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RESEARCH PAPER

Warranty Cost Models for a Repairable Multi-Component Product Protected by Lemon Laws with Failure Interaction

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ABSTRACT
This study develops warranty cost models for repairable products subject to Lemon Laws, encompassing Critical and Non-Critical components forming a multi-component system. Failures can arise naturally or be induced by other components (i.e., failure interaction), defining a lemon if recurrent failures reach a threshold (k) during the warranty period. A lemon declaration triggers a refund or replacement by the manufacturer. Four warranty cost models are proposed from the manufacturer's standpoint, considering failure mechanisms. Increasing failure thresholds in the warranty scheme substantially decreases warranty cost rates. For instance, a threshold (k) of 5 in refund and replacement schemes yields the lowest cost rates of 33.7159 and 25.8249, respectively. Failure interactions escalate total warranty costs; for instance, in a refund scheme (k = 5), costs with failure interaction reach 31.0169 compared to 28.7603 without. Similar trends apply to replacement schemes. Moreover, a lower warranty cost rate will extend the period, indicating regulation fulfillment due to a closer warranty period to the Lemon period. Sensitivity analysis also underscores the role of higher reliability in reducing warranty costs and complying with Lemon Laws. Finally, maintenance strategies and product reliability are emphasized to fulfill Lemon Laws with minimal costs, i.e., fewer warranty claims.

KEYWORDS: Warranty costs; Lemon laws; Refund; Replacement; Failure interaction; Reliability.

1. Introduction

Nowadays, in many marketplaces (e.g., USA, Canada, Europe, Australia, China, Singapore, and South Korea), the warranted products are also protected by Lemon Laws. These laws, a testament to the evolution of consumer protection, aim to protect consumers from recurring product failures occurring under warranty. Connecticut, a pioneer in consumer rights, was the first state in the USA to enact the Lemon Laws in 1982, and all states in the USA have adopted the Lemon Laws since 1987. Before the Lemon Laws' enactment, consumers frequently encountered frustration and expense more costs to rectify repetitive product failures [1]. A product with recurrent failures during warranty is deemed a defective product (or a lemon) since the performance of the product is unsatisfactory (or its quality is substandard). As a result, the manufacturer needs to refund the total price or replace the product with a new unit should recurrent failures happen.

The Lemon laws have been a popular subject of study among researchers, but their works primarily focus on these laws' legal and economic aspects. The legal aspect of Lemon laws pertains to the legal provisions governing lemon law warranties, which outline the rights and remedies available to consumers when they purchase a defective product that turns out to be a lemon. The legal framework guiding the dispute resolution process between the manufacturer, dealer, and consumer can be found in [1–5]. Whilst Centner et al. [6] have examined the automobile lemon laws to determine the economic efficiency and value of the laws.

However, the papers mentioned above do not study the Lemon laws' effect on warranty cost, which is in the interest of manufacturers. The

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study of the lemon laws focusing on a micro level (or a firm level) dealing with the warranty cost analysis is confined. The works of warranty cost analysis for a repairable product under the protection of the Lemon laws can be divided into two groups as follows: (i) the case where lemon is invoked by only the failures number reaching the threshold (i.e., k), and (ii) the one where lemon is triggered by either (a) recurrent failures occur (i.e., the failures number exceeds the predefined threshold) or (b) the length of out-of-service is more than 30 days because the dealer/manufacturer did not satisfactorily repair the failed product.

For group (i), Iskandar & Husniah [7] observed the cost of Lemon laws one-dimensional warranty, in which a product turns out to be a lemon when the failure number exceeds the threshold (i.e., k). Park et al. [8] studied the case of the Korean lemon laws. Wang et al. [9] proposed a warranty cost model similar to a Lemon Laws warranty with a refund scheme (i.e., the manufacturer will return the sales price to the consumer if the number of failures under a warranty period exceeds a pre-specified threshold). Furthermore, the work of Iskandar et al. [7] has been extended to remanufactured products [10] and used products [11]. The further extension includes the study of Lemon Laws warranty for a multi-component product [12] and two-dimensional warranties [13].

For group (ii), Husniah et al. [14] developed warranty cost models for a one-dimensional warranty and examined refund and replacement schemes. A simulation method was required to obtain the expected total warranty cost (EWC), as the expression for the EWC involves some complex integral equations. All the papers above assume the products are a single-component system using a black-box approach for modeling failure (i.e., do not consider the inner structure). The products (e.g., automobile and electronic products) are multi-component systems; hence, the inner structure of the product needs to be considered before modeling the product failure. The literature on multi-component system failure modeling has drawn much interest. The models can be divided into two categories, namely (i) models with no failure interaction (i.e., independent failure) and (ii) models with failure interaction (i.e., dependent failure).

For category (i), where component failures are independent (or have no failure interaction), Bai et al. [15] and Park [16] developed a series-parallel component configuration for warranty cost analysis of a multi-component system. Wu [17] proposed a failure process modeling using exponential smoothing of intensity function for a series multi-component system and claimed that the exponential smoothing is the most suitable approach to modeling the failure process compared to the Non-homogeneous Poisson Process (NHPP) and Renewal Process. Moreover, Piroozbakht et al. [18] developed a Remaining Useful Life (RUL) model for Micro-Electro-Mechanical-System (MEMS) using a general path process to model the degradation path. The study assumes that the shock that occurred was not affected by the system's state (i.e., independent). Meanwhile, Fallahnezad et al. [19] proposed an exponential distribution to model the failure process in a multi-component system with economic dependency.

For category (ii), where there is a failure interaction between components, the failure of one component possibly affects another. For instance, in the automobile case, a gearbox bearing failure may deteriorate the shaft performance [20], and the oil valve failure may also cause the wheel brake system to fail [21]. Liu et al. [22] developed failure interaction models for a series and parallel system under a renewing free-replacement warranty (RFRW). They consider that a component can induce failures in another component whenever it fails. Zhang et al. [23,24] use stochastic dependence to model a series of multi-component products. Luo et al. [25] modeled failure interaction between the software and hardware of a product sold with a non-renewing free replacement warranty. Wang et al. [26] studied the warranty cost for an extended warranty considering failure interaction.

Accordingly, this study considers a product as a multi-component system and models failure using a white-box approach considering the inner structure of the product. Husniah et al. [12] observed a Lemon Laws, a multi-component system under the protection of the Lemon laws in which the product obtained EWC for a refund case. Hence, this study extends the study of Husniah et al. [12] and finds the optimal lemon period for both refund and replacement cases. The extension involves developing a warranty formed as a replacement of a failed item with a new one. Moreover, this study also extends the failure models by considering the failure interaction mechanism between two component groups (i.e., critical and non-critical components).

The paper is organized as follows. Section 1 explains the issue of the implementation of Lemon Laws and provides some literature reviews related to this study. Section 2 presents a problem description comprising product failure mechanism and warranty policy. This study examines two cases of Lemon Laws warranty, which consist of refund and replacement cases. Then, Section 3 provides a model formulation to obtain the expected warranty costs, the expected warranty cost rate, and the optimization function. Section 4 deals with numerical examples illustrating both cases' expected warranty cost and the optimal lemon period. Finally, the conclusion, with a brief discussion of this study and topics for further research, is provided in Section 5.

2. Problem Description

In this study, we delve into the realm of repairable multi-component products (e.g., automobiles and electronic products) safeguarded by a warranty, focusing on the Lemon laws. These laws, a crucial pillar of consumer protection, offer additional security to consumers who may have purchased an inferior product. They mandate that the manufacturer must refund or replace the product in the event of recurrent failures. While beneficial for consumers, this condition can significantly increase warranty servicing costs and erode the manufacturer's profit. In cases where lemon laws come into play, the manufacturer must obtain EWC and strategically determine the optimal lemon period to manage warranty costs efficiently. This paper, therefore, aims to derive the expression of EWC and identify the optimal lemon period for refund and replacement cases, providing practical insights for manufacturers and policymakers.

2.1. Product failure mechanism

This study considers a product a multi-component system consisting of Critical (C) and Non-Critical (NC) Components. The product can fail due to either C or NC component failures. Regarding the failure process, the product components may fail independently (due to natural failure) or dependently (due to induced failure). In this study, two types of failure processes are considered - i.e., (i) natural failure and (ii) induced failure. In (i), both C and NC component failures are independent. In contrast, in (ii), the C component failure may occur naturally or induced by the NC component (i.e., one-way failure interaction). Thus, a white box approach is utilized to model the product failure; hence, one needs to consider the inner structure of a product.

All studies mentioned in the first section (i.e., category (ii)—models with failure interaction) consider that the failure interaction follows the type I interaction, where the failure of one component may trigger a failure of another component with probability p or have no effect with probability *1-p* [27,28]. This research, however, introduces a fresh perspective on understanding failure interaction, presenting a novel approach that deviates from the conventional type I interaction.

For products covered by the Lemon laws, Alifin et al. [29] and Iskandar et al. [30] studied warranty cost analysis considering the failure interaction of type I. In Alifin et al. [29]The failure interaction transpires in a one-way mechanism—i.e., only non-critical components can induce failures in the critical component. While Iskandar et al. [30] consider a two-way failure interaction -i.e., the failure interaction occurs mutually between critical and non-critical components. Unlike Husniah et al. [12], the study by Alifin et al. [29] and Iskandar et al. [30] discovered that a more significant failure threshold reduces the warranty cost and prolongs the optimal lemon period.

