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Optimal Stopping Policy for Multivariate Sequences; a Generalized Best Choice Problem

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Asset allocation;
Optimal stopping rule;
Stochastic dynamic programming;
Bayesian decision theory;
Mixed models

ABSTRACT

In the classical versions of "Best Choice Problem", the sequence of offers is a random sample from a single known distribution. We present an extension of this problem in which the sequential offers are random variables but from multiple independent distributions. Each distribution function represents a class of investment or offers. Offers appear without any specified order. The objective is to accept the best offer. After observing each offer, the decision maker has to accept or reject it. The rejected offers cannot be recalled again. In this paper, we consider both cases of known and unknown parameters of the distribution function of the class of next offer. Two optimality criteria are considered, maximizing the expected value of the accepted offer or the probability of obtaining the best offer. We develop stochastic dynamic programming models for several possible problems, depending on the assumptions. A monotone case optimal policy for both criteria is proved. We also show that the optimal policy of a mixed sequence is similar to the one in which offers are from a single density.

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1. Introduction

In many decision situations such as an investment, selling an asset or seeking a job, the decision-maker (DM) receives a sequence of offers one at a time. After the evaluation of each offer, DM must decide whether to take the offer on hand or reject it and waiting for next better ones. If the decision is irrevocable, the question is when to make the positive decision of accepting an offer. If the decision is made too early in the search process, then DM may miss some better future offer. However, if it is made too late, DM may have already passed over the best opportunities.

This class of sequential search and selection problems is variously referred to as secretary problem, the job search problem, candidate problem, best choice problem, parking spot problem, beauty contest problem, house-selling problem, optimal stopping problem. Basic assumptions of these series of sequential decision problems are as follows. [1].

1- One by one, a sample of N measurements is drawn from a population with continuous distribution function $F(x)$. The continuity assures that the probability ties are 0.
2- The total number of measurements, N, is known to DM a priori.
3- After each draw, the DM, who may (or may not) know the distribution function $F(x)$ and its parameters value, is informed of its value x_j (or its relative rank), whereupon DM must decide whether or not to choose that draw.

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4- No recall is allowed; that is, a draw once rejected may not be chosen later on.
5- DM is very particular and will be satisfied with nothing but the very best.

Thus, the problem is to find a rule or strategy that maximizes the probability of successfully choosing the 'best choice'. Owing to Assumption 5, the sequential decision problem is widely referred to as the best choice problem. In a managerial decision situation, the sequentially observed random variable X_j in assumption 3 may represent the j*th* bid in the house-selling problem, the value of the j*th* offer in the job search problem, the market price of an asset on the jth day, or the rate of return associated with the jth investment opportunity in the investment problem.

The cumulative distribution function F(x) in assumptions 1 and 3 may or may not be known to DM a priori. Thus, depending on the assumption made on the distribution function F(x), the best choice problems can be categorized as three following cases.

- In the no information case, on one hand, we assume that the distribution function F(x) is completely unknown to DM and thereby the DM's decision is solely based on the relative rankings of the choices that have been observed so far. The no information case has been widely known as the secretary problem, in which an executive must make an irrevocable decision from a pre-specified number of applicants who are interviewed for a secretarial position.
- In the full information case, on the other hand, we assume that the distribution function F(x) and its parameter values are fully known to DM a priori. In most full information cases, the sequentially observed variables X_j are assumed to be independent, identically distributed (i.i.d.) random observations from a common distribution, F(x).
- An intermediate problem of partial information case occurs when observations are taken from a known distribution, but containing one or more unknown parameters. An appropriate approach is to use Bayes formula to update the estimation of its parameters at the same time as deciding whether to stop or continue the search process, [2].

In this paper, however, we consider a different type of “full information” case in which the values of offer are from two (or more) different known independent distribution, like $F_X(x)$ and $F_Y(y)$. The incoming offers follow a random pattern in the arriving sequence. In this case, we prove under some assumptions the optimal strategy is the same as optimal stopping policy with just one distribution.

Since its appearance in the literature in the early 1960s, the best choice problem has been extended and generalized in many different directions by releasing some of the above mentioned five basic assumptions. The extensions reported in the literature include the infinite time horizon, unknown number of applicants, random arrivals of applicants, partial recall of rejected applicants, multiple selections, possibility of rejected offers, selection and assignment, discounting of the payoff, minimizing the expected rank, partial information case.

Kang [3] presents an optimal stopping problem with recall where a fee must be paid to accept the best offer which has so far appeared.

Kawai and Tamaki [4] consider a version of the secretary problem in which one is allowed to make one choice and regard the choice as successful when the chosen applicant is either the best or the second best among all *N* applicants, where *N* is a random variable with known distribution. Porosinski [5] characterizes a class of distributions of *N*, for a full-information best choice problem with a random number of objects *N* for which the so-called monotone case occurs.

