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Stochastic programming and robust optimization approaches for the product mix problem under uncertainty (case study: lubricant refinery)

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uncertainty,
robust model,
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lubricants industry

ABSTRACT

Determining optimal product mix under uncertain demand and capacity is a critical challenge in the lubricant industry. This study proposes four MILP models: deterministic, robust scenario-based, downside-risk two-stage stochastic, and CVaR two-stage stochastic. All models incorporate real-world constraints including multi-period, multi-product settings, dual-warehouse inventory, backlog/lost-sale shortages, and mandatory production of unprofitable products. Using real data from a major Iranian lubricant refinery, the models improve profit from the company's actual 300 million monetary units (MU) to 535 (deterministic) and 504 million MU (CVaR). Among stochastic models, the robust approach provides the highest worst-case profit and most stable performance. This is the first systematic comparison of these three stochastic approaches in the lubricant industry under identical constraints, demonstrating the value of uncertainty-aware production planning.

1. Introduction

Determining the optimal product mix, including selecting products and their quantities based on available resources, is a critical decision in manufacturing that directly impacts profitability. This problem, known as the "product mix" decision, becomes increasingly complex as the number of products and resources grows, often requiring advanced optimization methods (Chaharsooghi and Jafari, 2007). Traditional methods are effective for small-scale problems, but large-scale problems often necessitate heuristic approaches. Optimal product mix decisions aim to maximize profit while considering constraints like resources, market demand, and sales (Sk Ahad Ali and Jay Lee, 2003). Understanding product mix optimization is essential for companies to meet customer needs, manage inventory effectively, and enhance overall operational efficiency (Ezra et al., 2023).

This research proposes four multi-period, multi-material, and multi-product MILP models using Gurobi's exact solution method, incorporating deterministic, robust, and stochastic approaches. Additionally, uncertainties, such as demand and capacity, are addressed to aid decision-making and prevent excessive costs (Karakas et al., 2010). The models are validated through a case study of one of Iran’s largest lubricant companies, including the sales agent’s warehouse variable. Following the formulation of the deterministic model, three uncertainty-based models were formulated: a scenario-based model, a two-stage stochastic model with downside risk, and a two-stage stochastic model using Conditional Value at Risk (CVaR). These models were then compared to see how well they handled uncertainty.

The overview of this research is shown in Figure 1.

The main novelties of this study are threefold. First, while previous research has applied either robust or stochastic methods separately, we provide a systematic comparative analysis of three distinct uncertainty-handling approaches (robust scenario-based, downside-risk two-stage stochastic, and CVaR two-stage stochastic) under identical data and constraints in the lubricant industry. Second, our model incorporates two types of inventory warehouses (company and sales agent) and distinguishes between backlog and lost-sale shortages, which reflects the actual practice in lubricant distribution. Third, we include mandatory production of unprofitable products imposed by regulatory bodies – a realistic constraint often ignored in the literature. Using real operational data from one of Iran's four major lubricant refineries, we quantify the value of stochastic solution (VSS) and the expected value of perfect information (EVPI) for this industry.

The rest of this paper is presented as follows. Section 2 presents a literature review of models and data types were mapped. In section 3, a detailed statement of the problem to be studied and deterministic model, robust model, downside risk two-stage stochastic model and CVaR model are presented. In section 4, the models are solved with the real data of a lubricant manufacturing company and Gurobi solver. In the fifth section, the

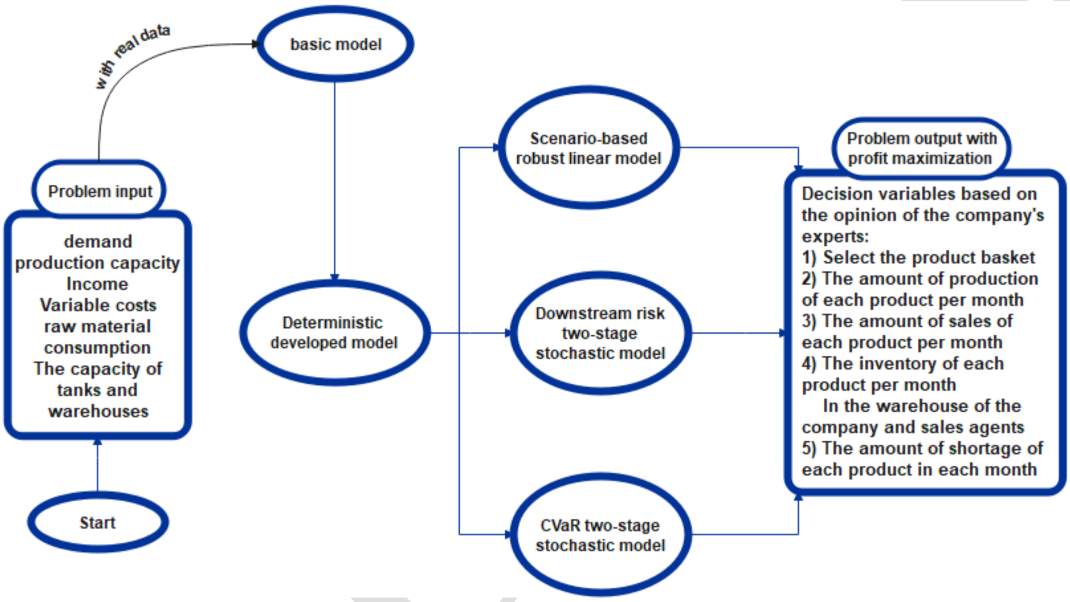

Figure 1. Overview of this research

output results of the models are given and the models have been validated using metrics and finally, the models have been compared with each other. In the section sixth, conclusions and future suggestions are given.

2. Literature Review

The product mix optimization problem, central to operations and production management, has been explored extensively using deterministic, robust, and stochastic approaches. This section synthesizes findings from past research, highlighting key methodologies and their relevance to addressing uncertainty in production planning.

Deterministic models assume fixed parameters, enabling straightforward optimization of production plans. Sarker and Khan (2002) developed a model to minimize costs for multi-raw material, single-stage systems. Moon and Hwang (1992) proposed integrated production - inventory models for multi-product facilities, emphasizing efficiency in resource allocation. Chen and Chen (2005) extended these works by introducing replenishment models that optimized multi-layer production chains. Cakravastia and Takahashi (2004) developed a procurement - production - distribution framework, focusing on profit maximization and supplier negotiations. While effective for stable conditions, deterministic models often fail to account for uncertainty in demand and resource availability.
Robust optimization focuses on creating solutions that perform well under uncertain parameters. Mulvey et al. (1995) introduced scenario-based robust optimization techniques, balancing feasibility and optimality under worst-case scenarios. Vaziri et al. (2018) applied robust models in multi-period manufacturing, incorporating machine allocation and inventory constraints.