Therefore, this study will investigate a warranted multi-component product under the Lemon Laws' protection. The product is a repairable item grouped into Critical (C) and Non-Critical (NC) components. A lemon is declared when the number of C or NC component failures corresponds to the pre-specified threshold (*k*). If the lemon is stated, then the dealer/manufacturer will provide an (i) refund (i.e., return the sales price) or (ii) replacement scheme (i.e., the product is replaced with a new unit). Furthermore, this study considers that failure of a component (i) occurs naturally (there is no failure interaction) or (ii) is induced by other component failures (there is failure interaction). The Lemon laws period is assumed to be equal to or less than the base warranty period (i.e., (i) $W_L = W$ or (ii) $W_L < W$). As it is found in the USA, $W_L = W$ = 12 months, and in Singapore, W_L (6 months) < W (12 months).

2.2. Warranty policy

Suppose that a repairable multi-component product (e.g., automobile products) is sold with a warranty for period W. This warranted product is also protected by the Lemon laws with period W_L. It is assumed that $W_L = W$. The dealer or the manufacturer provides a free-of-charge minimal repair whenever a product fails during the base warranty period (W). This study considers that a lemon is invoked by $\boldsymbol{k}^{\text{th}}$ successive failures of either C or NC components in (0, W_L). If a product is declared in lemon condition, then the dealer/manufacturer must (i) refund (i.e., similar to the sales price) or (ii) replace the product with a new one. As a result, four warranty cost models will be developed, as displayed in Table 1.

Tab. 1. Warranty cost models developed in this study

Warranty Schemes	Failure Mechanism	Warranty Period	Model Number
Refund	Independent	$W_L = W$	1
	Dependent		2
Replacement	Independent		3
	Dependent		4

2.3. Research gap

According to the above sections, the study of Lemon Laws concerning warranty costs and product failure mechanisms is still rare. Moreover, most other research studies neglected the product's inner structure, often adopting a 'black-box approach', which refers to a method that treats the product as a black box, focusing on its inputs and outputs without considering its internal workings. Finally, the contributions of this paper are two folds - (i) to extend the models of Husniah et al. [12] to the case of replacement and (ii) to examine the effects of independent and dependent failure (i.e., type I failure interaction [27,28]) on the total warranty cost and the optimal lemon period from the manufacturer's perspective. This study considers the manufacturer's perspective due to the enactment of Lemon Laws, which have binding legal force [1]. Hence, the manufacturer has no option but to obey it. The manufacturer could evaluate the policy through the warranty models constructed in this study by comparing the obtained optimal warranty period with the regulated Lemon Laws period. Thus, the manufacturer could find a way to adjust their policy by evaluating the optimal lemon period considering several aspects (e.g., costs, reliability, and quality). The comparison between this research and other relevant studies is summarized in Table 2.

Tab. 2. Comparison with some related studies

	Warranty scheme	Product Perspectives	Maintenance policy	Failure mechanism	Failure process modelling	Objective function	Decision variable
[7]	Refund according to Lemon Laws ($W_L \le W$ and $W_L = W$)	Single-component system	Minimal repair (CM)	Independent failure	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	-	-
[8]	Refund according to two-dimensional Lemon Laws warranty	Multi-component system	Imperfect repair (PM) and minimal repair (CM)	Independent failure	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	Minimizing cost per time unit	Age or usage interval
[9]	Free replacement warranty	Single-component system	-	Independent failure	Weibull distribution	Minimizing warranty cost	Renewal period
[12]	Refund according to Lemon Laws ($W_L \le W$ and $W_L = W$)	Multi-component system	Minimal repair (CM)	Independent failure	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	Minimizing cost per time unit (i.e., expected warranty cost rate)	Lemon period (W_L^*)
[16]	Imperfect repair (repair service warranty)	Multi-component system	Imperfect repair (CM)	Independent failure	Weibull distribution	-	-
[22]	Free replacement warranty	Multi-component system	Replacement or perfect repair (CM)	Mutual failure interaction	Exponential distribution	-	-

	Warranty scheme	Product Perspectives	Maintenance policy	Failure mechanism	Failure process modelling	Objective function	Decision variable
[25]	Free replacement warranty	Multi-component system	Minimal repair (CM)	Mutual failure interaction	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	Minimizing warranty cost	Warranty period and product price
[29]	Refund and replacement according to Lemon Laws ($W_L \leq W$)	Multi-component system	Minimal repair (CM)	One-way failure interaction	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	Minimizing cost per time unit (i.e., expected warranty cost rate)	Lemon period (W_L^*)
[30]	Refund and replacement according to Lemon Laws ($W_L < W$)	Multi-component system	Minimal repair (CM)	Mutual failure interaction	Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)	Minimizing cost per time unit (i.e., expected warranty cost rate)	Lemon period (W_L^*)
This study	*Refund and replacement according to Lemon Laws ($W_L = W$)*	*Multi-component system*	*Minimal repair (CM)*	*One-way failure interaction*	*Weibull (time-to-the-first-failure) and Non-Homogeneous Poisson Process (recurrent failure)*	*Minimizing cost per time unit (i.e., expected warranty cost rate)*	*Lemon period (W_L^*)*

2.4. Notations and assumptions

The notations used in this study are presented in Table 3. While the following assumptions will be used in the model formulation:

(1) The product under study is a multi-component repairable system comprising critical (C) and non-critical (NC) components. The component grouping is not arbitrary but based on a logical foundation maintenance and warranty cost incurred by the dealer/manufacturer. This approach ensures that the C component, more likely to produce a higher cost than the NC, is given due consideration.

(2) The times to the first failure distributions of C and NC components follow the Weibull distribution. This distribution is not only robust but also versatile, capable of generating various shapes of probability curves according to two parameters - scale (α) and shape parameters (β). This flexibility allows it to represent several shapes of failure rate functions such as increasing, decreasing, bathtub-shaped, and U or V-shaped [31], enhancing the adaptability of our model.

(3) Minimal repair actions rectify all failures before reaching the kth failure. This study considers this repair approach so that the post-repair action intensity rate is unchanged, and the counting process can be represented by the Non-homogeneous Poisson Process (NHPP).

(4) The C component failure may occur naturally or be induced by a natural failure of the NC component [20] [21].

Tab. 3. Mathematical notations used in this study

Notation/Abbreviation		Description
CM	:	Corrective maintenance
PM	:	Preventive Maintenance
C	:	Critical component
NC	:	Non-Critical component
EWC	:	Expected total warranty cost
EWR	:	Expected warranty cost rate
W	:	Base warranty period

Notation/Abbreviation		Description
W_L	:	Lemon laws period
c_p	:	Product sales price
c_m	:	Manufacturing cost
c_{rc}	:	The repair cost of the C component
c_{rn}	:	The repair cost of NC component
$C_1(W;k)$	:	Total warranty cost for refund scheme ($W_L = W$) with independent failure case
$C_2(W;k)$	:	Total warranty cost for replacement scheme ($W_L = W$) with independent failure case
$\tilde{C}_1(W;k)$	:	Total warranty cost for refund scheme ($W_L = W$) with dependent failure case
$\tilde{C}_2(W;k)$	:	Total warranty cost for replacement scheme ($W_L = W$) with dependent failure case
k	:	The number of failure threshold
L_W	:	Warranty length
$U_n[V_n]$	:	The time instance of the n^{th} failure of the C[NC] component
$\Gamma_k = \min(U_k, V_k)$	:	The time instance where the product turns out to be a lemon
$N_1(t)[N_2(t)]$	:	The number of natural failures in C[NC] component over $[0,t)$
$\tilde{N}_1(t)[\tilde{N}_2(t)]$	:	The number of induced failure in C[NC] component $[0,t)$
$G_n(t)[H_n(t)]$	:	Cumulative distribution function (CDF) of C[NC] component (independent failure case)
$g_n(t)[h_n(t)]$	:	Probability density function (PDF) of C[NC] component (independent failure case)
$\bar{G}_n(t)[\bar{H}_n(t)]$	:	Survival function of C[NC] component (independent failure case)
$\tilde{G}_n(t)$	:	Cumulative distribution function (CDF) of C component (dependent failure case)
$\tilde{g}_n(t)$	:	Probability density function (PDF) of C component (dependent failure case)
$\bar{\tilde{G}}_n(t)$	:	Survival function of C component (dependent failure case)
$\lambda_1(t)[\lambda_2(t)]$	:	Weibull intensity function of C[NC] component
$\Lambda_1(t)[\Lambda_2(t)]$	:	Cumulative hazard function of C[NC] component
$\alpha_c[\alpha_{nc}]$	:	Weibull scale parameter of C[NC]
β	:	Weibull shape parameter
$p_1[p_2]$	:	The Probability that the C[NC] component failures induce NC[C] component to fail

3. Model Formulation

3.1. Failure modeling

In failure modeling, this study describes the case for independent failure and later for dependent failure.

