k}k_M^4 + r_M\bar{r}^4k_M - \underline{r}\bar{r}^4k_M - r_M\bar{r}^4\underline{k} + \underline{r}\bar{r}^4\underline{k} - 4r_M\bar{r}^3k_M\bar{k} + 4\underline{r}\bar{r}^3k_M\bar{k} + 4r_M\bar{r}^3\underline{k}\bar{k} - 4\underline{r}\bar{r}^3\underline{k}\bar{k} + 6r_M\bar{r}^2k_M\bar{k}^2 - 6r_M^2\bar{r}k_M\bar{k}^2 - 6\underline{r}\bar{r}^2k_M\bar{k}^2 - 6r_M\underline{r}^2\underline{k}\bar{k}^2 + 6r_M^2\bar{r}\underline{k}\bar{k}^2 + 6\underline{r}\bar{r}^2\underline{k}\bar{k}^2 + 6\underline{r}r_M^2k_M\bar{k}^2 + 6\underline{r}^2\bar{r}k_M\bar{k}^2 - 6\underline{r}^2r_M k_M\bar{k}^2 - 6\underline{r}r_M^2\underline{k}\bar{k}^2 - 6\underline{r}^2\bar{r}\underline{k}\bar{k}^2 + 6\underline{r}^2r_M\underline{k}\bar{k}^2)$
7	$\frac{1}{24(\underline{k}-k_M)(\underline{r}-r_M)(r_M-\bar{r})(k_M-\bar{k})}(-4r_M^3\bar{r}\underline{k}\bar{k} + 4r_M^3\bar{r}\underline{k}k_M - 4\underline{r}\bar{r}\underline{k}^3 + 4\underline{r}\bar{r}\underline{k}^3k_M + 4\underline{r}r_M\underline{k}^3\bar{k} - 4\underline{r}r_M\underline{k}^3k_M + 4\underline{r}r_M^3\underline{k}\bar{k} - 4\underline{r}r_M^3\underline{k}k_M + 6\underline{r}^2\bar{r}\underline{k}^2\bar{k} - 6\underline{r}^2\bar{r}\underline{k}^2k_M - 6\underline{r}^2r_M\underline{k}^2\bar{k} + 6\underline{r}^2r_M\underline{k}^2k_M - 4r_M\bar{r}k_M^3\bar{k} + 4\underline{r}\bar{r}k_M^3\bar{k} + 4r_M\bar{r}\underline{k}k_M^3 - 4\underline{r}\bar{r}\underline{k}k_M^3 + 6r_M^2\bar{r}k_M^2\bar{k} - 6\underline{r}^2\bar{r}k_M^2\bar{k} - 6\underline{r}r_M^2k_M^2\bar{k} + 6\underline{r}^2r_M k_M^2\bar{k} - 6r_M^2\bar{r}\underline{k}k_M^2 + 6\underline{r}^2\bar{r}\underline{k}k_M^2 + 6\underline{r}r_M^2\underline{k}k_M^2 - 6\underline{r}^2r_M\underline{k}k_M^2 + r_M^4\bar{r}\bar{k} - r_M^4\bar{r}k_M + \bar{r}\underline{k}^4\bar{k} - \bar{r}\underline{k}^4k_M - r_M\underline{k}^4\bar{k} + r_M\underline{k}^4k_M - \underline{r}r_M^4\bar{k} + \underline{r}r_M^4k_M + r_M k_M^4\bar{k} - \underline{r}k_M^4\bar{k} - r_M\underline{k}k_M^4 + \underline{r}\underline{k}k_M^4 + r_M\bar{r}^4k_M - \underline{r}\bar{r}^4k_M - r_M\bar{r}^4\underline{k} + \underline{r}\bar{r}^4\underline{k} - 4r_M\bar{r}^3k_M\bar{k} + 4\underline{r}\bar{r}^3k_M\bar{k} + 4r_M\bar{r}^3\underline{k}\bar{k} - 4\underline{r}\bar{r}^3\underline{k}\bar{k} + 6r_M\bar{r}^2k_M\bar{k}^2 - 6r_M^2\bar{r}k_M\bar{k}^2 - 6\underline{r}\bar{r}^2k_M\bar{k}^2 - 6r_M\underline{r}^2\underline{k}\bar{k}^2 + 6r_M^2\bar{r}\underline{k}\bar{k}^2 + 6\underline{r}\bar{r}^2\underline{k}\bar{k}^2 + 6\underline{r}r_M^2k_M\bar{k}^2 + 6\underline{r}^2\bar{r}k_M\bar{k}^2 - 6\underline{r}^2r_M k_M\bar{k}^2 - 6\underline{r}r_M^2\underline{k}\bar{k}^2 - 6\underline{r}^2\bar{r}\underline{k}\bar{k}^2 + 6\underline{r}^2r_M\underline{k}\bar{k}^2)$