This method proves particularly useful in the lubricant industry, where fluctuating supply and demand necessitate robust decision-making frameworks.
Stochastic programming incorporates probabilistic demand and capacity data, allowing for adaptive planning. Birge and Louveaux (2011) formalized two-stage stochastic frameworks widely applied in inventory and transportation systems. Aljazzar et al. (2016) extended these models to multi-product environments, while Gharaei and Pasandideh (2017)

optimized inventory in multi-level supply chains under stochastic conditions. Shafiee et al. (2021) applied stochastic methods in energy systems, demonstrating their adaptability in volatile industries. Such methods are highly relevant for lubricant manufacturing, where demand uncertainty impacts production schedules.

Recent work by Covei (2025) investigates a regime-switching stochastic production planning problem driven by economic cycles, capturing demand volatility through a Markov chain and multi-dimensional Brownian motion. The model is formulated via a quadratic cost functional and solved using Hamilton–Jacobi–Bellman PDEs, yielding optimal adaptive control strategies for inventory and production decisions. In parallel, Schlenkrich et al. (2025) apply a progressive hedging algorithm to a multi-stage capacitated lot-sizing problem under uncertain demand, incorporating setup carry-over constraints and achieving near-optimal costs (within 1 %) with significantly reduced computation times.

Mohammadpour Larimi et al. (2025) addressed disruption risks in inventory-location problems for perishable goods using lateral transshipment, offering robust strategies adaptable to uncertain production systems. Tsega et al. (2025) proposed the C-SCOM framework to evaluate supply chain performance in manufacturing, emphasizing flexibility and responsiveness under uncertainty.

Recent studies combine robust and stochastic approaches to address multi-dimensional uncertainties. Díaz-Madroñero et al. (2017) integrated production and transportation decisions in stochastic supply chains, achieving cost-efficiency. Abrha Honiey et al. (2022) combined seasonal forecasting with inventory optimization, balancing accuracy with adaptability. Hybrid models offer a comprehensive framework, blending the strengths of deterministic, robust, and stochastic approaches to optimize production planning in uncertain environments.

Despite advancements, there is limited research on integrating multi-product, multi-material systems with stochastic demand in the lubricant industry. Comparative analyses of robust and stochastic models for this sector are scarce, leaving a gap in understanding trade-offs between these methodologies.

For models that utilize deterministic data, several contributions stand out. Sarker and Khan (Khan and Sarker, 2002) developed a model aimed at minimizing system costs using a periodic delivery policy for a multi-raw material, single-stage production, single-plant, single-product system. Similarly, Moon and Hwang (1992) proposed an integrated production-inventory model for managing multiple products in a single facility. Chen and Chen (2005) extended this by developing a joint replenishment and channel coordination model for multi-layer production and replenishment, considering multiple items. In another work, Cakravastia and Takahashi (2004) introduced a multi-objective model focused on maximizing profits and minimizing negotiation gaps with suppliers. Lo et al. (2007) addressed energy production planning in their model, while Balkhi (2009) proposed a multi-item integrated production-inventory system. Cao, Chai, and Liu (2011) developed an IPP model for discrete production environments using a nonlinear programming approach. Additionally, Díaz-Madroñero, Mula, and Peidro (2017) contributed a transportation decision model, and Vaziri et al. (2018) implemented an IPP model for a multi-period, multi-product manufacturing system, accounting for machine allocation and inventory constraints. Gharaei, Hoseini Shekarabi, and Karimi (2020) investigated a joint economic measurement model in a multi-level, multi-product supply chain, while Benbouja et al. (2021) focused on procurement, production, and distribution planning, considering inventory levels and transportation in the supply chain. Dolgui and Ivanov (2021) applied an IPP inventory model to address joint economic order quantities for raw materials, production batches, and buyers' orders. More recently, Abrha Honiey et al. (2022) optimized an inventory control system through integrated seasonal forecasting and integer programming, and Nemati et al. (2022) introduced a deterministic mixed-integer linear programming (MILP) model for multi-material and multi-product systems using deterministic data in lubricant industry.

Table 1. Review of some articles containing the integrated model of production and inventory in recent years

The author	Year	objective function					solution method							Case study
		Minimum cost	Maximum profit	Maximum income	Minimum inventory level	Minimum shortage	Exact	Metaheuristic	Heuristic	Simulation	Hybrid	Stochastic		
(Chen, 2018)	2018		❅				❅							food industry
(Fauza *et al.*, 2018)	2018		❅					❅						food industry
(Fauza *et al.*, 2018)	2018		❅					❅						food industry

Table 1 (continued)

(Fauza, Prasetyo and Amanto, 2018a)	2018		❅					❅					food industry
(Karabağ and Tan, 2019)	2019		❅							❅			food industry
(Kaur *et al.*, 2020)	2020		❅				❅						production
(Phong and Yenradee, 2020)	2020	❅						❅					food industry
(Ibrahim *et al.*, 2020)	2020					❅				❅			food industry
(Ibrahim *et al.*, 2020)	2020		❅					❅					production
(Kafiabad, Kazemi Zanjani and Nourelfath, 2020)	2020	❅					❅						Gas turbine engines
(Masudin *et al.*, 2021)	2021		❅					❅					food industry
(Ghasemkhani *et al.*, 2022)	2022		❅								❅		Chemical factory
(Nemati, Kargari and Nikbakhsh, 2022)	2022		❅				❅						Lubricant industry
(Ezra et al., 2023).	2023		❅				❅						Bakery industry
Rashidi et) (al., 2024	2024		❅									❅	Semiconductor industry
(Ayman R. M et al., 2024)	2024	❅							❅				small manufacturing
(Troncoso, N et al., 2024)	2024	❅							❅				product–mold–machine manufacturing
This paper	2025		❅				❅					❅	Lubricant industry

For models involving fuzzy data, Maity (2014), Das et al. (2015), and Kumar et al. (2016) developed nonlinear models focused on single-raw material and single-product systems. In the realm of stochastic data, Jauhari (2012) proposed a nonlinear model for a single-raw material, single-product system, while Sana et al. (2014) developed a linear model incorporating multiple raw materials and products. Aljazzar et al. (2016), AlDurgam et al. (2017), and Islam et al. (2017) all contributed nonlinear models in this domain. Gharaei and Pasandideh (2017), along with Shafiee et al. (2021), extended these works by proposing nonlinear models for systems involving multiple raw materials and products.

Although previous studies have examined deterministic, robust, and two-stage stochastic MILP models for multi-product, multi-material systems under uncertainty, their application in the lubricant industry remains underexplored. These models, when integrating real-world constraints such as inventory management and market demands, have the potential to enhance production planning in this sector. Furthermore, the lubricant industry's unique constraints and variables have not been holistically addressed within a single model, and comparative evaluations of different stochastic approaches for this context are significantly absent from existing research.