Tamaki [6] considers an infinite version of secretary problem. From an infinite stream of applicants, *m* applicants should be chosen and assigned to *m* positions.

Chun [7] considers the problem of selecting the single best choice when several groups of choices are presented sequentially for evaluation. In the group interview problem, he assumes that the value of choices is random observations from a known distribution function and derives the optimal search strategy that maximizes the probability of selecting the best among all choices. Chun [8] derives a simple selection rule called the optimal partitioning strategy in which the decision-maker divides the entire groups into two disjoint sets. After evaluating the choices in the first set, the relatively best available choice is chosen for the first time in the second set.

Bearden et al. [9] present a generalization of secretary problem in which applicants are characterized by multiple attributes and then present a procedure for numerically computing the optimal search policy and test it in two experiments with incentive-compatible payoffs. Chun [1] proposes a third approach (in addition to dynamic programming and Markovian approaches) to a generalized version of the best choice problem, based on the theory of information economics. Freeman [10] and Ferguson [11] present excellent review papers elaborating some extensions of this problem.

In this paper, we propose a novel approach that is based on the concept of mixture models and illustrate how to derive the optimal strategy for a generalized version of the best choice problem.

In section 2, we formulate and obtain an optimal strategy for the situation that considers a mixed sequence of offers coming from two (or more) different known distributions by applying dynamic programming approach and assuming that that DM has prior information of the probability of the next offer. In Section 3, we relax this assumption and formulate the problem within dynamic programming framework involving learning. Section 4 deals with introducing the concept of mixed models and shows that mixed

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sequences are reducible to well studied univariate sequences. In section 5 we illustrate our discussion. Finally, our results are summarized in concluding section.

2. A mixed Sequence with the Known A: Probability of the Distribution

Consider an investor who has one chance to invest her money in different types of assets, e.g. stocks, bonds, real estates and etc. She does not have enough cash to create a portfolio of assets. Therefore, she tries to select the best single offer to invest. Offers come sequentially in time. She doesn't know what would be the type of next offer or its value. Each time, after receiving an offer (type and value), she can either accept it and invest or reject it and keep on selecting.

More generally, consider a Decision Maker (DM) who receives a sequence of offers of two types (or classes). Each offer is a nonnegative random variable which comes either from distribution function X or Y. (The general case with more than two distributions is also discussed at the end of this section). Let X_j and Y_j be the j^{th} offer (or observation) of classes I or II (or drawn from distributions F_X and F_Y), respectively. X_js and Y_js are assumed to be i.i.d. A probable sequence, for example, can be X Y Y Y X X Y X … X X Y.

Suppose DM receives N offers, where N is a prior information to DM. However, the number of offers from each class is not known in advance. Let n and m represent the number of offers from classes I or II, respectively. Then, N=n+m. Furthermore, DM knows that the next offer (random variable) is from distribution F_X with a predetermined probability p and as a result, from distribution has F_Y with probability (1-p). After observing the offer, DM becomes aware of its value as well as its true distribution. Then, DM can either accept it and terminate the process or reject it and keep on selecting. An offer must be selected finally. No recall is allowed. We formulate the problem to satisfy two different criteria.

2.1. First Criterion: Maximizing the Expected Value of the Accepted Offer

We use the following notations to formulate the problem in the framework of stochastic dynamic programming:

Let V(k, v |Z) be the maximal expected value of the accepted offer if the number of remaining offers is k (including the present offer which has not been decided yet), and the class of distribution and value of the presented offer are Z and v, respectively. Note that Z is either X or Y and is known just after the offer is presented. Then,

$$V(k,v|z) = \max(\int_u V(k-1,u|X)dF_x(u)$$
$$+(1-p)\int_u V(k-1,u|Y)dF_Y(u) \qquad (1)$$

with boundary condition: V(1, v |Z) = v.

If DM accepts this offer, he gains value v and the process terminates. However, if the offer is rejected, then he receives another offer with probability p from F_X and with probability (1-p) from F_Y distribution.

Proposition1. The optimal policy is as follows:
There are increasing numbers $0 \le a_1 \le a_2 \le \cdots \le a_N$ such that if there are remaining k offers, then accept the present one if its value is at least a_k.

***Proof*:** The present offer is accepted if $V(k,v|Z) = v \ge a_k$. Therefore, if v, the value of present offer, is at least a_k, DM accepts it. It is obvious that for $k > 1$,

$$a_k = p\int_u V(k-1,u|X)dF_X(u)$$
$$+(1-p)\int_u V(k-1,u|Y)dF_Y(u)$$

From the boundary condition, $V(1,v|Z) = v,\ a_1$ is equal to zero. For a_2:

$$a_2 = p\int_u V(1,u|X)dF_X(u) + (1-p)$$
$$\int_u V(1,v|Y)dF_Y(u) = p\int_u udF_X(u)$$
$$+(1-p)\int_u udF_Y(u) = pE(X) + (1-p)E(Y)$$