8	$\frac{1}{24(\underline{r}-r_M)(k_M-\bar{k})}(4\underline{r}r_M^2\bar{k} - \underline{k}k_M^2\bar{k} + 4\underline{r}k_M^2\bar{k} + 4\underline{r}\underline{k}^2\bar{k} - 4\underline{r}\underline{k}^2k_M - 4\underline{r}k_M^2\underline{k} - 4r_M^2\bar{r}\bar{k} - \underline{r}r_M^2\bar{r} - \underline{r}r_M\bar{r}^2 - 6\underline{r}^2k_M\bar{k} - 6\underline{r}^2\underline{k}\bar{k} + 6r_M\bar{r}\bar{k}^2 - 6\underline{r}r_M\bar{k}^2 - 6\underline{r}\bar{r}\bar{k}^2 - \underline{k}^2k_M\bar{k} + 6\underline{r}^2\underline{k}k_M + 4\underline{r}\bar{r}^2\bar{k} - 4r_M\bar{r}^2\bar{k} + \underline{k}k_M^3 - \underline{k}^3\bar{k} + \underline{k}^3k_M - k_M^3\bar{k} - \underline{r}r_M^3 + r_M^3\bar{r} + r_M\bar{r}^3 - \underline{r}\bar{r}^3 + 6\underline{r}^2\bar{k}^2 + \underline{k}^2k_M^2 + 4\underline{r}\underline{k}k_M\bar{k} + 4\underline{r}r_M\bar{r}\bar{k} + r_M^2\bar{r}^2)$
9	$\frac{1}{24(r_M-\bar{r})(k_M-\bar{k})(\underline{r}-r_M)}(\bar{r}\underline{k}k_M^2\bar{k} - r_M\underline{k}k_M^2\bar{k} - 4\underline{r}\bar{r}k_M^2\bar{k} + 4\underline{r}r_M k_M^2\bar{k} + 4\underline{r}\bar{r}\underline{k}^2k_M - 4\underline{r}r_M\underline{k}^2k_M + 4\underline{r}r_M\underline{k}k_M\bar{k} - 4\underline{r}\bar{r}\underline{k}k_M\bar{k} - \bar{r}\underline{k}k_M^3 + r_M\underline{k}k_M^3 + \bar{r}\underline{k}^3\bar{k} - r_M\underline{k}^3\bar{k} - \bar{r}\underline{k}^3k_M + r_M\underline{k}^3k_M + \bar{r}k_M^3\bar{k} - r_M k_M^3\bar{k} - 4r_M^3\bar{r}\bar{k} + 4\underline{r}r_M^3\bar{k} - 4r_M\bar{r}\bar{k}^3 + 4\underline{r}\bar{r}\bar{k}^3 + 6r_M^2\bar{r}\bar{k}^2 - 6\underline{r}^2\bar{r}\bar{k}^2 - 6\underline{r}r_M^2\bar{k}^2 + 6\underline{r}^2r_M\bar{k}^2 - \bar{r}\underline{k}^2k_M^2 + r_M\underline{k}^2k_M^2 + r_M^4\bar{r} - \underline{r}r_M^4 + r_M\bar{k}^4 - \underline{r}\bar{k}^4 + 4\underline{r}\bar{r}\underline{k}k_M^2 - 4\underline{r}r_M\underline{k}k_M^2 + 6\underline{r}^2\bar{r}\underline{k}\bar{k} + 6\underline{r}^2\bar{r}k_M\bar{k} + 6\underline{r}^2\bar{r}\underline{k}\bar{k} - 6\underline{r}^2r_M k_M\bar{k} - 6\underline{r}^2r_M\underline{k}\bar{k} + \bar{r}\underline{k}^2k_M\bar{k} - r_M\underline{k}^2k_M\bar{k} + 6\underline{r}^2r_M\underline{k}k_M - 6\underline{r}^2\bar{r}\underline{k}k_M)$
10	$\frac{1}{24(r_M-\bar{r})}(-\underline{k}k_M^2 + k_M^2\bar{k} + 4\bar{r}\underline{k}k_M - 4\bar{r}k_M\bar{k} - \underline{k}^2k_M + k_M\bar{k}^2 - 6\underline{r}\bar{r}\underline{k} - 6r_M\bar{r}\underline{k} + 6\underline{r}r_M\underline{k} - \underline{k}^3 + 4\bar{r}\underline{k}^2 + 6r_M\bar{r}\bar{k} - 6\underline{r}r_M\bar{k} + 6\underline{r}\bar{r}\bar{k} - 4\bar{r}\bar{k}^2 + \bar{k}^3$