3. Modeling for the product mix problem

This section presents the mathematical formulation for solving the multi-period, multi-product product mix problem in a manufacturing environment with inventory and shortage considerations. The primary goal is to determine the optimal product mix and production rates for each period to maximize profit margins.

3-1. Basic Assumptions

The following assumptions are considered for the proposed model:

1. **Objective Function**: The objective of the model is to maximize the profit margin by optimizing the product mix and production rate across multiple periods.

2. **Product Selection**: The composition of the product mix in each period is represented by a

binary decision variable (0 or 1), indicating whether a product is included in the production plan for that period.

3. **Production Rate**: The optimal production rate for each product is determined for every period, taking into account available resources and demand.

4. **Inventory System**: Products can be stored for future periods, allowing for inter-period supply management through an inventory system.

5. **Shortages and Backlogs**:

   - The model accommodates both lost shortages (for perishable or time-sensitive products such as antifreeze) and backlogs, where unmet demand is carried over to subsequent periods.

   - Backlogged demand is supplied in future periods when resources become available.

6. **Forecasting Period**: The product portfolio is forecasted on a monthly basis, with demand being a stochastic variable, estimated using forecasting tools prior to running the mathematical model. This demand serves as an input parameter.

7. **Time Horizon**: The model can be applied over various time horizons, including multi-period scenarios (e.g., seasonal or annual).

8. **Stochastic Demand**: Demand for each product is uncertain and modeled stochastically, based on forecasts. This stochasticity reflects real-world variability and is incorporated into the optimization model.

9. **Pricing**: Product prices are externally regulated, determined by order and under the supervision of the consumer protection organization.

10. **Profit Calculation**: Profit for each product is calculated as the difference between the selling price and the cost of production.

11. **Cost of Production**: The total cost of producing each unit of a product is influenced by global price fluctuations for lube cuts (raw materials used in lubricant production), driven by the global demand for lubricants and international oil prices, as well as domestic exchange rate fluctuations.

12. **Unprofitable Products**: Some products may be loss-making, yet the company is required to produce them due to obligations imposed by a sponsoring organization. These products are included in the production plan despite their negative contribution to profit.

13. **Production and Sales Programs**: The production schedule is distinct from the sales schedule, allowing for flexibility in managing production and distribution.

14. **Initial and Final Balance**: Inventory balances at the start of the first period and at the end of the last period are both set to zero to ensure that no leftover products remain in storage at the end of the planning horizon.

15. **Cost of Shortages for Unprofitable Products**: The cost of shortages for unprofitable products is assumed to be zero, meaning that failing to meet demand for these products incurs no additional penalty.

16. **Maintenance Costs**:

   - Maintenance costs for inventory consist of two components: fixed and variable.

   - The variable maintenance cost for unprofitable products is assumed to be zero, reflecting the reduced priority placed on their storage and upkeep.

The proposed model comprehensively represents a real-world production environment by integrating various constraints and uncertainties, including stochastic demand, regulatory pricing constraints, inventory management, and the production of both profitable and non-profitable products. The following sections provide a detailed exposition of the objective function formulation, the associated constraints, and the optimization approach utilized to solve the mixed-integer linear programming (MILP) models.

3-2. Model Inputs

The model's inputs include demand (represented as an interval), production capacity, revenue, variable costs, raw material consumption, and the storage capacities of tanks and warehouses. The variables, sets, and parameters used in the model are outlined below.

Variables

$X_{j,t}$: The optimal production rate of product j in period t

$Q_{j,t}$: The amount of shipment (sale) of product j in period t

$I1_{j,t}$: The inventory of product j at the end of period t in the company's warehouse

$I2_{j,t}$: Inventory of product j at the end of period t in the warehouse of the sales agent

$y_{j,t}$: Whether product j is produced in period t or not (binary variable)

$S_{j,t}$: Shortage of product j in period t

ε: Amount of raw material not consumed

W: Fixed model variable

Sets

j: Index of products (j = 1, 2, ..., n)

i: Index of primary sources (i = 1, 2, ..., m)

t: Time period index (t = 1, 2, ..., T)

r: Factor index (r = 1, 2, ..., R)

s: Scenario index (s = 1, 2, ..., S)

k: category products index (k = 1, …, L)

Parameters

Rt_j: Product conversion factor j (Ratio)

$a_{i,j}$: The consumption of raw material i in product j

V_j: The packaging volume of product j for its storage

$batch_j$: Minimum order size of product j for production

Pen: The selling penalty rate exceeds the demand of period t

$market_{kt}$: The production share of category products k in period t for our company

P_j: Product production capacity j

$b_{i,t}$: Raw material available i in period t

$D_{j,t}^{\min}$: Minimum demand for product j in period t

$\widetilde{D}_{j,t}$: Maximum demand for product j in period t

cp: The volume of base oil tanks

H1: Product storage capacity in the company's warehouse

$H2_r$: The storage capacity of the product in the seller's warehouse

Ic_{jt}: Sales revenue per unit of product j in period t

LC^{jt}: Lob out cost per unit of product j in period t

Add_{jt}: The cost of additives per unit of product j in period t

VC_{jt}: The variable cost of each unit of product j in period t

$MARR_t$: Minimum attractive rate of return in period t

FixCost: Fixed cost of maintenance

Z_L: Acceptable profit level

R: Maximum level of acceptable risk

R_s: Downside risk under scenario s

P_s: The probability of scenario s

q_s: CVaR model

VaR: Value at risk

CVaR_min: Minimum expected profit in the limit CVaR

M: A big number

$Rate_t$: Deficiency penalty rate in period t

$\delta_{j,t}$: Binary parameter type of shortage of product j (backlost or lost)

3-2-1. Parameter Estimation and Justification

The following key parameters are estimated using company data and expert judgment, supported by sensitivity analysis (Section 5.1).

MARR (Minimum Attractive Rate of Return): Set to 3% per period based on the company's average return on assets over the past three years (2.8%) and the central bank announced inflation rate (3.2% on average). The value was validated by financial managers.

Pen (penalty rate for exceeding demand) and Rate (shortage penalty): These are directly extracted from the company's contracts with sales agents. Any shipment exceeding the agent's demand is penalized at 2% of the selling price (Pen = 0.02 × Ic_jt). Shortages (unmet demand) are penalized at 5% of the selling price (Rate =

0.05 × Ic_jt) for non-critical products; for mandatory products, the penalty is zero by regulation.

α and CVaR_min: The confidence level α = 0.9 is selected following standard practice in financial risk management. CVaR_min = 535 million Rials is initially set as the deterministic model's profit. These values are tested in a sensitivity analysis (Section 5.1) over α ∈ [0.7,0.95] and CVaR_min ∈ [400,600] million Rials; the model's ranking of alternatives does not change within ±15% of these values.