3.1.1. Independent failure

In this case, each failure of C and NC components occurs independently (i.e., no failure interaction between both components). Let $N_1(t)[N_2(t)]$ be the number of failures of the C[NC] component over [0, t). All failures until $k-1$ are rectified by minimal repair, and the repair times are assumed to be small relative to the mean time between failures. Hence, $N_1(t)[N_2(t)]$ follows the Non-Homogeneous Poisson Process (NHPP) with the intensity function $\lambda_1(t)[\lambda_2(t)]$. Let $U_{n_1}[V_{n_2}]$ be the time of the n^{th} failure of the C[NC] component. The times to the first failure $U_1[V_1]$ are supposed to follow the Weibull distribution function, which is widely used in some literatures [31][32]. Then, the cumulative distribution function of $U_{n_1}[V_{n_2}]$ is given by $G_{n_1}(t)[H_{n_2}(t)]$. The probability of ***n*** successive failures for C[NC] components over (0, t] is given by,

$$P(N_1(t)=n_1)=G_{n_1}(t)-G_{n_1+1}(t) \quad [P(N_2(t)=n_2)=H_{n_2}(t)-H_{n_2+1}(t)] \tag{1}$$

Where,

$$P(U_{n_1}\le t)=G_{n_1}(t)=1-\sum_{i=0}^{k-1}e^{-\Lambda_1(t)}\frac{\Lambda_1(t)^i}{i!};$$

$$P(V_{n_2}\le t)=H_{n_2}(t)=1-\sum_{i=0}^{k-1}e^{-\Lambda_2(t)}\frac{\Lambda_2(t)^i}{i!}$$

Whilst, the intensity function ($\lambda_1(t)[\lambda_2(t)]$) and the cumulative intensity function ($\Lambda_1(t)[\Lambda_2(t)]$) are given by,

$$\lambda_1(t)=\frac{\beta}{\alpha_c}\left(\frac{t}{\alpha_c}\right)^{\beta-1}\left[\lambda_2(t)=\frac{\beta}{\alpha_{nc}}\left(\frac{t}{\alpha_{nc}}\right)^{\beta-1}\right];$$

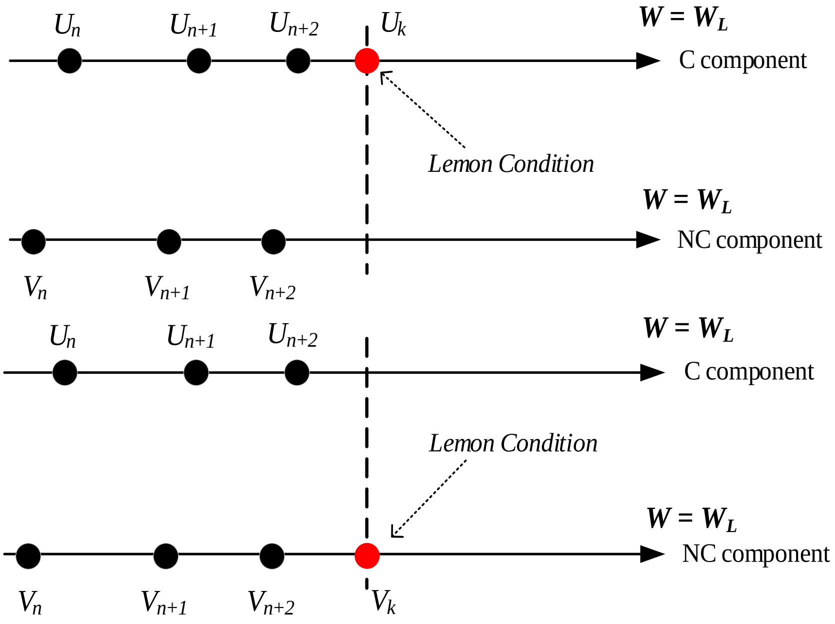
$$\Lambda_1(t)=\int_o^t\lambda_1(\tau)d\tau\left[\Lambda_2(t)=\int_o^t\lambda_2(\tau)d\tau\right]$$

Then, the probability that $P(U_{n_1}>t)\left[P(V_{n_2}>t)\right]$ is given by,

$$P(U_{n_1}>t)\left[P(V_{n_2}>t)\right]=\bar{G}_{n_1}(t)\left[\bar{H}_{n_2}(t)\right] =1-G_{n_1}(t)\left[1-H_{n_2}(t)\right] \tag{2}$$

3.1.2. Dependent failure

Here, it is considered that failure interaction is one-way -i.e., NC component failure will induce the C component to fail – but the other way around will not be possible. Hence, the failure of the C component may form naturally or as an induced failure, while the NC component is all-natural. The failure threshold for a lemon occurred is only based on the number of C component failures, then the number of failures of the NC component can be greater than k – i.e., $N_2(t)=n_2>k$. Let p_2 be the probability of the NC component inducing a failure to the C component or otherwise, $q_2 = 1 - p_2$ (no induced failure). This type of failure interaction is called a type-1 in Murthy et al. [27,28]. Furthermore, let $\tilde{N}_2(t)$ be the number of induced failures of the C component caused by the NC component over [0, t). Then, the total number of the C component failures is given by $N_1(t)+\tilde{N}_2(t)$, which is the sum of two stochastic failure processes (i.e., natural, $N_2(t)$ and induced failure, $\tilde{N}_2(t)$) over [0, t) with intensity function $\lambda_1(t)+p_2\lambda_2(t)$. If U_k is the time instance of a declared lemon condition, then two ways can occur in the interval $[t, t+\delta t)$ as follows (See Figure 2).



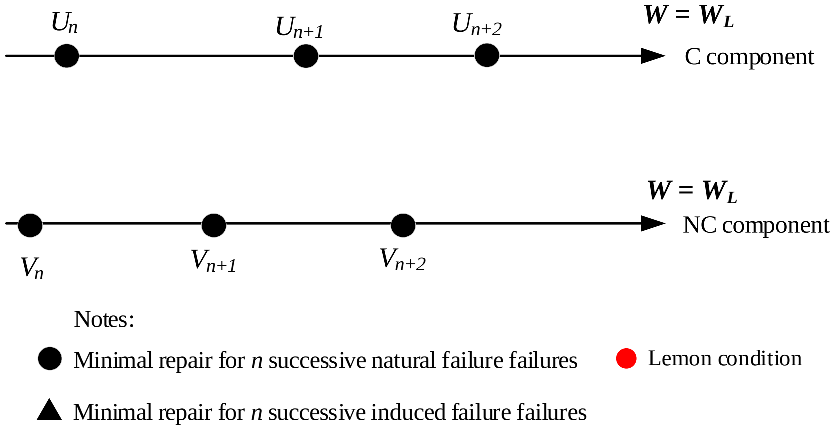


Fig. 1. Possible lemon conditions occurred for independent failure case

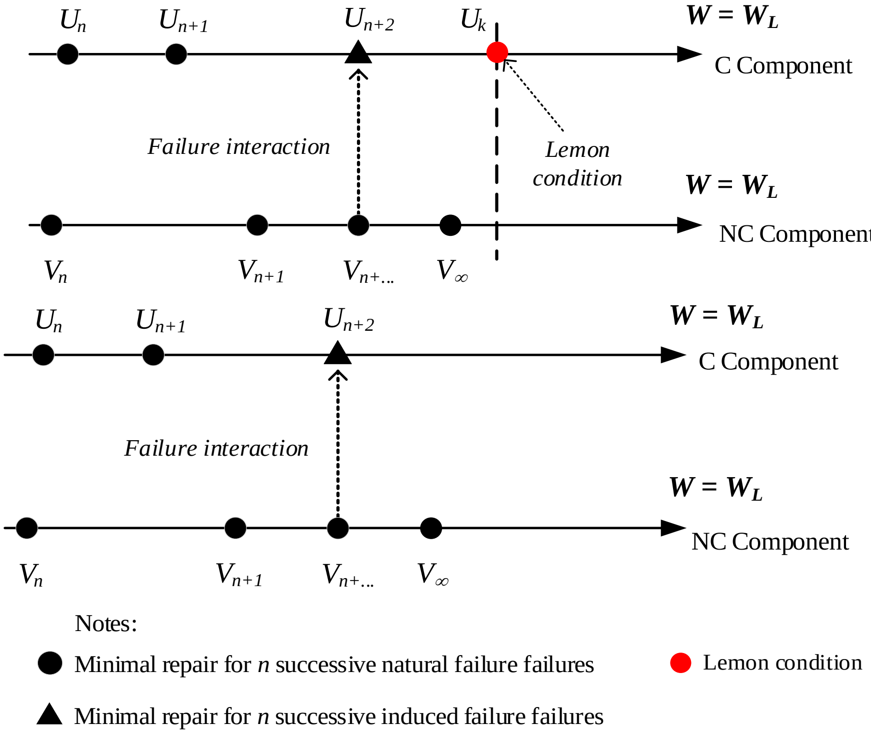


Fig. 2. Possible lemon conditions occurred for dependent failure case

(1) $N_1(t)=n_1 \le k-1, \tilde{N}_2(t)=\tilde{n}_2=k-1-n_1$, where $N_1(t)+\tilde{N}_2(t)=k-1$ and the C component fails in $[t,t+\delta t)$ (due to natural failure) or

(2) $N_1(t)+\tilde{N}_2(t)=k-1$ and the NC component fails in $[t,t+\delta t)$ inducing the C component to fail (due to induced failure).