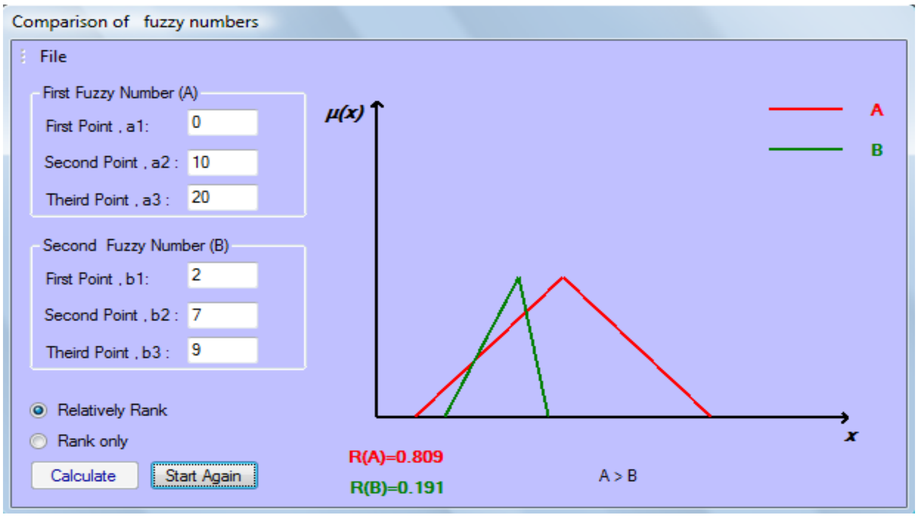

Fig. 4. Fuzzy inequality and comparison software.