R (maximum acceptable risk for downside-risk model): Initially set to 30 million Rials based on expert opinion (maximum allowed deviation from target profit). Sensitivity analysis shows that the model becomes feasible only when R ≥ 192 million Rials; this is discussed in the results.

3-3. Proposed Deterministic Mathematical Model

In this section, a deterministic mathematical model is proposed to optimize production and inventory management under known and fixed parameters. The model focuses on determining optimal production rates, inventory levels, and shipment quantities, considering constraints like demand, production capacities, and cost structures. This approach aims to minimize costs while ensuring efficient resource allocation and satisfying customer demand over the planning horizon.

The objective function seeks to maximize the profit margin. The first term represents the total income from sales. The second term accounts for production costs, including lube-cut, additives, and variable costs. The third term covers the maintenance costs, consisting of capital costs and fixed storage costs. The fourth term reflects the penalty costs incurred from sending more than the demand to the sales agents' warehouse. The fifth term represents the cost of shortages, while the sixth term accounts for the penalty related to unutilized raw materials. This formulation is mathematically expressed in Equation (1).

$$Maximize Z = \sum_{t=1}^{T}\sum_{j=1}^{n}\left[Ic_{jt}.Q_{jt} - \left(LC_{jt} + Add_{jt} + VC_{jt}\right)X_{jt}\right] - \left\{\begin{aligned} &\sum_{t=1}^{T}\sum_{j=1}^{n}\left[MARR_t\left(LC_{jt} + Add_{jt} + VC_{jt}\right) + Fix\,Cos\,t\right]I1_{jt} \\ &+ \sum_{t=1}^{T}\sum_{j=1}^{n} Pen\left[Ic_{jt}\right]I2_{jt} + \sum_{t=1}^{T}\sum_{j=1}^{n} Rate\left[Ic_{jt}\right]s_{jt} + \varepsilon.M \end{aligned}\right\} \quad (1)$$

The raw material availability constraint ensures that the total raw materials used in production do not exceed the supply available in each period. This constraint is mathematically defined in Equation (2).

$$b_{it} - \varepsilon \le \sum_{j=1}^{n} a_{ij}.X_{jt} \le b_{it}, \quad i = 1,\dots m;\ t = 1,\dots,T \quad (2)$$

The inventory balance constraint ensures that the inventory levels at the company's warehouse are updated based on production, shipments, and existing stock at each time period. This constraint is mathematically defined in Equation (3).

$$I1_{jt} = I1_{j,t-1} + X_{jt} - Q_{jt}, \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (3)$$

The constraint related to the inventory balance of sales agents ensures that the stock levels at the sales agents' warehouses are adjusted based on the amount of shipments and the demand in each period. This constraint is mathematically defined in Equation (4).

$$I2_{jt} - \delta_{jt}s_{jt} = I2_{j,t-1} + Q_{jt} - \left(D_{jt} + s_{j,t-1}\right), \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (4)$$

The sales ceiling constraint limits the quantity of each product sold in each period, ensuring it does not exceed the maximum allowable sales. This constraint is mathematically defined in Equation (5).

$$Q_{jt} \le I1_{jt-1} + X_{jt}, \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (5)$$

The minimum production constraint ensures that the production of each product meets or exceeds the minimum demand in each period. This constraint is mathematically defined in Equation (6).

$$D_{jt}^{min}\, y_{jt} \pounds X_{jt}, \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (6)$$

The constraint on maximum production across the time horizon limits total production based on the cumulative demand for future periods and any shortages from previous periods. This constraint is mathematically defined in Equation (7).

$$X_{jt} \le \sum_{t=t'}^{T}\widetilde{D}_{jt} + s_{j,t-1}, \quad j = 1,2,\cdots,n;\ t' = 1,\dots,T \quad (7)$$

The total sales constraint ensures that by the end of the planning horizon, the cumulative sales do not exceed the total demand across all periods. This constraint is mathematically defined in Equation (8).

$$\sum_{t=1}^{T} Q_{jt} \le \sum_{t=t'}^{T}\widetilde{D}_{jt}, \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (8)$$

The constraint on production capacity ensures that the production of each product remains within the company's production limits. This constraint is mathematically defined in Equation (9).

$$X_{jt} \le P_{jt}.y_{jt}, \quad j = 1,2,\cdots,n;\ t = 1,\dots,T \quad (9)$$

The minimum production constraint ensures that the production of each product meets the minimum requirements imposed by the Consumer Protection Organization and aligns with the company's policies. This constraint is mathematically defined in Equation (10).

$$\sum_{j=1}^{n} X_{jt}.Rt_j \le CP, \quad t = 1,\dots,T \quad (10)$$

The raw material tank capacity constraint ensures that the amount of lube-cut (the main raw material) stored does not exceed the available tank capacity. This constraint is mathematically defined in Equation (11).

$$\sum_{j=1}^{n} I1_{jt}.V_j \leq H1 \ , \qquad t = 1, \dots, T \qquad (11)$$

The company's final product storage constraint ensures that the total inventory of finished products remains within the company's warehouse capacity. This constraint is mathematically defined in Equation (12).

$$\sum_{j=1}^{n} I2_{jt}.V_j \leq H2_r \ , \qquad t = 1, \dots, T; r = 1,2, \dots, R \quad (12)$$

The storage capacity constraint for sales agents ensures that the inventory of finished products stored in the agents' warehouses does not exceed their storage limits. This constraint is mathematically defined in Equation (13).

$$batch_j y_{jt} \leq X_{jt}, \qquad j = 1,2, \cdots, n; \ t = 1, \dots, T \quad (13)$$

The production category constraint ensures that production adheres to the company's defined product categories, and this constraint is mathematically defined in Equation (14).

$$\sum_{k=1}^{L} X_{kt}.Rt_j \leq market_{kt}, \quad j = 1,2, \cdots, n; \ t = 1, \dots, T \quad (14)$$

The market share constraint ensures that production aligns with the market share allocated to each of the four lubricant companies. This constraint is mathematically defined in Equation (15).

$$D_{jt}^{min} \leq X_{jt} \ , \qquad j = 1,2, \cdots, n; \ t = 1, \dots, T \qquad (15)$$

The inventory balance constraint ensures that the company's inventory is empty at the beginning of the first period and at the end of the last period. This constraint is mathematically defined in Equation (16).

$$I1_{j0} = I1_{jT} = 0 \qquad j = 1,2, \cdots, n \qquad (16)$$

Similarly, the inventory balance constraint for sales agents ensures that their warehouses are empty at the beginning of the first period and at the end of the last period. This constraint is mathematically defined in Equation (17).

$$I2_{j0} = I2_{jT} = 0 \qquad j = 1,2, \cdots, n \qquad (17)$$

The shortage constraint ensures that there is no shortage of products at the beginning of the first period. This constraint is mathematically defined in Equation (18).