To show the sequence $a_1,\dots,a_N$ is monotone, we use induction on k. It is obvious that,

$$a_2 = pE(X) + (1-p)E(Y) \ge a_1 = 0.$$

Now, let $V(k,v|Z) \ge a_{k,}$. then,

$$a_{k+1} = p\int_u V(k,u|X)dF_X(u) + (1-p)$$
$$\int_u V(k,u|Y)dF_Y(u) = \mathrm{p}\int_\mathrm{u} \max(\mathrm{u},\mathrm{a_k})\mathrm{dF_x(u)}$$
$$+(1-p)\int_u max\,(u,a_k)\,dF_Y(u) +$$
$$(1-p)\int_u \max(u,a_k)dF_Y(u) \ge$$
$$p\int_u a_k dF_X(u) + (1-p)\int_u a_k dF_Y(u)$$
$$= pa_k + (1-p)a_k = a_k$$

In case of more than two distributions (classes), the same result can be obtained. Suppose offers come from a family of q known distribution $\{F_i\}_{i=1}^q$, and DM knows that the next offer belongs to F_i family with pre-known probability p_i, while $\sum_{i=1}^q p_i = 1$. The obtained

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"monotone case" policy can be generalized for the sequences with q distribution:

$$\left\{ \forall k, \exists a_k \middle| \begin{array}{c} if\ in\ each\ stage\ v \geq a_k, \\ then\ choose\ the\ present\ offer \\ and\ stop, \\ else\ continue \end{array} \right\}$$

where,

$$a_k = \sum_{i=1}^{q} p_i \int_u V(k-1, v|Z_i) F_{Z_i}(u) \quad (2)$$

It is obvious that in this situation the maximal expected value in stage k would be:

$$V(k, v|Z) = max\left(v, \sum_{i=1}^{q} p_i \int_u V(k-1, v|Z_i) F_{Z_i}(u)\right) \quad (3)$$

B: Second Criterion: Maximizing the Probability of Obtaining the Best Offer

Suppose DM wants to maximize the probability of acquiring the best offer, instead of maximizing its expected value, under the same assumptions of the previous section. To formulate this problem, extra information needed tob included in the state vector. By our notation $v_m^{Z'}$ indicates the value (m) as well as the distribution (Z') of the best unaccepted previous offer. In bivariate sequences, Z' is either X or Y.

Let $\Psi\left(k, v, v_m^{Z'} \middle| Z\right)$ denote the maximal probability of obtaining the best offer when the value and distribution of the present offer are m and F_Z, respectively and the number of remaining offers is k.

There are two possible situation: $v < v_m^{Z'}$ or $v \geq v_m^{Z'}$.

$$\Psi\left(k, v, v_m^{Z'} \middle| Z\right) = \begin{cases} \text{reject the present offer and continue} & \\ \text{with the next offer} & v < v_m^{Z'} \\ max\left(\begin{array}{c}\text{accept the present offer and} \\ \text{it is the best offer,} \\ \text{reject it and continue}\end{array}\right) & v \geq v_m^{Z'} \end{cases} \quad (4\text{-a})$$

Let $P_{AB}(v)$ denote the expected probability that the present offer is the best one, when $v \geq v_m^{Z'}$.Then, by conditioning on the number of remaining offer with F_X distribution.

$$P_{AB}(v) = \sum_{i=0}^{k-1} P_{AB}\left[v\middle|\begin{array}{c} i \text{ offers come from } F_X \text{ and} \\ k-1-i \text{ offers come from } F_y) \end{array}\right] \quad (4\text{-b})$$

Hence,

$$P_{AB}(v) = \sum_{i=0}^{k-1} \binom{k-1}{i} p^i (1-p)^{k-1-i} \ F_X(v)^i F_Y(v)^{k-1-i} \quad (4\text{-c})$$

Finally, we have,

$$\Psi\left(k, v, v_m^{Z'} \middle| Z\right) = \begin{cases} p\int_u \Psi\left(k-1, u, v_m^{Z'}|X\right) dF_X(u) + & \\ (1-p)\int_u \Psi\left(k-1, u, v_m^{Z'}|Y\right) dF_Y(u) & , v < v_m^{Z'} \\ max\left(\begin{array}{c} \sum_{i=0}^{k-1}\binom{k-1}{i} p^i (1-p)^{k-1-i} F_X(v)^i F_Y(v)^{k-1-i}, \\ p\int_u \Psi(k-1, u, v|X) dF_X(u) + \\ (1-p)\int_u \Psi(k-1, u, v|Y) dF_Y(u) \end{array}\right) & , v \geq v_m^{Z'} \end{cases} \quad (5)$$