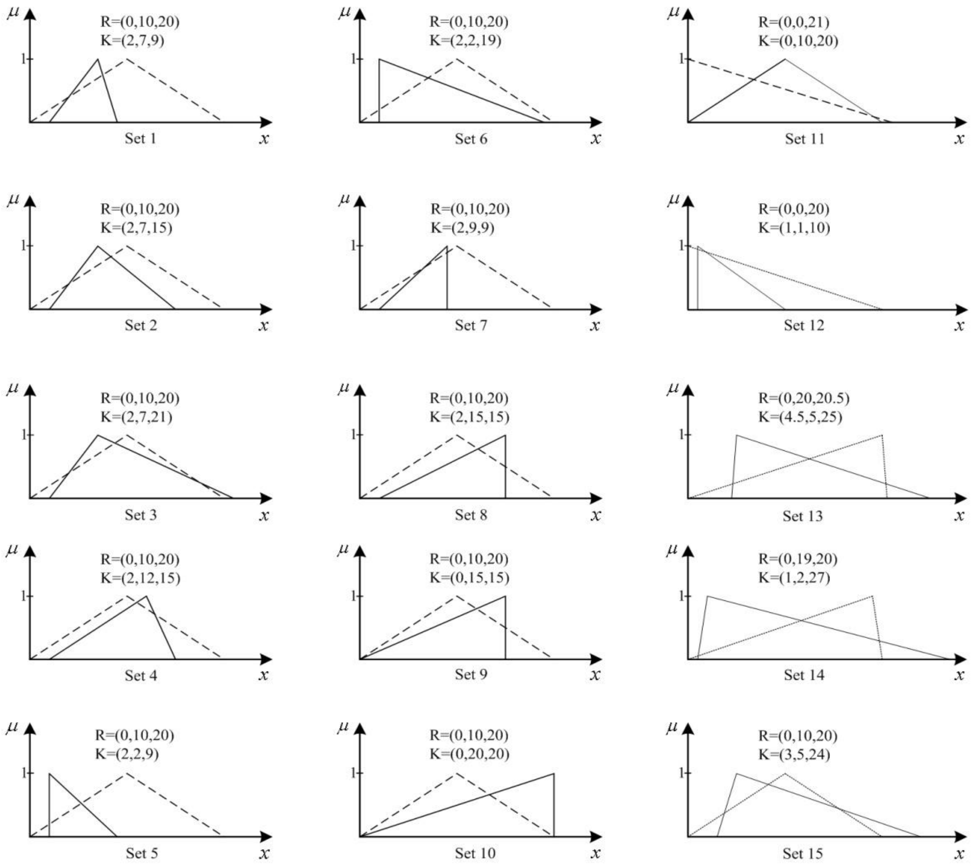

Fig. 5. Triangular fuzzy number sets in the numerical example

Tab. 2 . Comparative examples

Set	Cheng [17]	Chu and Tsao [18]	Wang et al. [39]	Wang and Lee [27]	Deng et al. [19]	Nasseri and Sohrabi [40]	Modarres and Sadi-Nezhad [7]	The proposed method
1	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.828} \zeta_K$	$\zeta_R \succ_{0.81} \zeta_K$
2	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.767} \zeta_K$	$\zeta_R \succ_{0.655} \zeta_K$
3	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R \prec_{0.665} \zeta_K$	$\zeta_R \succ_{0.511} \zeta_K$
4	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \prec_{0.514} \zeta_K$	$\zeta_R \succ_{0.52} \zeta_K$
5	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.896} \zeta_K$	$\zeta_R \succ_{0.893} \zeta_K$
6	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.89} \zeta_K$	$\zeta_R \succ_{0.664} \zeta_K$
7	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.802} \zeta_K$	$\zeta_R \succ_{0.764} \zeta_K$
8	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec_{0.679} \zeta_K$	$\zeta_R \prec_{0.558} \zeta_K$
9	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R = \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \prec_{0.654} \zeta_K$	$\zeta_R \prec_{0.511} \zeta_K$
10	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec_{1} \zeta_K$	$\zeta_R \prec_{0.708} \zeta_K$
11	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec_{0.854} \zeta_K$	$\zeta_R \prec_{0.688} \zeta_K$
12	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.91} \zeta_K$	$\zeta_R \succ_{0.651} \zeta_K$
13	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.559} \zeta_K$	$\zeta_R \succ_{0.631} \zeta_K$
14	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ \zeta_K$	$\zeta_R \succ_{0.629} \zeta_K$	$\zeta_R \succ_{0.666} \zeta_K$
15	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec \zeta_K$	$\zeta_R \prec_{0.811} \zeta_K$	$\zeta_R \prec_{0.521} \zeta_K$

However, In the one hand, all of these methods, except our method and the one proposed by Modarres and Sadi-Nezhad [7], are not able to calculate the degree of membership of inequalities. In the other hand, Modarres and Sadi-Nezhad's method [7] suffers from some drawbacks.

For example, in set 4, applying Modarres and Sadi-Nezhad's method [7] leads to an incorrect ranking order: $\zeta_R \prec \zeta_K$, which is contrary to the ranking order: $\zeta_R \succ \zeta_K$ obtained by the other methods. Moreover, based on Modarres and Sadi-Nezhad's method [7], we have $\zeta_K \succ \zeta_R$ with the degrees of membership 0.665 and 0.654 in sets 3 and 10 respectively. However the proposed method gives almost the same results with the other fuzzy ranking techniques. Besides, in set 10, Modarres and Sadi-Nezhad's method [7] yields the ranking $\zeta_R \underset{1}{\prec} \zeta_K$, but it is intuitively admitted that ζ_K is not completely bigger than ζ_K .

6. Conclusion

In this paper we propose a method for capturing fuzzy membership degree of fuzzy inequalities. The motivation for this work was to extend a non-counterintuitive methods in the domain of fuzzy inequalities giving fuzzy inequalities $\lesssim_{?}$ and $\gtrsim_{?}$ where $\boxed{?}$ gets a real numbers from [0, 1]. Also the proposed method in this paper can be extended also to fuzzy equalities $\eqsim_{?}$.

The main potential application of this method lies in decision making analysis problems where the preferences of decision elements (i.e. decision criteria and decision alternatives) expressed in vague terms must be examined, and moreover, in fuzzy mathematical programming with fuzzy inequality constraints. The proposed method showed good features when it was examined against Wang and Kerre's [10] essential and reasonable properties for the ordering of fuzzy quantities.

The closed form derivation of the method and also its written compute codes were developed based on this method for triangular fuzzy numbers. For other fuzzy numbers like trapezoidal ones, similar algorithm can be developed.

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