If $\tilde{g}_k(t)$ is the probability density function (PDF) of U_k, then $\tilde{g}_k(t)$ can be obtained through the following theorems.

Theorem 1

If the $N_1(t)$ and $N_2(t)$ follow the NHPP with intensity function given by $\lambda_1(t)$ and $\lambda_2(t)$, respectively, then the PDF, $\tilde{g}_k(t)$, is given by,

$$\tilde{g}_k(t)=\frac{\left[\Lambda_1(t)+p_2\Lambda_2(t)\right]^{k-1}\left[\lambda_1(t)+p_2\lambda_2(t)\right]e^{-\left[\Lambda_1(t)+p_2\Lambda_2(t)\right]}}{(k-1)!} \quad (3)$$

Proof

In the dependent failure case, the probability density function (PDF) $\tilde{g}_k(t)$ can be obtained using a conditional approach involving two steps. First, define the function $\tilde{g}_k(t|N_1(t)=n_1, N_2(t)=n_2)$ where $0 \le n_1 \le k-1,\ k-1-n_1 \le n_2 < \infty$ and then remove the condition of $N_1(t)$ and $N_2(t)$. Note that the distribution for $\tilde{N}_2(t)=k-1-n_1$ given $N_2(t)$ is Binomial (i.e., the success or failed induced failure attempts). Hence,

$$\tilde{g}_k(t|N_1(t)=n_1, N_2(t)=n_2) = \binom{\infty}{k-1-n_1} p_2^{k-1-n_1}(1-p_2)^{n_2-(k-1-n_1)}[\lambda_1(t)+p_2\lambda_2(t)] \quad (4)$$

The following are how to simplify Eq. (4). Removing the conditioning on $N_2(t)$ yields,

$$\tilde{g}_k(t|N_1(t)=n_1) = \left\{ \sum_{n_2=k-1-n_1}^{\infty} \binom{n_2}{k-1-n_1} p_2^{k-1-n_1}(1-p_2)^{n_2-(k-1-n_1)} P(N_2(t)=n_2) \right\}[\lambda_1(t)+p_2\lambda_2(t)]$$

Removing the conditioning on $N_1(t)$ yields,

$$\tilde{g}_k(t) = \sum_{n_1=0}^{k-1}\left[\frac{e^{-\Lambda_1(t)}\Lambda_1(t)^{n_1}}{n_1!}\left\{\sum_{n_2=k-1-n_1}^{\infty}\binom{n_2}{k-1-n_1}p_2^{k-1-n_1}(1-p_2)^{n_2-(k-1-n_1)}\frac{e^{-\Lambda_2(t)}\Lambda_2(t)^{n_2}}{n_2!}\right\}\right][\lambda_1(t)+p_2\lambda_2(t)]$$

Hence,

$$\tilde{g}_k(t) = \sum_{n_1=0}^{k-1}\left[\frac{e^{-\Lambda_1(t)}\Lambda_1(t)^{n_1}}{n_1!}\left\{\frac{\Lambda_2(t)^{k-1-n_1}e^{-\Lambda_2(t)}p_2^{k-1-n_1}}{[k-1-n_1]!}\sum_{n_2=k-1-n_1}^{\infty}\frac{[\Lambda_2(t)(1-p_2)]^{n_2-(k-1-n_1)}}{[n_2-(k-1-n_1)]!}\right\}[\lambda_1(t)+p_2\lambda_2(t)]\right.$$

After some mathematical manipulation, a simplified form of $\tilde{g}_k(t)$ is given by,

$$\tilde{g}_k(t) = \sum_{n_1=0}^{k-1}\left[\frac{e^{-\Lambda_1(t)}\Lambda_1(t)^{n_1}}{n_1!}\left\{\frac{[p_2\Lambda_2(t)]^{k-1-n_1}e^{-\Lambda_2(t)}}{[k-1-n_1]!}e^{\Lambda_2(t)[1-p_2]}\right\}\right][\lambda_1(t)+p_2\lambda_2(t)]$$

Finally, as $e^{\lambda} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$, then,

$$\tilde{g}_k(t) = \frac{\left[\Lambda_1(t)+p_2\Lambda_2(t)\right]^{k-1}[\lambda_1(t)+p_2\lambda_2(t)]e^{-[\Lambda_1(t)+p_2\Lambda_2(t)]}}{(k-1)!} \quad (5)$$

The Cumulative Distribution Function (CDF) of U_k, $\tilde{G}_k(t)$, is as follows.

$$\tilde{G}_k(t) = 1-\sum_{i=0}^{k-1}\frac{e^{-[\Lambda_1(t)+p_2\Lambda_2(t)]}\left[\Lambda_1(t)+p_2\Lambda_2(t)\right]^i}{i!} = 1-\frac{\Gamma\left(k,\Lambda_1(t)+p_2\Lambda_2(t)\right)}{(k-1)!}, \quad (6)$$

The $\tilde{G}_k(t)$ can be obtained using the properties of incomplete gamma function given by,

$$\Gamma(k,x) = (k-1)!e^{-x}\sum_{i=0}^{k-1}\frac{x^i}{k!} = \int_x^{\infty} y^{k-1}e^{-y}dy$$

which is related to,

$$\tilde{G}_k(t) = \int_0^t \tilde{g}_k(x)dx$$

Hence, the probability of the C component failures considering the failure interaction is given by,

$$\Pr\left\{N_1(t)+\tilde{N}_2(t)=n_1\right\} = \tilde{G}_{n_1}(t)-\tilde{G}_{n_1+1}(t) = \frac{e^{-[\Lambda_1(t)+p_2\Lambda_2(t)]}\left[\Lambda_1(t)+p_2\Lambda_2(t)\right]^{n_1}}{n_1!} \quad (7)$$

Since the failure of the NC component is all-natural, the CDF of the NC component is given by Eq. (1), then the probability that $U_{n_1} > t[V_{n_2} > t]$ is given by,

$$P(U_{n_1} > t)\left[P(V_{n_2} > t)\right] = \bar{\tilde{G}}_{n_1}(t)\left[\bar{H}_{n_2}(t)\right] = 1-\tilde{G}_{n_1}(t)\left[1-H_{n_2}(t)\right] \quad (8)$$

<u>Theorem 2</u>

If the $N_1(t)$ and $N_2(t)$ follow NHPP and HPP with intensity function given by $\lambda_1(t)$ and $\lambda_2(t)=\lambda_2$, respectively, then the PDF, $\tilde{g}_k(t)$, is given by,

$$\tilde{g}_k(t) = \frac{\left[\Lambda_1(t)+p_2\lambda_2 t\right]^{k-1}[\lambda_1(t)+p_2\lambda_2]e^{-[\Lambda_1(t)+p_2\lambda_2 t]}}{(k-1)!} \quad (9)$$

<u>Proof:</u>

The proof is straightforward –i.e., from Eq. (5) with $\lambda_1(t)$ and $\lambda_2(t)=\lambda_2$ then Eq. (9) is obtained.

<u>Theorem 3</u>

If the $N_1(t)$ and $N_2(t)$ follow HPP and NHPP with intensity function given by $\lambda_1(t)=\lambda_1$ and $\lambda_2(t)$ respectively, then the PDF, $\tilde{g}_k(t)$, is given by,

$$\tilde{g}_k(t) = \frac{\left[\lambda_1 t+p_2\Lambda_2(t)\right]^{k-1}[\lambda_1+p_2\lambda_2(t)]e^{-[\lambda_1 t+p_2\Lambda_2(t)]}}{(k-1)!} \quad 10)$$

<u>Proof:</u>

The proof is straightforward –i.e., from Eq. (5) with $\lambda_1(t)=\lambda_1$ and $\lambda_2(t)$ then Equation Eq. (10) is obtained.

3.2. Warranty cost modeling

In warranty cost modeling, this study constructs the model for the refund scheme and, later, the replacement scheme.

3.2.1. Refund scheme

In this scheme, the manufacturer is required to refund the sales price if the product turns out to be a lemon (i.e., the number of failures reaches the threshold - k) in $(0,W_L]$, where $W_L = W$. Let Γ_k be the time instance of a lemon declared. Then, if $\Gamma_k \le W_L$, the product is declared as a lemon, where $\Gamma_k = \min(U_k, V_k)$. In this scheme, note that the warranty period will be terminated immediately at Γ_k. Hence, the expected warranty cost depends on Γ_k and it is given by $E[C_i(.) \mid \Gamma_k]$ which has two elements - i.e., E[No-Refund], $\Gamma_k \le W_L$ and E[Refund], $\Gamma_k > W_L$. These will be discussed in the following sections.