When considering a general sequence with more than two distributions, the formulation would be:

$$\Psi\left(k, v, v_m^{Z'}|Z\right) = \begin{cases} \sum_{i=1}^{q} p_i \int_u \Psi\left(k-1, u, v_m^{Z'}|Z_i\right) dF_{Z_i}(u) & , v < v_m^{Z'} \\ max\left(\begin{array}{c} \sum_{i_1}^{k-1} \sum_{i_2}^{k-1-i_1} \sum_{i_{q-1}}^{k-1-\sum_{j=1}^{q-1} i_j} \binom{k-1}{i_1, \dots, i_q} \prod_{k=1}^{q} [p_k F_{Z_k}(v)]^{i_k}, \\ \sum_{i=1}^{q} p_i \int_u \Psi(k-1, u, v|Z_i) dF_{Z_i}(u) \end{array}\right) & , v \geq v_m^{Z'} \end{cases} \quad (6)$$

with boundary condition:

$$\Psi\left(1, v, v_m^{Z'} \middle| Z\right) = \begin{cases} 0 & v < v_m^{Z'} \\ 1 & \text{else} \end{cases}. \quad (7)$$

Proposition 2. The optimal policy has the following form:

At each stage, say k, if the present offer is not as high as the value of the best of previous ones, *i.e.* .$v < v_m^{Z'}$), reject it and continue. Otherwise, accept this offer and terminate the process if $v \geq s_k$, where s_k is a predetermined number.

***Proof*:** It is obvious that when $v < v_m^{Z'}$, DM does not accept the offer because this offer is not the best one.

For the case $v \geq v_m^{Z'}$, let

$EC(v) = \sum_{i=1}^{q} p_i \int_u \Psi(k-1, u, v|Z_i) dF_{Z_i}(u)$,

indicate the expected probability of obtaining the best offer when DM should proceed. On the other hand, $P_{AB}(v)$ is an increasing function of v, defined as follows.

$$P_{AB}(v) = \sum_{i_1}^{k-1} \sum_{i_2}^{k-1-i_1} \dots \sum_{i_{q-1}}^{k-1-\sum_{j=1}^{q-1} i_j} \binom{k-1}{i_1, \dots, i_q} \prod_{k=1}^{q} [p_k F_{Z_k}(v)]^{i_k}$$

The reason is that it is a polynomial function of $F_{Z_k}(v)^{i_k}$ which is obviously increasing in v. Furthermore, it can be proved by induction on k that EC(v) is a decreasing[1] function of v , see Appendix A.

[1] *Throughout this paper, decreasing means not increasing.*

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It is easy to comprehend why EC(v) is decreasing. As much as the value of the present offer, v , increases, finding one offer with higher value than v becomes less likely.
The higher the value of the present offer makes it less probable to obtain "the best offer" among next ones. Therefore, *EC(v)* must decrease in *v*.
The boundary conditions are:
$P_{AB}(0)=0$, $P_{AB}(v)_{v\to\infty}=1$, $EC(v)_{v\to 0}=\Delta>0$, and $EC(v)_{v\to\infty}=0$. (Note that $EC(v)$ and $P_{AB}(v)$ are defined on the domain $[v_m^{Z'},\infty]$, but because it is easier to compare them on the boundary 0, we consider the dummy boundary 0 and then reconsider $v_m^{Z'}$).

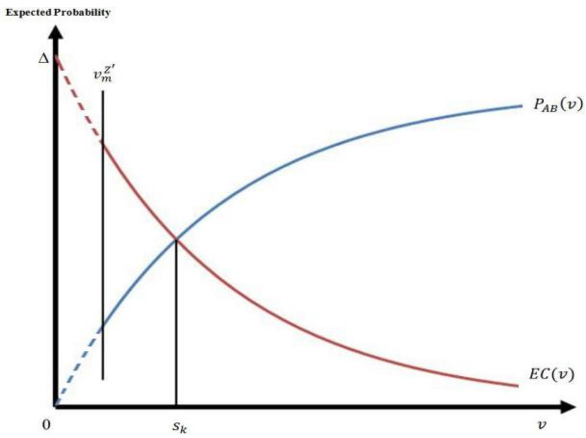

Fig. 1. Accepting and rejecting areas

Figure 1 show the existance of s_k. If $P_{AB}(v)$ and $EC(v)$ intersect in a value greater than $v_m^{Z'}$, with values higher than s_k, $P_{AB}(v)$ is higher than $EC(v)$, i.e. it is the time to accept the offer.

3. A Practical Case; Mixed Sequences with Bayesian Updates

In previous situations, from DM viewpoint, the random variable that determines the class of next offer is Bernoulli (for bivariate sequence) or polynomial (for multivariate sequence).
This random variable was assumed to be completely known to DM, i.e. he knows p or p_i. However, in real world this is not always true. Consider the case of investor mentioned in Section 2. If this investor doesn't know what would be the class of next offer and with what probability, then she can only say that different offers come with an order of polynomial distribution with unknown parameters.
At the beginning, this investor can put equal chances for the next offer to be from any class of assets (or distributions). However, as the offers come, investor can have a better image of the next ones, i.e. she can update the chances based on the previous offers of different class of assets that have been observed so far.
This problem can be formulated within the framework of stochastic dynamic programming involving learning.