$$s_{j0} = 0 \qquad j = 1,2, \cdots, n \qquad (18)$$

The binary variable constraint controls the selection of the product mix, ensuring that only valid product choices are made. This constraint is mathematically defined in Equation (19).

$$y_{jt} = \begin{cases} = 1 \ \ If\ product\ j\ is\ produced\ in\ period\ t \\ = 0 \ \ If\ product\ j\ is\ not\ produced\ in\ period\ t \end{cases} \quad (19)$$

The shortage type constraint categorizes the type of shortage for each product, ensuring appropriate classification. This constraint is mathematically defined in Equation (20).

$$\delta_j = \begin{cases} = 1 \ If\ shortage\ of\ product\ j\ is\ not\ \text{missing} \\ = 0 \ If\ shortage\ of\ product\ j\ is\ \text{missing} \end{cases} \quad (20)$$

The model enforces variable constraints, requiring production variables to be integer and non-negative, product selection variables to be binary, and non-utilization of raw material variables to be continuous real values. These are formulated as follows:

$$Q_{jt}, I1_{jt}, I2_{jt}, s_{jt}, X_{jt} \geq 0 \ \ and\ Integer, \qquad \forall j, t \qquad (21)$$

3-4. Proposed Two-Stage Stochastic Mathematical Model

The two-stage stochastic mathematical model has its origins in the foundational work of George B. Dantzig in the 1950s, where he introduced innovative stochastic programming methods to address optimization under uncertainty (Dantzig, 1955). Initially designed for decision-making in uncertain conditions, this model structure has evolved into a critical tool in fields such as production planning, supply chain management, and energy systems. The two-stage model includes two phases: the first stage involves making initial decisions on production planning and resource allocation before uncertain parameters are realized, while the second stage allows for adaptive decision-making. In this latter phase, decisions are updated based on observed outcomes, such as production rates and inventory levels, enabling better responsiveness to market fluctuations. In this way, the two-stage model effectively manages uncertainties and provides the flexibility needed for adjusting decisions (Rashidi et al., 2024). In the following, the variables and parameters of the second stage are all indexed s, the linear model is proposed as follows.

$$\begin{aligned} MaximizeZ = \sum_{t=1}^{T}\sum_{j=1}^{n} \Big[& Ic^{s}{}_{jt}.Q^{s}{}_{jt} \\ & - \left(LC^{s}{}_{jt} + Add^{s}{}_{jt} + VC^{s}{}_{jt}\right)X_{jt}\Big] \end{aligned}$$

$$- \left\{ \begin{aligned} & \sum_{t=1}^{T}\sum_{j=1}^{n} \left[MARR_t \left(LC^{s}{}_{jt} + Add^{s}{}_{jt} + VC^{s}{}_{jt}\right) + Fix\,Cos\,t\right] I1^{s}{}_{jt} \\ & + \sum_{t=1}^{T}\sum_{j=1}^{n} Pen\left[Ic^{s}{}_{jt}\right] I2^{s}{}_{jt} + \sum_{t=1}^{T}\sum_{j=1}^{n} Rate\left[Ic^{s}{}_{jt}\right] s^{s}{}_{jt} + \varepsilon.M \end{aligned} \right\} \quad (22)$$

$$b_{it} - \varepsilon \leq \sum_{j=1}^{n} a_{ij}\,.X_{jt} \leq b_{it}, \quad i = 1, \dots m; \ t = 1, \dots, T \quad (23)$$

$$I1^{s}{}_{jt} = I1^{s}{}_{j,t-1} + X_{jt} - Q^{s}{}_{jt} \ ,$$

$j = 1,2, \cdots, n;\ t = 1, \dots, T;\ s = 1, \dots, S$ (24)

$I2^s{}_{jt} - \delta^s{}_{jt} s^s{}_{jt} = I2^s{}_{j,t-1} + Q^s{}_{jt} - (D^s{}_{jt} + s^s{}_{j,t-1})$,

$j = 1,2, \cdots, n;\ t = 1, \dots, T;\ s = 1, \dots, S$ (25)

$Q^s{}_{jt} \le I1^s{}_{jt-1} + X_{jt},$

$j = 1,2, \cdots, n;\ \ t = 1, \dots, T;\ \ s = 1, \dots, S$ (26)

$D^{min}_{jt} y_{jt} \text{£}\ X_{jt}$,

$j = 1,2, \cdots, n;\ \ t = 1, \dots, T;\ \ s = 1, \dots, S$ (27)

$X_{jt} \le \sum_{t=t'}^{T} \widetilde{D^s}_{jt} + s^s{}_{j,t-1}$,

$j = 1,2, \cdots, n;\ \ t' = 1, \dots, T;\ \ s = 1, \dots, S$ (28)

$\sum_{t=1}^{T} Q^s{}_{jt} \le \sum_{t=t'}^{T} \widetilde{D^s}_{jt}$,

$j = 1,2, \cdots, n;\ t = 1, \dots, T; s = 1, \dots, S$ (29)

$X_{jt} \le P^s{}_{jt}.y_{jt}$,

$j = 1,2, \cdots, n;\ t = 1, \dots, T; s = 1, \dots, S$ (30)

$\sum_{j=1}^{n} X_{jt}.Rt_j \le CP$, $t = 1, \dots, T$ (31)

$\sum_{j=1}^{n} I1^s{}_{jt}.V_j \le H1$, $t = 1, \dots, T;\ s = 1, \dots, S$ (32)

$\sum_{j=1}^{n} I2^s{}_{jt}.V_j \le H2_r$,

$t = 1, \dots, T; r = 1,2, \dots, R; s = 1, \dots, S$ (33)

$batch_j y_{jt} \le X_{jt},\ \ j = 1,2, \cdots, n;\ t = 1, \dots, T$ (34)

$\sum_{k=1}^{L} X_{kt}.Rt_j \le market^s{}_{kt},$

$j = 1,2, \cdots, n;\ \ t = 1, \dots, T;\ \ s = 1, \dots, S$ (35)

$D^{min}_{jt} y_{jt} \text{£}\ X_{jt}$,
$j = 1,2, \cdots, n;\ t = 1, \dots, T; s = 1, \dots, S$ (36)

$I1^s{}_{j0} = I1^s{}_{jT} = 0$ $j = 1,2, \cdots, n;\ \ s = 1, \dots, S$ (37)

$I2^s{}_{j0} = I2^s{}_{jT} = 0$ $j = 1,2, \cdots, n;\ \ s = 1, \dots, S$ (38)