(1) Independent Failure (Model 1)

Here, failures of C and NC components occur naturally, or there is no failure interaction between the two components (independent failure). It is assumed that a lemon condition can be triggered by either C or NC components, whichever reaches the threshold first. Hence, the product turns out to be a lemon if $\Gamma_k \le W_L$, where $\Gamma_k = \min(U_k, V_k)$. To obtain $E[C_1(W;k) \mid \Gamma_k]$, all conditions related to how lemon occurred must be considered – (i.e., a lemon declared or no lemon declared), and these possible conditions are as follows (See Figure 1 for k = 4).

i. Lemon due to C component, $(\Gamma_k = U_k, N_1(W) = k, N_2(W) \le k-1)$
ii. Lemon due to NC component, $(\Gamma_k = V_k, N_2(W) = k, N_1(W) \le k-1)$
iii. No lemon occurs, $(\Gamma_k = W, N_1(W) \le k-1, N_2(W) \le k-1)$

The respective illustrations for those conditions are presented in Figure 1.

Let $C_1(W;k)$ be the total warranty cost for the refund scheme. Based on the three conditions above, $E[C_1(W;k) \mid \Gamma_k]$, the expected total warranty cost conditional on Γ_k is given by,

$$E\left[C_1(W;k) \middle| \Gamma_k\right] = \begin{cases} A_1, & \text{if } \Gamma_k = U_k \le W, V_k > U_k \\ B_1, & \text{if } \Gamma_k = V_k \le W, U_k > V_k \\ C_1, & \text{if } \Gamma_k > W \ (or\ U_k > W, V_k > W) \end{cases} \tag{11}$$

where,

$$A_1 = \left[c_{rc}(k-1) + c_p + \sum_{n_2=1}^{k-1} n_2 P\left\{ N_2(W) = n_2 \right\} c_{rn} \right]$$

$$B_1 = \left[c_{rn}(k-1) + c_p + \sum_{n_1=1}^{k-1} n_1 P\left\{ N_1(W) = n_1 \right\} c_{rc} \right]$$

$$C_1 = \left[\sum_{n_1=1}^{k-1} n_1 P\left\{ N_1(W) = n_1 \right\} c_{rc} + \sum_{n_2=1}^{k-1} n_2 P\left\{ N_2(W) = n_2 \right\} c_{rn} \right]$$

The A_1 occurs for the lemon condition due to the C component. This can be indicated by the emergence of minimal repair costs (c_{rc}) following the $(k-1)^{\text{th}}$ failure and the sales price (c_p). This pattern also applies to the B_1, where the lemon occurs due to the NC component. Meanwhile, the C_1 represents the condition where no lemon condition occurs (i.e., no sales price returned by the manufacturer). Furthermore, removing the conditioning yields,

$$\begin{aligned} E\left[C_1(W;k)\right] = {} & \left[A_1\right] \int_0^W \bar{H}_k(x) g_k(x) dx + \\ & \left[B_1\right] \int_0^W \bar{G}_k(x) h_k(x) dx + \\ & \left[C_1\right] \bar{G}_k(W) \bar{H}_k(W) \end{aligned} \tag{12}$$

where,

- $P\{N_1(W) = n_1\} = G_{n_1}(W) - G_{n_1+1}(W)$, and $P\{N_2(W) = n_2\} = H_{n_2}(W) - H_{n_2+1}(W)$
- $P(U_k \le W, V_k > U_k) = \int_0^W \bar{H}_k(x) g_k(x) dx$, $P(V_k \le W, U_k > V_k) = \int_0^W \bar{G}_k(x) h_k(x) dx$, and $P(U_k > W, V_k > W) = \bar{G}_k(W) \bar{H}_k(W)$

(2) Dependent Failure (Model 2)

In this case, a lemon is invoked only by the number of the C component failures reaching the threshold in $(0,W_L]$. The failure of the C component can occur naturally or induced by the NC component. If U_k is the time of the k^{th} failure of the C component, then all possible conditions related to a lemon (or no lemon) in $(0,W_L]$ are as follows and illustrated in Figure 2 (Note that $W_L = W$).

i. Lemon due to C component, $(\Gamma_k = U_k < W$, $N_1(W) + \tilde{N}_2(W) = k$, $N_1(W) < \infty)$
ii. No lemon occurs, $(\Gamma_k = U_k > W$, $N_1(W) + \tilde{N}_2(W) \le k-1$, $N_2(W) < \infty)$

Then, Figure 2 illustrates the conditions in the dependent failure case (i.e., for k = 4).

Let $E\left[\tilde{C}_1(W;k) \middle| U_k\right]$ be the expected total warranty cost conditional on U_k. Hence,

$$E\left[\tilde{C}_1(W;k)\middle|U_k\right]=\begin{cases}A_2, & \text{if } U_k\le W\\ B_2, & \text{if } U_k>W\end{cases} \quad (13)$$

where,

$$A_2=\left[c_{rc}(k-1)+(c_p)+c_{rn}E[N_2(W)=n_2]\right]$$

$$B_2=\left[c_{rc}E[N_1(W)+\tilde{N}_2(W)=n_1\mid n_1\le k-1] + c_{rn}E[N_2(W)=n_2]\right]$$

Since there are only two possible lemon conditions in the dependent failure case, the A_2 represents the lemon due to the C component (i.e., indicated by the $(k-1)^{\text{th}}$ minimal repair cost and the sales price incurred by the manufacturer). Whilst, the B_2 shows that there is no lemon condition. The k does not limit the NC component failures in the dependent failure case. Hence, removing the condition yields,

$$E\left[\tilde{C}_1(W;k)\right]=\left[[c_{rc}(k-1)+c_p]+c_{rn}\sum_{n_1=1}^{\infty}n_2P\{N_2(W)=n_2\}\right] + \left[\sum_{n_1=0}^{k-1}c_{rc}n_1P\{N_1(W)+\tilde{N}_2(W)=n_1\} + c_{rn}\sum_{n_1=1}^{\infty}n_2P\{N_2(W)=n_2\}\right]$$

$$E\left[\tilde{C}_1(W;k)\right]=\left[[c_{rc}(k-1)+c_p]\tilde{G}_k(W)+[c_{rn}\int_0^W\Lambda_2(t)dt]\tilde{g}_k(W)\right] + \left[\sum_{n_1=0}^{k-1}c_{rc}n_1\{\tilde{G}_{n_1}(W)-\tilde{G}_{n_1+1}(W)\} + [c_{rn}\Lambda_2(W)]\bar{\tilde{G}}_k(W)\right]$$

$$E\left[\tilde{C}_1(W;k)\right]=\left[c_{rc}\sum_{n_1=1}^{k-1}n_1\tilde{G}_{n_1}(W)+(c_p)\tilde{G}_k(W) + [c_{rn}\int_0^W\lambda_2(t)dt]\bar{\tilde{G}}_k(W)\right] \quad (14)$$

3.2.2. Replacement scheme

In the replacement scheme, if the product turns out to be a lemon at the time Γ_k where $\Gamma_k\le W$, the manufacturer must replace the failed product with a new one. A replacement unit comes with a new warranty. Hence, under this scheme, the manufacturer needs to provide warranty service for a longer period ($>W$) and this, in turn, may cause a higher expected total warranty cost compared with that for the refund scheme. As in the refund case, the expected total warranty cost conditional on Γ_k has two elements -i.e., E[No-Replacement], $\Gamma_k>W$ and E[Replacement], $\Gamma_k\le W$. These will be discussed in the following two sections.

(1) Independent Failure (Model 3)

As in the independent failure case for the refund scheme, in this, the lemon can be triggered either by (i) the C component or (ii) the NC component. A lemon is declared if $\Gamma_k\le W$, where $\Gamma_k=\min(U_k,V_k)$. Using the conditional approach as in the refund scheme, the expected total warranty cost conditional on Γ_k for the replacement scheme is given by,

$$E\left[C_2(W;k)\middle|\Gamma_k\right]=\begin{cases}A_3, & \text{if } \Gamma_k=U_k\le W, V_k>U_k\\ B_3, & \text{if } \Gamma_k=V_k\le W, U_k>V_k\\ C_3, & \text{if } \Gamma_k>W \text{ (or } U_k>W, V_k>W)\end{cases} \quad (15)$$

where,

$$A_3=\left[c_{rc}(k-1)+c_m+\sum_{n_2=1}^{k-1}n_2P\{N_2(W)=n_2\}c_{rn}\right]$$

$$B_3=\left[c_{rn}(k-1)+c_m+\sum_{n_1=1}^{k-1}n_1P\{N_1(W)=n_1\}c_{rc}\right]$$

$$C_3=\left[\sum_{n_1=1}^{k-1}n_1P\{N_1(W)=n_1\}c_{rc}+\sum_{n_2=1}^{k-1}n_2P\{N_2(W)=n_2\}c_{rn}\right]$$

Note that:

$$P\{N_1(W)=n_1\}=G_{n_1}(W)-G_{n_1+1}(W)$$

$$P\{N_2(W)=n_2\}=H_{n_2}(W)-H_{n_2+1}(W)$$

Theorem 4

Removing the conditioning and considering the renewal event that occurs at Γ_k (i.e., on A_3 and B_3 conditions) the expected total warranty cost for model 3 is given by,

$$E\left[C_2(W;k)\right]=\{A_3\}\left(1+\frac{\int_0^W\bar{H}_k(x)g_k(x)dx}{\bar{G}_k(W)\bar{H}_k(W)}\right) + \{B_3\}\left(1+\frac{\int_0^W\bar{G}_k(x)h_k(x)dx}{\bar{G}_k(W)\bar{H}_k(W)}\right) + \{C_3\}\bar{G}_k(W)\bar{H}_k(W) \quad (16)$$