Lemma1. Let p be an unknown parameter of a Bernoulli distribution which is determined by sampling a uniform distribution. If k and l are the number of independent experiment and successful outputs up to this point respectively, then, its expected value is $\bar{p}=\frac{l+1}{k+2}$.

Proof. We can compute $f(x|k,l)$, the posterior density of p, by Bayes' law:

$$f(x|k,l)=\frac{g(k,l|x)f(x)}{g(k,l)}=\frac{x^l(1-x)^{k-l}f(x)}{\int_0^1 z^l(1-z)^{k-l}f(z)dz}$$

where $g(k,l)$ is the probability of obtaining l successes from k independent Bernoulli experiments. Then,

$$\bar{p}=\int_0^1 xf(x|k,l)dx=\frac{\int_0^1 x^{l+1}(1-x)^{k-l}f(x)}{\int_0^1 z^l(1-z)^{k-l}f(z)dz}$$
$$=\frac{l+1}{k+2}.$$

Consider a bivariate sequence. Let the number of offers from F_x and F_y distributions up to this point (including the present one) be S_x and S_y, respectively. Then, by Lemma 1, $\bar{p}=\frac{S_x+1}{S_x+S_y+2}$. Let $V(S_x,S_y,v|Z)$ and $\Psi(S_x,S_y,v,v_m^{Z'}|Z)$ denote the maximal expected value and maximal expected probability of accepting the best offer, respectively. Then,

$$V(S_x,S_y,v|Z)=max\begin{pmatrix} v,\frac{S_x+1}{S_x+S_y+2}\int_u V(S_x+1,S_y,u|X)dF_x(u)+\\ \frac{S_y+1}{S_x+S_y+2}\int_u V(S_x,S_y+1,u|Y)dF_y(u)\end{pmatrix}$$

with boundary conditions:

$$V(S_x,N-S_x,v|Z)=V(N-S_y,S_y,v|Z)=v$$

and

$$\Psi\left(S_x,S_y,v,v_m^{Z'}\middle|Z\right)=$$
$$\begin{cases} \frac{S_x+1}{S_x+S_y+2}\int_u \Psi\left(S_x+1,S_y,u,v_m^{Z'}\middle|X\right)dF_X(u)+ \\ \quad\frac{S_y+1}{S_x+S_y+2}\int_u \Psi\left(S_x,S_y+1,u,v_m^{Z'}\middle|Y\right)dF_Y(u) \\ \qquad ,v<v_m^{Z'} \\ \max\begin{pmatrix} \sum_{k=0}^{N-(S_x+S_y)}\binom{N-(S_x+S_y)}{k}\left(\frac{S_x+1}{S_x+S_y+2}\right)^k\times \\ \left(\frac{S_y+1}{S_x+S_y+2}\right)^{N-(S_x+S_y)-k} F_X(v)^kF_Y(v)^{N-(S_x+S_y)-k}, \\ \frac{S_x+1}{S_x+S_y+2}\int_u \Psi\left(S_x+1,S_y,u,v\middle|X\right)dF_X(u)+ \\ \frac{S_y+1}{S_x+S_y+2}\int_u \Psi\left(S_x+1,S_y,u,v\middle|Y\right)dF_Y(u)\end{pmatrix} \\ \qquad ,v\geq v_m^{Z'}\end{cases} \quad (8)$$

with boundary conditions:

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$$\Psi\left(S_x, N-S_x, v, v_m^{Z'} \middle| Z\right) = \Psi\left(N-S_y, S_y, v, v_m^{Z'} \middle| Z\right) = \begin{cases} 0 & v < v_m^{Z'} \\ 1 & v \geq v_m^{Z'} \end{cases}$$

4. Mixed Sequence Models

Mixture distributions comprise a finite or infinite number of components, possibly of different distributional types, that can describe different features of data. They, thus facilitate much more careful description of complex systems, as evidenced by the enthusiasm with which they have been adopted in such diverse areas such as astronomy, ecology, bioinformatics, computer science, ecology, economics, engineering, robotics and biostatistics. For instance, in genetics, location of quantitative traits on a chromosome and interpretation of microarrays both relate to mixtures, while, in computer science, spam filters and web context analysis start from a mixture assumption to distinguish spams from regular emails and group pages by topic, respectively [12].