$s^s{}_{j0} = 0$ $j = 1,2, \cdots, n;\ \ s = 1, \dots, S$ (39)

$$y_{jt} = \begin{cases} 1 & If\ product\ j\ is\ produced\ in\ period\ t \\ 0 & If\ product\ j\ is\ not\ produced\ in\ period\ t \end{cases} \quad (40)$$

$$\delta^s{}_j = \begin{cases} 1 & If\ shortage\ of\ product\ j\ is\ not\ missing\ in\ scenario\ s \\ 0 & If\ shortage\ of\ product\ j\ is\ missing\ in\ scenario\ s \end{cases} \quad (41)$$

$Q^s{}_{jt}, I1^s{}_{jt}, I2^s{}_{jt}, s^s{}_{jt}, X_{jt} \ge 0\ \ and\ Integer, \forall j, t, s$ (42)

3-5. Proposed Two-Stage Stochastic Mathematical Model of Downside Risk

Suppose that ZL is the acceptable profit level (expected), R is the maximum level of acceptable risk (for a one-way deviation from ZL) and Ps is the probability of each scenario occurring, and for each scenario, if the profit obtained is greater than ZL, it is desirable in our opinion, and if at the very least, we have faced a risk. If we represent the downside risk under scenario s relative to the target ZL by Rs, we will have the following relations, the rest of the constraints will be the same as the developed deterministic model. Rs is set only if the model profit is less than the acceptable profit margin and we want to minimize the mathematical expectation of Rs. In fact, Rs is the amount of transgression from ZL, and the more a scenario is less than ZL, the more unfavorable it is, and it states that the average distance of scenarios from ZL should be reduced as much as possible, which is shown as follows:

$$MaximizeZ = \sum_{s=1}^{S} \cdot \sum_{t=1}^{T} \sum_{j=1}^{n} P_s \big[Ic_{jts}.Q_{jts} - (LC_{jts} + Add_{jts} + VC_{jts}) X_{jt} \big] - \left\{ \sum_{s=1}^{S} \cdot \sum_{t=1}^{T} \sum_{j=1}^{n} P_s \big[MARR_t (LC_{jts} + Add_{jts} + VC_{jts}) + Fix\,Cos\,t \big] I1_{jts} + Pen\big[Ic_{jts}\big] I2_{jts} + Rate\big[Ic_{jts}\big] s_{jts} + \varepsilon.M \right\} \quad (43)$$

Subject to:

$$R_s \le z_L - \sum_{s=1}^{S} \cdot \sum_{t=1}^{T} \sum_{j=1}^{n} P_s \big[Ic_{jts}.Q_{jts} - (LC_{jts} + Add_{jts} + VC_{jts}) X_{jt} \big] - \left\{ \sum_{s=1}^{S} \cdot \sum_{t=1}^{T} \sum_{j=1}^{n} P_s \big[MARR_t (LC_{jts} + Add_{jts} + VC_{jts}) + Fix\,Cos\,t \big] I1_{jts} + Pen\big[Ic_{jts}\big] I2_{jts} + Rate\big[Ic_{jts}\big] s_{jts} + \varepsilon.M \right\} \quad (44)$$

$s = 1, \dots, S$

$$\sum_{s=1}^{S} P_s R_s \le R \quad (45)$$

$Eq(23 - 42)$

3-6. Proposed Robust Mathematical Model

The objective function of the robust model is the maximization of the worst possible scenario. Only one constraint is added to the deterministic model and the rest of the constraints of the robust model are the same as the multi-period deterministic model, with the difference that all variables and parameters (except for the production variable) have indexed scenarios. The scenario-based robust peer model includes three absolute, deviant and relative approaches, which are done here with the relative approach of modeling. This approach is more interesting in the literature and here it had better result than the other approaches.

$$MaxZ_D = Min \underset{s=1,2,\ldots,S}{Max} \{(\sum_{t=1}^{T}\sum_{j=1}^{n}[Ic_{jt}^s.Q_{jt}^s - (LC_{jt}^s + Add_{jt}^s + VC_{jt}^s)X_{jt}^s] - \sum_{t=1}^{T}\sum_{j=1}^{n}[MARR_t(LC_{jt}^s + Add_{jt}^s + VC_{jt}^s) + Fix\,Cos\,t]I1_{jt}^s - \sum_{t=1}^{T}\sum_{j=1}^{n}Pen[Ic_{jt}^s]I2_{jt}^s - \sum_{t=1}^{T}\sum_{j=1}^{n}Rate[Ic_{jt}^s]s_{jt}^s) - \varepsilon.M\}/Z_s^*, s\ ,1,2,\ldots,S \qquad (46)$$

$Subject\ to:$

$Eq(23-42)$

A member variable of real numbers (w) is defined which this time is maximized in the objective function. Then, in constraints such as relation 46, the larger w is equal to the objective function that is supposed to be close to the optimal value and is calculated divided by the optimal objective function of the previously calculated deterministic model. The rest of the constraints are the same as the deterministic model (Mulvey, Vanderbei and Zenios, 1995). The linearized relative scenario-based robust model is given in Eq. (47).

$MaxW \qquad W \in R$

$Subject\ to:$

$$W \le \{(\sum_{t=1}^{T}\sum_{j=1}^{n}[Ic_{jt}^s.Q_{jt}^s - (LC_{jt}^s + Add_{jt}^s + VC_{jt}^s)X_{jt}^s] - \sum_{t=1}^{T}\sum_{j=1}^{n}[MARR_t(LC_{jt}^s + Add_{jt}^s + VC_{jt}^s) + Fix\,Cos\,t]I1_{jt}^s - \sum_{t=1}^{T}\sum_{j=1}^{n}Pen[Ic_{jt}^s]I2_{jt}^s - \sum_{t=1}^{T}\sum_{j=1}^{n}Rate[Ic_{jt}^s]s_{jt}^s - \varepsilon.M)\}/Z_s^*, s\ ,1,2,\ldots,S \qquad (47)$$

$Subject\ to:$

$Eq(23-42)$

3-7. Proposed CVaR Two-stage Stochastic Mathematical Model

The added constraints to the deterministic model are given in Eq.s (48-50) where Eq. 49 calculates the value of the objective function under scenario s to fit into the constraint of Eq. 50. We assume that the sum of the probabilities of all scenarios that benefit less than VaR is α. Relationship 50 states that the profitability of the model under the scenarios that get a value lower than VaR in α-1 percent of the time should be greater than CVaR_min. The amount of CVaR_min is determined by an expert, which is usually calculated by subtracting the acceptable profit level from the value of VaR and the amount of CVaR_min is obtained. The rest of the constraints are the same as the deterministic model.

$$Maximize\ Z = \sum_{s=1}^{S}\sum_{t=1}^{T}\sum_{j=1}^{n}P_s\left[Ic_{jts}.Q_{jts} - \left(LC_{jts} + Add_{jts} + VC_{jts}\right)X_{jt}\right] - \left\{\begin{array}{l}\sum_{s=1}^{S}\sum_{t=1}^{T}\sum_{j=1}^{n}P_s\left[MARR_t(LC_{jts} + Add_{jts} + VC_{jts}) + Fix\,\mathrm{Cos}\,t\right]I1_{jts} \\ + Pen\left[Ic_{jts}\right]I2_{jts} + Rate\left[Ic_{jts}\right]s_{jts} + \varepsilon.M\end{array}\right\} \qquad (48)$$