Proof

Here, if the lemon is declared or $\Gamma_k\le W$, the manufacturer will replace the failed product with a new one. Consequently, the warranty period will be renewed immediately. The renewal process occurs in conditions A_3 and B_3. Assume that $R(W)$ is the number of the renewal process that occurred during the warranty period. Hence, for cost components A_3 and B_3, the probability of $R(W)$ are given by,

$$P\{R_{A_3}(W)=r_{A_3}\} = P(\Gamma_k>W)[P(U_k\le W,V_k>U_k)]^{r_{A_3}} \qquad r_{A_3}=0,1,2...n$$

$$P\{R_{B_3}(W)=r_{B_3}\}$$
$$=P(\Gamma_k>W)[P(V_k\le W,U_k>V_k)]^{r_{B_3}} \qquad r_{B_3}=0,1,2...n$$

Both $P\{R_{A_3}(W)=r_{A_3}\}$ and $P\{R_{B_3}(W)=r_{B_3}\}$ follow geometric distribution, then the expected average number of the renewal process is given by,

$$E[R_{A_3}(W)]=\frac{P(U_k\le W,V_k>U_k)}{P(\Gamma_k>W)},$$
$$E[R_{B_3}(W)]=\frac{P(V_k\le W,U_k>V_k)}{P(\Gamma_k>W)}$$

Then, applying the equation above, the expected total warranty cost for the replacement scheme is given by,

$$E\left[C_2\left(W;k\right)\right]=\left[(A_3)(1+E[R_{A_3}(W)=r_{A_3}])\right]+\left[(B_3)(1+E[R_{B_3}(W)=r_{B_3}])\right]+\left[(C_3)P(U_k>W,V_k>W)\right]$$

Hence,

$$E\left[C_2\left(W;k\right)\right]=\{A_3\}\left(1+\frac{\int_0^W\bar{H}_k(x)g_k(x)dx}{\bar{G}_k(W)\bar{H}_k(W)}\right)+\{B_3\}\left(1+\frac{\int_0^W\bar{G}_k(x)h_k(x)dx}{\bar{G}_k(W)\bar{H}_k(W)}\right)+\{C_3\}\bar{G}_k(W)\bar{H}_k(W)$$

(2) Dependent Failure (Model 4)

Here, as in the refund case, the lemon condition is only caused by recurrent failures of the C component. Hence, if the lemon occurs at the time U_k, then the failed product is replaced by a new one, and the warranty period restarts (as it is viewed as a renewal event). Similar to the refund scheme with dependent failure case, two conditions need to be considered - i.e., (i) a lemon occurs and (ii) no lemon happens. Hence, the expected total warranty cost conditional on U_k is given by,

$$E\left[\tilde{C}_2\left(W;k\right)\middle|U_k\right]=\begin{cases}A_4, & \text{if } U_k\le W\\ B_4, & \text{if } U_k>W\end{cases} \qquad (17)$$

where,

$$A_4=\left[c_{rc}(k-1)+c_m+c_{rn}E[N_2(W)=n_2]\right]$$
$$B_4=\left[c_{rc}E[N_1(W)+\tilde{N}_2(W)=n_1\,|\,n_1\le k-1]+c_{rn}E[N_2(W)=n_2]\right]$$

Theorem 5

Removing the conditioning and considering the renewal event that occurs at U_k (i.e., on A_4 condition) the expected total warranty cost for model 4 is given by,

$$E\left[\tilde{C}_2\left(W;k\right)\right]=\frac{\left[c_{rc}\sum_{n_1=1}^{k-1}n_1\tilde{G}_{n_1}(W)+(c_m)\tilde{G}_k(W)+[c_{rn}\int_0^W\lambda_2(t)dt]\bar{\tilde{G}}_k(W)]\right]}{\int_0^W\bar{\tilde{G}}_k(x)dx} \qquad (18)$$

Proof

In this case, the lemon is declared if $U_k\le W$, the failed product will be replaced with a new one by the manufacturer, and the warranty period will be renewed. Since the lemon condition occurs to the C component, the renewal process will only apply to condition A_4. Similar to Theorem 4, $R(W)$ is the number of the renewal process that occurred during the warranty period. Thus, the probability of $R(W)$ for this case is given by,

$$P\{R_{A_4}(W)=r_{A_4}\}$$
$$=P(U_k>W)[P(U_k\le W)]^{r_{A_4}} \qquad r_{A_4}=0,1,2...n$$

$P\{R_{A_4}(W)=r_{A_4}\}$ follows geometric distribution, then the expected average number of the renewal process is given by,

$$E[R_{A_4}(W)]=\frac{P(U_k\le W)}{P(U_k>W)}$$

Applying the equation above yields,

$$E\left[C_2\left(W;k\right)\right]=\left[(A_4)(1+E[R_{A_4}(W)=r_{A_4}])\right]+(B_4)$$

Removing the conditioning yields,

$$E\left[\tilde{C}_2\left(W;k\right)\right]=\left[[c_{rc}(k-1)+c_m]+c_{rn}\sum_{n_1=1}^{\infty}n_2\Pr\{N_2(W)=n_2\}\right](1+E[R_{A_4}(W)=r_{A_4}])+\left[\sum_{n_1=0}^{k-1}c_{rc}n_1\Pr\{N_1(W)+\tilde{N}_2(W)=n_1\}+c_{rn}\sum_{n_1=1}^{\infty}n_2\Pr\{N_2(W)=n_2\}\right]$$

Then,

$$E\left[\tilde{C}_2\left(W;k\right)\right]=\left[[c_{rc}(k-1)+c_m]\tilde{G}_k(W)+[c_{rn}\int_0^W\Lambda_2(t)dt]\tilde{g}_k(W)\right]\left(1+\frac{P(U_k\le W)}{P(U_k>W)}\right)+\left[\sum_{n_1=0}^{k-1}c_{rc}n_1\{\tilde{G}_{n_1}(W)-\tilde{G}_{n_1+1}(W)\}+[c_{rn}\Lambda_2(W)]\bar{\tilde{G}}_k(W)\right]$$

Finally, the expected total warranty cost is given by,

$$E\left[\tilde{C}_2(W;k)\right] = \frac{\left[c_{rc}\sum_{n_1=1}^{k-1} n_1\tilde{G}_{n_1}(W) + (c_m)\tilde{G}_k(W) + [c_{rn}\int_0^W \lambda_2(t)dt]\bar{\tilde{G}}_k(W)\right]}{\int_0^W \bar{\tilde{G}}_k(x)\,dx}$$

3.3. Optimization

Given the product reliability and the manufacturer's cost structure, the optimal lemon Laws period (W_L^*) that minimizes the expected warranty cost rate (*EWR*) will be constructed. The ***EWR*** is the ratio between the expected total warranty cost and the expected warranty length, $E[L_W]$. The optimal lemon period obtained is compared with the lemon Laws period, $\tilde{W}_L$ set by the regulation of a country. If W_L^* is close to the value of $\tilde{W}_L$ (i.e., 12 months), The product's reliability will meet the regulations satisfactorily. Otherwise, if it is well greater or smaller than $\tilde{W}_L$, then it indicates that the manufacturer must upgrade the product's reliability to control the warranty cost.

(1) Independent Failure

First, obtain $E[L_W]$ and then *EWR*. The warranty ends either at (i) Γ_k, if the products turns out to be a lemon or at W if no lemon occurs ($L_W = W$). Hence, L_W is given by,

$$E[L_W] = \begin{cases} \Gamma_k, \text{if } \Gamma_k \le W \\ W, \text{ if } \Gamma_k > W \end{cases} \quad (19)$$

Removing the conditioning yields,

$$E[L_W] = \int_0^W s f_\Gamma(s)ds + W\bar{G}_k(W)\bar{H}_k(W) \quad (20)$$

Where,

$$F_\Gamma(s) = P(\Gamma_k < s)$$
$$= \int_0^s \bar{H}_k(x)g_k(x)dx + \int_0^s \bar{G}_k(x)h_k(x)dx$$
$$f(s) = \frac{\partial F(s)}{\partial s}$$

Then, the expected warranty cost rate for refund and replacement schemes are given respectively,

$$EWR\left[C_1(W;k)\right] = \frac{E\left[C_1(W;k)\right]}{E[L_W]};$$
$$EWR\left[C_2(W;k)\right] = \frac{E\left[C_2(W;k)\right]}{E[L_W]} \quad (21)$$

As a result, the optimization functions for both schemes are as follows.

$$\underset{W_L}{Min}\left\{EWR\left[C_1(W;k)\right]\right\};$$
$$\underset{W_L}{Min}\left\{EWR\left[C_2(W;k)\right]\right\} \quad (22)$$

The optimal value of W_L (i.e., W_L^*) is the one that satisfies Eq. (22) – i.e., minimizing the warranty cost rate.