4.1. The Finite Mixture Framework

Definition 1. A **mixture of distributions** is any convex combination of them, *i.e.*,

$$\sum_{i=1}^{k} p_i f_{Y_i}(x), \sum_{i=1}^{k} p_i = 1, p_i \geq 0, \; k > 1 \qquad (9\text{-a})$$

In the **parametric mixture model**, the component distributions are from a parametric family, with unknown parameters θ_i:

$$f_X(x) = \sum_{i=1}^{n} p_i f_Y(x; \theta_i) \qquad (9\text{-b})$$

A **continuous mixture** is defined similarly:

$$f_X(x) = \int_\theta h(\theta f_Y(x;\theta) d\theta \qquad (9\text{-c})$$

where $h(\theta) \geq 0, \quad \forall \theta \in \boldsymbol{\Theta}$ and $\int_\theta h(\theta) d\theta = 1$.

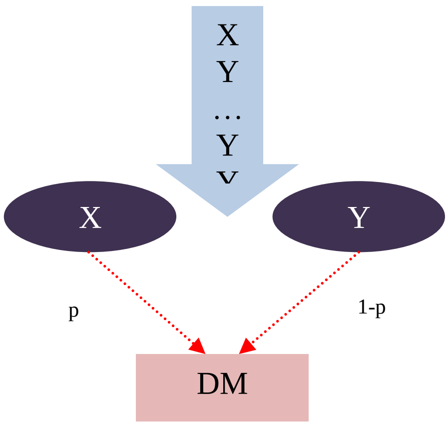

Fig. 2. Decision making model for DM, bivariate sequence

Now we can relate the mixed sequence of offers to the concept of mixture density. Consider the case introduced in section 2, when encountering a new offer, DM situation can be shown as in Figure 2.
It means that DM does not know from which distribution offers come, but he can imagine that offers are coming from a mixture of X and Y, see Figure 1. The mixture variable, Z, has the following density:

$$f_Z(\zeta) = p f_X(\zeta) + (1-p) f_Y(\zeta) \qquad \zeta > 0$$

Or

$$F_Z(\zeta) = p F_X(\zeta) + (1-p) F_Y(\zeta) \qquad \zeta > 0$$

Therefore, we can replace the model in Figure 3 with the following one:

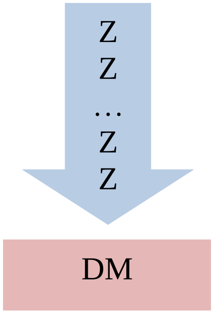

Fig. 3. Decision making model for DM after reducing to a mixed model

DM can assume that offers come from a **single known distribution,** namely F_Z, see Figure 3. But this reduced problem is not a new one. Gilbert and Mosteller [13] obtained the optimality policy structure of the maximal value-objective and maximal probability-objective cases, respectively.
In fact, when offers come from a unique distribution Z, then the stochastic dynamic programming formulation for the previous two mentioned objectives are:

i. maximal value-objective:

$$V(k,v) = \max\left(v, \int_u V(k-1,u) dF_Z(u)\right) \qquad (10)$$

with boundary condition: $V(1,v) = v$.

ii. maximal probability-objective:

$$\Psi(k,v,v_m) = \begin{cases} \int_u \Psi(k-1,u,v_m) dF_Z(u), & v < v_m \\ max(F_Z(v)^{k-1}, \int_u \Psi(k-1,u,v) dF_Z(u)), & v \geq v_m \end{cases} \qquad (11)$$

with boundary condition: $\Psi(1,v,v_m) = 1$.

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Optimal policy in the framework of mixture models

Consider a multivariate sequence of offers from q positive independent random variables $\{Z_i\}_{i=1}^{q}$ with distributions $\{F_i\}_{i=1}^{q}$. At each stage, one offer is presented to DM and he must decide either to accept it and terminates the process or to reject it. After the offer is presented, DM can observe its value as well as its distribution.

DM can have prior information regarding the probability of the family (or distribution) of the next offer or not.

If DM does not have the exact value of this probability, he can estimate it, as discussed before. The optimal policy for this situation is exactly like the optimal policy when offers come from just one distribution and its equivalent distribution is $F_Z(\zeta) = \sum_{i=1}^{q} p_i F_{Z_i}(\zeta)\,(\zeta > 0)$. ($p_i$ is the exact value of predetermined probability or its estimation.

For example in section 3, for $i = 2$, $p_i = p_2 = \frac{S_x+1}{S_x+S_y+2}$).

For the case considered in section 3, DM decides to continue: 1) when $v < v_m^{Z'}$ and 2) when $v \geq v_m^{Z'}$ and

$$\frac{S_x+1}{S_x+S_y+2}\int_u \Psi\big(S_x+1, S_y, u, v\big|X\big)dF_X(u)$$

$$+\frac{S_y+1}{S_x+S_y+2}\int_u \Psi\big(S_x+1, S_y, u, v\big|Y\big)dF_Y(u)$$

takes a greater value than

$$\sum_{k=0}^{N-(S_x+S_y)}\binom{N-(S_x+S_y)}{k}\left(\frac{S_x+1}{S_x+S_y+2}\right)^k\left(\frac{S_y+1}{S_x+S_y+2}\right)^{N-(S_x+S_y)-k}$$
$$\times F_X(v)^k F_Y(v)^{N-(S_x+S_y)-k}$$