$Subject\ to:$

$$q_s = \sum_{s=1}^{S}\sum_{t=1}^{T}\sum_{j=1}^{n}P_s\left[Ic_{jts}.Q_{jts} - \left(LC_{jts} + Add_{jts} + VC_{jts}\right)X_{jt}\right] - \left\{\begin{array}{l}\sum_{s=1}^{S}\sum_{t=1}^{T}\sum_{j=1}^{n}P_s\left[MARR_t(LC_{jts} + Add_{jts} + VC_{jts}) + Fix\,\mathrm{Cos}\,t\right]I1_{jts} \\ + Pen\left[Ic_{jts}\right]I2_{jts} + Rate\left[Ic_{jts}\right]s_{jts} + \varepsilon.M\end{array}\right\}$$

$$s = 1,\ldots,S \qquad (49)$$

$$CVaR\_\min \le VaR + \frac{1}{(1-\alpha)|S|}\sum_{s\in S}(q_s - VaR)^+ \qquad (50)$$

$Eq(23-42)$

4. Solving Models

The proposed model is solved with the real data of the company and according to the defaults of Section 3-1. The company that provides the required data for the problem is a lubricant manufacturing company that produces various types of lubricants, including base oil, diesel and industrial engine oils.

The input data related to the actual sales of the company includes 46 products, the average of the actual sales limits of the company as the demand ceiling of the model, MARR = 3% (related to variable cost of maintenance) and R = 5% (related to shortage) is considered. It is assumed that at least 3.2 million liters of general diesel family products and 6 million liters of product number 1 should be produced in each period. The fixed storage cost per liter is 0.000005 million MU. For the downside risk model, the acceptable profit level is 510 million MU and the maximum acceptable risk level is 30 million MU. Also, for the CVaR model, the value of α is equal to 0.9 and CVaR_min is equal to 535 million MU. Models are solved with pyomo library with Gurobi solver 9.5.1, the processor RYZEN 9 and RAM 16GB as system specifications.

4-1. Scenarios

The scenarios have been created with the help of real production data of several months of the production company and placed in the models. The numbers related to the final price of the products have been multiplied by an inflation factor, as well as the demand of the products in hypothetical numbers and by the opinion of the company's experts. Three scenarios have been produced, the first scenario is consistent with the company's real data, in the second scenario, income, costs and demand are increased by 10% compared to the first scenario, and in the third scenario, income, costs and demand are increased by 10% compared to the second scenario.

In the following, the outputs related to the amount of production in the first period, is given in figure 2.

5. Results and Discussion

The actual profit of the company is 300 million monetary units (MU). In Table 3 comparison of the profitability of all models with the actual profit of the company and also the increase in the profit margin of each model is given.

For the robust model, the stable response value index (VRS) has been used. The value of Z_{EEV} is the average value of the cost of using the average value answer and Z_{HN} is the answer of the here and now method. Finally, the value of the VRS index is calculated from equation 51 (Tarim, Manandhar and Walsh, 2006).

$$VRS = Z_{HN} - Z_{EEV} \quad (51)$$

Table 3. Comparison chart of the profit margin of all models with the actual profit of the company and increase in their profitability

Multi-period model with inventory system					modeling
The actual profit of the company	Deviation robust model	CVaR model	Stochastic model (downside risk)	Deterministic model	
300 million MU	**357 million MU**	**504 million MU**	**505 million MU**	**535 million MU**	profitability

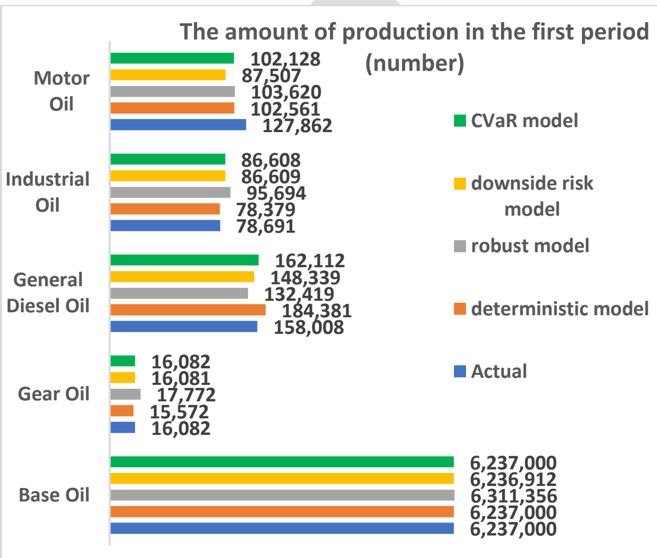

Figure 2. The amount of production of each product category in the first period (number)

The value of Z_{EEV} is (-488) and Z_{HN} is 357. Then the value of the VRS index will be 845 million MU that shows if we do not consider the uncertainty and use the average, we will lose about 237% profit in the long term, which indicates the very high importance of considering the uncertainty in this issue.

For stochastic models, The Value of Stochastic Solution (VSS) has been used. The value of the VSS index is calculated from equation 52 (Birge and Louveaux, 2011).

$$VSS = Z_{HN} - Z_{EEV} \quad (52)$$

For the two-stage stochastic model of downside risk, the value of Z_{EEV} is 386 and Z_{HN} is 504.08 million MU, considering the assumed parameters for this model, the problem becomes unjustified and the value of the parameter of the maximum level of acceptable risk should be increased from 50 million MU to 192 million MU, to be justified. The value of the VSS index assuming the new value of the R parameter will be equal to 118.08 million MU that shows if we consider the uncertainty and do not use the average, we will earn about 30.59% more profit in the long-term.

For CVaR two-stage stochastic model, the value of Z_{EEV} is 386 and Z_{HN} is 504.07 million MU, considering the assumed parameters for this model, the problem becomes unjustified and we must reduce the value of CVaR_min parameter from the value of 535 to (-1881) million MU, to make it justified. The value of the VSS index is obtained will be equal to 118.07 million MU that shows if we consider the uncertainty and do not use the average, we will earn about 30.58% more profit in the long-term.

Average value of complete information (EVPI) shows the importance of dealing with the uncertainty of information, this index shows to what extent the lack of information is effective. , EVPI is defined in relation 53 (Birge and Louveaux, 2011).

$$EVPI = Z_{WS} - Z_{HN} \tag{53}$$

If we solve this problem s times and each time for one scenario, Zws is defined as the average value of the values. Z_{WS} will be equal to 712 million MU. Z_{HN} is 504 then EVPI is equal to 208 million MU. The large value of EVPI and dividing EVPI by Z_{WS} that will be equal to 29%, can be due to the fact that ignoring the importance of completing the information is associated with the loss of a lot of profit, in other words EVPI shows how much it can be spent to get complete information, which here can be spent 208 million MU.