(2) Dependent Failure

As the approach used in the independent failure case, the warranty length (L_W) in this case is equal to (i) U_k if $U_k \le W$ or, (ii) W if no lemon occurs ($U_k > W$). Thus, L_W is given by,

$$E[L_W] = \begin{cases} U_k, \text{if } U_k \le W \\ W, \text{ if } U_k > W \end{cases} \quad (23)$$

Then,

$$E[L_W] = \int_0^W s f(s)ds + W\bar{\tilde{G}}_k(W)\bar{H}_k(W) \quad (24)$$

Where,

$$F(s) = P(U_k < s)$$
$$= \int_0^s \bar{\tilde{G}}_k(x)h_k(x)dx + \int_0^s \tilde{g}_k(x)h_k(x)dx,$$
$$f(s) = \frac{\partial F(s)}{\partial s}$$

The expected warranty cost rate for refund and replacement schemes, respectively, is given by,

$$EWR\left[\tilde{C}_1(W;k)\right] = \frac{E\left[\tilde{C}_1(W;k)\right]}{E[L_W]};$$
$$EWR\left[\tilde{C}_2(W;k)\right] = \frac{E\left[\tilde{C}_2(W;k)\right]}{E[L_W]} \quad (25)$$

Hence, the optimization functions for both schemes are as follows.

$$\underset{W_L}{Min}\left\{EWR\left[\tilde{C}_1(W;k)\right]\right\};$$
$$\underset{W_L}{Min}\left\{EWR\left[\tilde{C}_2(W;k)\right]\right\} \quad (26)$$

The optimal value of W_L (i.e., W_L^*) is obtained using Eq. (25).

4. Numerical Example

This study considers the case where $W_L = W$ (i.e., the Lemon laws period is equal to the base warranty period ($W_L = W$ = 12 months), and the product failure follows Weibull distribution with scale and shape parameters (α, β), and other parameter values given in Table 4.

Results (the expected warranty cost, *EWR*, and optimal lemon period) are obtained using numerical methods, and the codes are written in Matlab. Table 5 shows the results of the refund scheme for independent failure and dependent failure cases with k = 3, 4, and 5. These k values are determined according to some standard lemon regulations regarding failure thresholds in some countries [1] [30]. The plots of *EWR* for the independent failure and dependent failure cases with k = 3, 4, and 5 are shown in Figure 3.

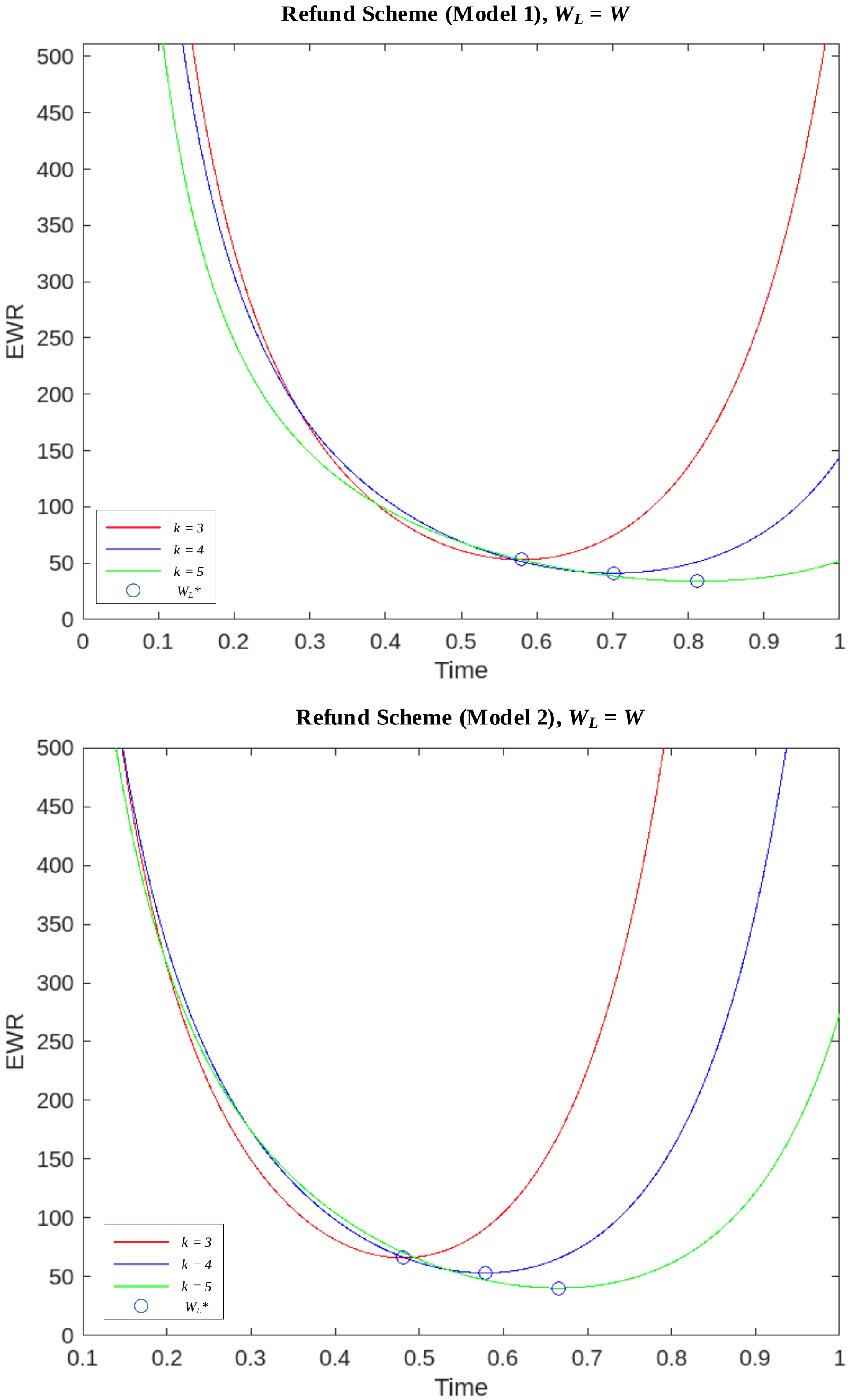


Fig. 3. Graphical illustration of warranty cost rate optimization (refund scheme)

Tab. 4. Parameter value used in this study

Parameter	c_p	c_{rc}	c_{rn}	c_m	β	W	p_2	α_c	α_{nc}
Value	100	$0.05c_p$	$0.025c_p$	$0.7c_p$	2	1	0.8	0.4	0.7

Tab. 5. Results for refund scheme

Independent failure case (Model 1)				
k	$E[C_1(W;k)]$	$E[L_W]$	$EWR[C_1(W;k)]$	W_L^*
3	32.3700	0.6080	53.2445	0.5790
4	30.2751	0.7366	41.1026	0.7015
5	28.7603	0.8530	33.7159	0.8124
Dependent failure case (Model 2)				
k	$E[\tilde{C}_1(W;k)]$	$E[L_W]$	$EWR[\tilde{C}_1(W;k)]$	W_L^*
3	33.9139	0.5101	66.4804	0.4807
4	33.2201	0.6252	53.1378	0.5782
5	31.0169	0.7658	40.5027	0.6652

Some findings for the refund scheme are as follows.

- The optimal lemon period (W_L^*) is inversely proportional to the objective function. The longer the lemon period, the lower the expected warranty cost rate. This finding is due to the extended lemon period indicating fewer warranty claims.
- As k increases, the expected total warranty cost ($E[C_1(W;k)]$, $E[\tilde{C}_1(W;k)]$) and expected warranty cost rate ($EWR[C_1(W;k)]$, $EWR[\tilde{C}_1(W;k)]$) decrease. This is because the number of lemons declared decreases as k increases.
- The optimal lemon period (W_L^*) gets longer for a larger k (failure thresholds), and this is as expected.
- The failure interaction causes the expected total warranty cost ($E[C_1(W;k)], E[\tilde{C}_1(W;k)]$) and expected warranty cost rate ($EWR[C_1(W;k)], EWR[\tilde{C}_1(W;k)]$) to increase as more product failures are likely to happen. This pattern is due to component failures caused by natural and induced failures.

Furthermore, Table 6 shows the results of the replacement scheme for independent failure and dependent failure cases with ***k*** = 3, 4, and 5. Figure 4 shows the plots of ***EWR*** for the independent failure and dependent failure cases with ***k*** = 3, 4, and 5. Findings are relatively similar to refund schemes, particularly regarding the objective function. However, the replacement scheme produces a longer expected warranty length ($E[L_W]$). This finding is due to the renewal event in every product replacement in this scheme.