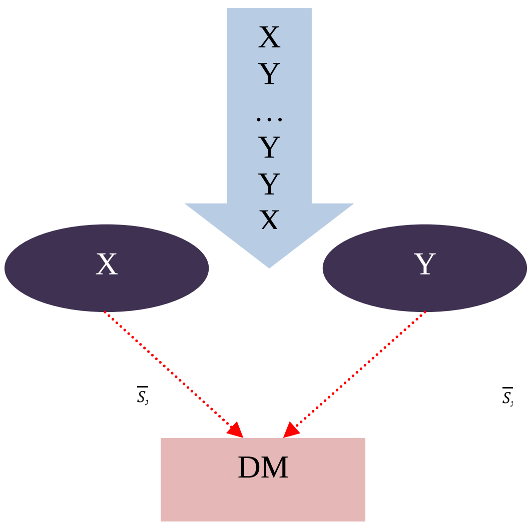

Fig. 4. Decision making model, bivariate sequence, involving learning

$$F_Z(\zeta) = \frac{S_x+1}{S_x+S_y+2}F_X(\zeta) +$$
$$\left(1-\frac{S_x+1}{S_x+S_y+2}\right)F_Y(\zeta)\;\;(\zeta > 0)$$

i.e. there is a mixed density from which offers come.

When DM decides to accept the offer, the maximal expected probability of obtaining the best offer is

$$P_{AB}(v)$$
$$=\sum_{k=0}^{N-(S_x+S_y)}\binom{N-(S_x+S_y)}{k}\left(\frac{S_x+1}{S_x+S_y+2}\right)^k\left(\frac{S_y+1}{S_x+S_y+2}\right)^{N-(S_x+S_y)-k}$$
$$\times F_X(v)^k F_Y(v)^{N-(S_x+S_y)-k}.$$

But, $P_{AB}(v)$ is just the binomial expansion of the expression

$$\left(\left(\frac{S_x+1}{S_x+S_y+2}\right)F_X(\zeta) + \left(1-\frac{S_x+1}{S_x+S_y+2}\right)F_Y(\zeta)\right)^{N-(S_x+S_y)}$$

or in our notation $F_Z(v)^{N-(S_x+S_y)}$. Therefore, if the offers come from a F_Z distribution, $v \geq v_m^{Z'}$, and DM accepts the present offer, $P_{AB}(v) = F_Z(v)^{N-(S_x+S_y)}$ shows the probability that this offer will have a higher value than those of $N-(S_x+S_y)$ remaining ones.

5. Results

In this section, we illustrate our results by presenting an example. Consider an investor who wants to invest in one of the 20 opportunities which occur sequentially in future. She knows the offers are either for real estate or precious metals. The value of an offer is a random variable with negative exponential distribution with mean θ_1=10 for real estate offers and θ_2=12 for precious metals offers. Therefore,

$$f_X(x) = \frac{1}{10}e^{-\frac{x}{10}}\;\;(x > 0)$$

And

$$f_Y(y) = \frac{1}{12}e^{-\frac{y}{12}}\;\;(y > 0).$$

Her broker estimates that 80 percent of the opportunities are for real estate, based on previous experiences.

As we discussed in section 2, in each stage of decision making, she should accept the present offer if the value of offer is greater than,

$$a_k = 0.8\int_u V(k-1, u|X)dF_X(u) +$$
$$0.2\int_u V(k-1, u|Y)dF_Y(u)$$
$$= 0.08\int_u V(k-1, u|X)e^{-\frac{u}{10}}du$$

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$$+0.017\int_u V(k-1,u|Y)e^{-\frac{u}{12}}du$$

Therefore, at each stage (arrival of a new offer), she should compare the value of the offer to a_k. The following procedure summarizes our discussion about the optimal policy in 3 last stages of the this example.

The Procedure of the Optimal Policy for Bivariate Sequence in 3 Last Tags

Satge 1, $\boldsymbol{a_1} = 0$,
Action: For each value v, accept the offer.

Satge 2, $\boldsymbol{a_2} = 0.8E(X) + 0.2E(Y) = 10.4$
Action: If $v \geq 10.4$ accept the offer, else proceed.

Satge 3,
$a_3 = 0.8\int_u \max\ \{u, 10.4\}e^{-0.1u}du$
$+0.2\int_u \max\ \{u, 10.4\}e^{-\frac{u}{12}}du = 146.7$

Action: If $v \geq 146.7$ accept the offer, else proceed.