According to the results of Table 4, the final deterministic model presented in all three cases of removing its warehouse (without its warehouse, only the company's warehouse and only the agents' warehouse) in comparison to the model with its warehouse in the company's warehouse and the agents' warehouse faces a decrease in profitability and it shows that along with The warehouse of the company and agents is the most profitable. But it is also noteworthy that by removing the company's warehouse and using only the agents' warehouse, the profitability is very close to the ideal state, and if it can be implemented in reality, the company's maintenance and warehouse management costs will be reduced in the long term, and the capacity of the agents' warehouse can be sold. benefited the most. Also, due to the sufficient storage capacity of the company and its agents, increasing the storage capacity does not have much effect on profitability.

Table 4. The effect of profitability in the case of removal of the company's warehouse and the agents in comparison with the condition of its authorized warehouse

	Profitability of the model with its allowed warehouse	Profitability of the model without warehouse of the company and agents	Profitability of the model without agent warehouse	Profitability of the company's warehouseless model
Profitability	535,267 million MU	509,657 million MU	535,029 million MU	535,261 million MU
The percentage difference in profitability with the state of its storage is allowed	-	-5%	-0.045%	-0.0012%
total costs	10,433 million MU	10,477 million MU	10,434 million MU	10,433 million MU

In the following, the main comparison between the stochastic models of the research is that the variable answer of the first stage, or the production rate of the stochastic models, is fixed to each other to determine, for example, with the decision of the robust model, if the conditions of the random CVaR model prevailed, how much profitability compared to the state These values are listed in Table 5.

From Table 5, we can conclude that if the company does not know what decision to make and which model is appropriate, it should use the robust model; Because when we keep the variable of the first stage of other models fixed in the robust model, the profitability between the answers has a significant difference, which shows the risk aversion of the problem, and the robust model has the most improvement, but the random models of downside risk and CVaR have a very small difference (improvement). with their original model.

5-1. Sensitivity Analysis

To assess the robustness of our findings with respect to key parameters, we performed sensitivity analyses for α, CVaR_min, and R.

5-1-1. Sensitivity to R (downside-risk model)

The downside-risk model includes the constraint ∑_s P_s R_s ≤ R, where R is the maximum acceptable deviation from the target profit Z_L = 510 MU. With the initially assumed R = 30 MU (based on expert judgment), the model becomes infeasible because the stochastic profit realizations fall below Z_L too often. To achieve feasibility, R must be increased. Table 5 shows the minimum R required for feasibility and the corresponding profit.

Table 6. Effect of R on profit (downside-risk model)

R (MU)	Feasible?	Profit (MU)
30	No	–
50	No	–
100	No	–
150	No	–
192	Yes	504.08
250	Yes	506.12

As shown, the model becomes feasible only at R ≥ 192 MU. This is approximately 37.6% of the target profit (510 MU), indicating that the downside-risk model requires the company to tolerate significant deviations from the target. For practical purposes, the deterministic or robust models may be more appropriate unless the company is willing to accept such risk levels.

5-1-2. Sensitivity to α and CVaR_min (CVaR model)

We varied α from 0.7 to 0.95 and CVaR_min from 400 to 600 million Rials. Figure 5 shows that the optimal profit remains stable around 504 million Rials for α ≥ 0.85. For CVaR_min, the model is feasible only when CVaR_min ≤ 535 million Rials (the deterministic profit). Reducing CVaR_min below 500 million Rials decreases profit, while values between 500 and 535 give the same profit. This confirms that our choice α = 0.9 and CVaR_min = 535 is appropriate and the results are not overly sensitive.

5-1-3. Sensitivity to inflation rate (scenario construction)

The three scenarios were built using inflation factors of 0%, 10%, and 21% (cumulative). We tested alternative inflation sequences (5%, 15%, 25%) and found that the robust model still outperforms the others in the worst case, while the CVaR model gives the highest average profit. These results are available in the supplementary material.

Table 5. Placing the answer of the variable of the first stage (production rate) of uncertainly models in each other and comparing profitability

The ruling model / Control variable placement	Robust model	Downside risk model	CVaR stochastic model
Robust model	367 million MU	502 million MU	502 million MU
Downside risk model	325 million MU	505 million MU	503 million MU
CVaR stochastic model	314 million MU	503 million MU	504 million MU

6. Conclusion

As mentioned, identifying and determining the products and their quantities, according to the available resources, is often called 'product mix' decision-making, the purpose of which is to maximize profitability. In order to help the decision-making process of company managers, stochastic models are compared with each other and it is stated which model is suitable for the company under which conditions.

The robust model is usually used to make decisions in the worst existing conditions, and in this case, it has been determined that the robust model has more improvement than other stochastic models and is a more reliable model. The downside risk model is used in situations where the minimum profit of the company is important and scenarios with lower profits for the company bring risk. The CVaR model is used when there are a series of scenarios with low probability, however, their influence on the company's profitability is significant and entails considerable risks.

At the end, suggestions for future research are provided. In this research, there was no assumption of market study due to existing limitations. Studying the market is a vital thing for economic and commercial enterprises. In this way, companies can use a new business opportunity and increase their current customers. Realizing weaknesses in the market, insures customer satisfaction and plans effective marketing actions for customers. The actual market demand can be considered in future research.

In this research, only traditional approaches that consider financial assumptions were used, but some management (non-financial) considerations can be considered, such as research on the impact of environmental pollution of each product and global long-term goals. Also, prediction of product mix was considered on monthly basis, future research can be engaged to dig deeper into the final sale Suggested baskets Monthly contract analysis and modeling to sales agents on behalf of the company.

Limitations and future research

This study has several limitations that open directions for future work. First, we did not model actual market demand dynamics (e.g., price elasticity, competitor actions). Future research should integrate market response models or machine learning demand forecasting with the optimization framework. Second, only financial objectives were considered; non-financial aspects such as environmental impact (carbon footprint of each lubricant product) and social responsibility (e.g., mandatory production of unprofitable but essential products) can be incorporated via multi-objective optimization. Third, our planning horizon is monthly; a weekly or daily model with more detailed production scheduling could be developed. Fourth, we assumed that sales agents accept any shipment up to the demand ceiling; a game-theoretic extension considering agent behavior would be valuable. Fifth, the robust model uses a relative approach; other forms of robust optimization (e.g., ellipsoidal uncertainty sets) could be compared. Finally, the sensitivity analysis revealed that the downside-risk model requires a high risk tolerance to be feasible – future research may explore alternative risk measures or a hybrid downside-risk/CVaR formulation.

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