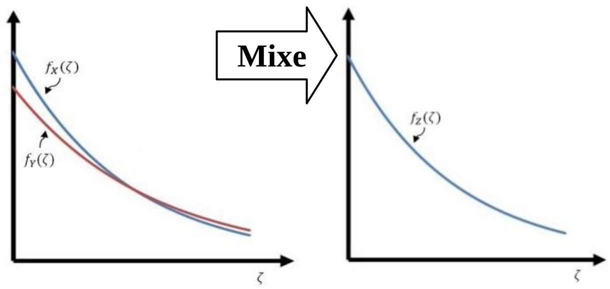

Fig. 5. Generating a mixed sequence from two components

This situation is similar to one in which offers have the following mixed density:

$$f_Z(\zeta) = 0.8f_X(\zeta) + 0.2f_Y(\zeta) = \frac{8}{100}e^{-\frac{\zeta}{10}} + \frac{2}{120}e^{-\frac{\zeta}{12}} \quad \zeta > 0$$

For this univarite sequence, at each stage, DM should compare the value of present offer with a boundary number b_k that is,

$$b_k = \int_u V(k-1,u)\left(\frac{8}{100}e^{-\frac{u}{10}} + \frac{2}{120}e^{-\frac{u}{12}}\right)du$$

At stage k, if v is greater than b_k, the investor should accept the offer, else, try the next opportunity. The two important results that can be inferred from the discussion are:
1) For all value of k, b_k,= a_k.
2) DM's knowledge about the distribution of the present offer doesn't affect her optimal policy.

Concluding Remarks and Further Researches

"Best choice problem or secretary problem" had been exposed to many research experiences to make its assumptions more compatible with real world observations. This particular field of study has experienced rapid growth and extensive application to a variety of decision problems. In this paper, we showed that under some assumptions, DM's optimal policy is irrelevant to the existing of two or more distributions in the upcoming sequence. In other words, DM is indifferent to the distinct classes of offers and considers them under a unit framework. Now, the question is how the different families of offers can affect DM's decisions. The probable solution may be hidden under the chronological order of revealing the distribution of the next offer. This is the subject of our next works.

Appendix A

$EC(v) = \sum_{i=1}^{q} p_i \int_u \Psi(k-1,u,v|Z_i)dF_{Z_i}(u)$ is a decreasing function of $\boldsymbol{v}$.

Lemma2. $g(R,x)$ is a real function of R and x; $G(R) = \int_a^b g(x,R)dx$ is decreasing function of R, if $g(x,R)$ has the same property.

Proof:

$$\frac{\partial G(R)}{\partial R} = \frac{\partial \int_a^b g(x,R)dx}{\partial R} = \int_a^b \frac{\partial g(x,R)}{\partial R}dx$$

If for each value of x, $g(x,R)$ decreases in R ($\frac{\partial g(x,R)}{\partial R} \leq 0, \forall x$), then

$$\int_a^b \frac{\partial g(x,R)}{\partial R}dx \leq 0 \Rightarrow \frac{\partial G(R)}{\partial R} \leq 0 \Rightarrow G(R)$$

is decreasing in R.

Therefore, it is sufficient to show that $\Psi(k-1,u,v|Z_i)$ is a decreasing function of v (Here, $G = \int_u \Psi(k-1,u,v|Z_i)\,dF_{Z_i}(u)$ and $g = \Psi(k-1,u,v|Z_i)$).
For $k = 1$, the result follows. Suppose for
Then,

$$\Psi(n=k,u,v|Z_i) = \begin{cases} \sum_{j=1}^{q} p_j \int_{u'} \Psi(k-1,u',v|Z_j)dF_{Z_j}(u'), & u < v \\ max\begin{pmatrix} \sum_{i_1}^{k-1}\sum_{i_2}^{k-1-i_1}\sum_{i_{q-1}}^{k-1-\sum_{j=1}^{q-1} i_j}\binom{k-1}{i_1,\ldots,i_q}\prod_{k=1}^{q}[p_k F_{Z_k}(u)]^{i_k}, \\ \sum_{j=1}^{q} p_j \int_u \Psi(k-1,u',u|Z_i)dF_{Z_j}(u') \end{pmatrix}, & u \geq v \end{cases}$$

where$u \geq v$,

$$max\begin{pmatrix} \sum_{i_1}^{k-1}\sum_{i_2}^{k-1-i_1}\ldots\sum_{i_{q-1}}^{k-1-\sum_{j=1}^{q-1} i_j}\binom{k-1}{i_1,\ldots,i_q}\prod_{k=1}^{q}[p_k F_{Z_k}(u)]^{i_k}, \\ \sum_{j=1}^{q} p_j \int_u \Psi(k-1,u',u|Z_i)dF_{Z_j}(u') \end{pmatrix}$$

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is a constant function in v. when v increases until $u < v$,
$\Psi(k, u, v|Z_i) = \sum_{j=1}^{q} p_j \int_{u'} \Psi(k-1, u', v|Z_j) dF_{Z_j}(u')$
but according to induction hypothesis,
$\sum_{j=1}^{q} p_j \int_{u'} \Psi(k-1, u', v|Z_j) dF_{Z_j}(u')$ decreases in v, so the result follows.

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