Tab. 6. Results for replacement scheme

Independent failure case (Model 3)				
k	$E[C_2(W;k)]$	$E[L_W]$	$EWR[C_2(W;k)]$	W_L^*
3	41.8424	1.4289	29.2831	0.6548
4	41.2656	1.4233	28.9930	0.8061
5	39.1077	1.3703	28.5400	0.9381
Dependent failure case (Model 4)				
k	$E[\tilde{C}_2(W;k)]$	$E[L_W]$	$EWR[\tilde{C}_2(W;k)]$	W_L^*
3	49.9265	1.8497	26.9919	0.7101
4	47.9533	1.8085	26.5153	0.8422
5	45.7533	1.7717	25.8249	0.9559

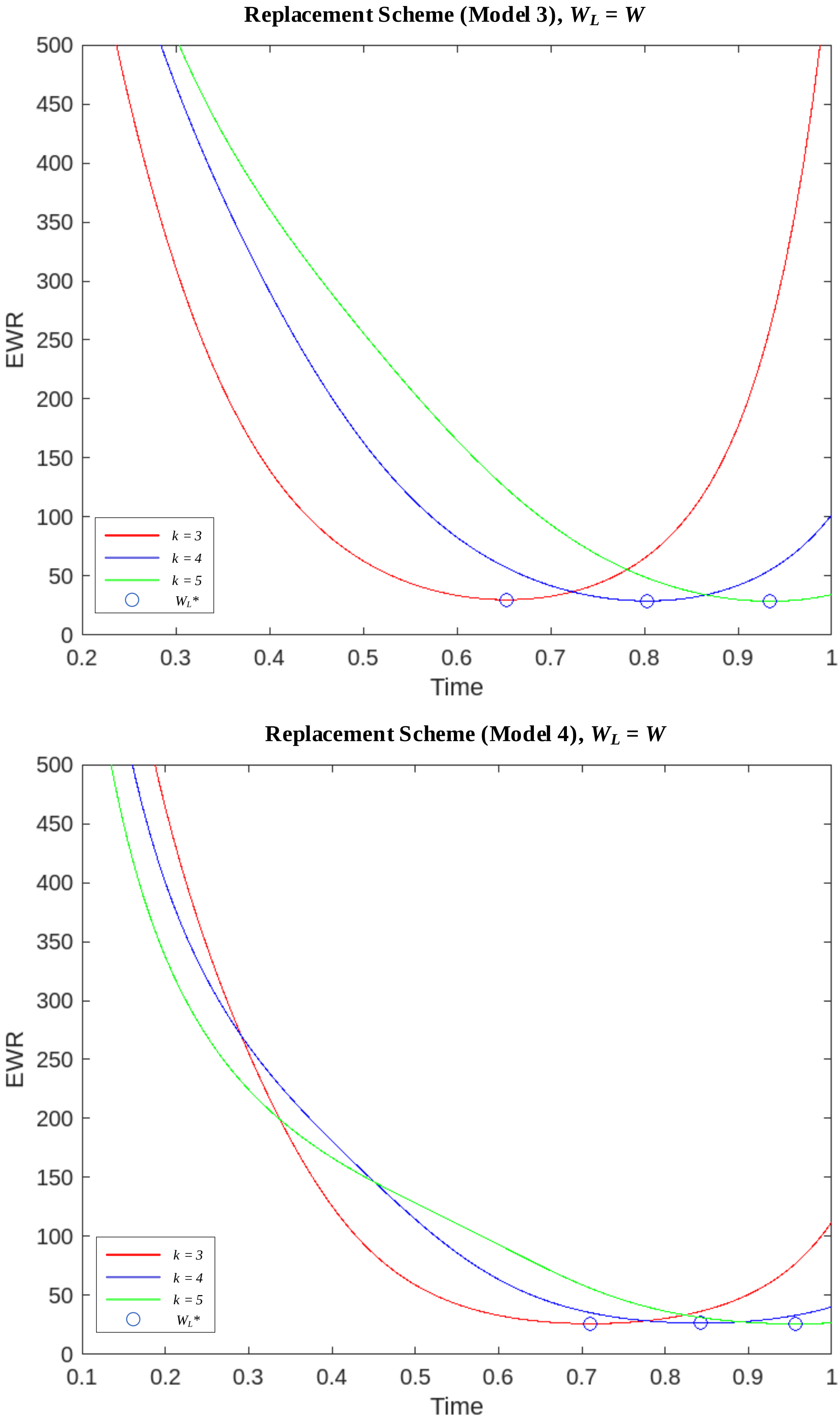


Fig. 4. Graphical illustration of warranty cost rate optimization (replacement scheme)

4.1. Sensitivity analysis

Tables 7 and 8 show the results of refund and replacement schemes, respectively, for four sets of scale parameter values (α_c, α_{nc}), which represents the reliability of C and NC components for k = 3. Note that α_c, α_{nc} = 0.4 and 0.7 for low (L) and high (H) reliability, respectively. As seen in Tables 7 and 8, the optimal lemon period (W_L^*) gets closer to $\tilde{W}_L = 1$ as the reliability of C and NC components increases (i.e., HH combination). This indicates that the manufacturer needs to enhance the product reliability to comply with the Lemon laws regulation. Afterward, the results in Table 9 show that $W_L^* \geq 1$ for parameter values α_c = 1.150, α_{nc} = 1.150 (meaning that the reliability of C and NC are much improved than those in Tables 7 and 8).

Finally, the findings for the refund and replacement schemes are as follows.

Tab. 7. The effect high and low reliability on refund scheme

Reliability		Failure Thresholds (k)	Refund Scheme							
			Independent Failure (no failure interaction)				Dependent Failure (failure interaction)			
α_c	α_{nc}		$E[C_1(W;k)]$	$EWR[C_1(W;k)]$	W_L^*	$E[L_W]$	$E[\tilde{C}_1(W;k)]$	$EWR[\tilde{C}_1(W;k)]$	W_L^*	$E[L_W]$
H	H	3	25.8960	33.4684	0.7369	0.7737	26.6464	41.0411	0.6118	0.6493
H	L		27.3664	53.2555	0.4894	0.5139	28.6433	66.1373	0.4081	0.4331
L	H		36.1567	70.3615	0.4894	0.5139	38.8217	84.1253	0.4395	0.4615
L	L		37.5527	84.9310	0.4211	0.4422	40.6973	110.8677	0.3496	0.3671

Tab. 8. The effect high and low reliability on replacement scheme

Reliability		Failure Thresholds (k)	Replacement Scheme							
			Independent Failure (no failure interaction)				Dependent Failure (failure interaction)			
α_c	α_{nc}		$E[C_2(W;k)]$	$EWR[C_2(W;k)]$	W_L^*	$E[L_W]$	$E[\tilde{C}_2(W;k)]$	$EWR[\tilde{C}_2(W;k)]$	W_L^*	$E[L_W]$
H	H	3	34.3675	32.5230	0.8200	1.0567	39.9428	37.1942	0.8863	1.0739
H	H		35.5934	30.0747	0.5544	1.1835	42.5023	25.4097	0.5947	1.6727
L	L		46.6918	28.4603	0.5482	1.6406	54.5384	26.2232	0.6759	2.0798
L	L		48.2716	24.8978	0.4827	1.9388	57.3611	21.6412	0.5237	2.6506

Tab. 9. The results for α_c, α_{nc} = 1.150, which comply with the Lemon law regulation

Warranty Scheme	$E[C_1(W;k)]$	$EWR[C_1(W;k)]$	W_L^*	$E[L_W]$	Failure mechanism
Refund Scheme	13.4859	13.4859	1.0000	1.0000	Independent
	23.8789	23.8789	1.0000	1.0000	Dependent
	$E[C_2(W;k)]$	$EWR[C_2(W;k)]$	W_L^*	$E[L_W]$	
Replacement Scheme	29.6701	29.6701	1.0000	1.0000	Independent
	30.1067	30.1067	1.0000	1.0000	Dependent

- The lowest expected total warranty costs are when both components (i.e., Critical and Non-Critical) are in high reliability (i.e., HH combination). The expected total warranty costs are 25.8960 (no failure interaction) and 26.6464 (failure interaction) for the refund scheme and, 34.3675 (no failure interaction) and 39.9428 (failure interaction) for the replacement scheme.
- The highest expected total warranty costs are when both components (i.e., Critical and Non-Critical) have low reliability (i.e., LL combination) for both schemes. This means that product reliability plays a crucial part in determining warranty costs.
- The optimal lemon period (W_L^*) decreases when the level of reliability decreases -i.e., from HH to LL combinations. This is as expected since the lower reliability level results in a smaller lemon period to minimize the number of lemons declared. This pattern applies to both refund and replacement schemes.
- All optimal lemon periods obtained for both schemes, W_L^* are smaller than $\tilde{W}_L$ (=1) means that the product's reliability does not satisfactorily meet the regulation—more lemons are declared under warranty. In other words, the manufacturer needs to improve the reliability of the components to minimize the probability of a lemon occurrence and, hence, to control the warranty cost.
- The parameter values (α_c, α_{nc} = 1.150) results in W_L^* = 1 (see Table 9). This means that the product reliability is effective to face the Lemon laws regulation as $W_L^* = \tilde{W}_L$ = 1 (the Lemon laws period according to the regulation)

will result in the lower warranty cost. Hence, the value of W_L^* and the associated parameter values (α_c, α_{nc}) are in the manufacturer's interest to control the warranty cost when the warranted product is also protected by the Lemon laws.

5. Conclusion

In this study, the warranty cost models for repairable multi-component products protected by Lemon laws have been proposed. The study considers the failure interaction between critical and non-critical components and obtains the expected warranty costs for refund and replacement schemes. Given the product's reliability and the manufacturer's cost structure, the result also provides the optimal lemon period for the warranted product that the manufacturer can use to compare with the Lemon laws period stated in the regulation. Some significant findings of this study include: i) The optimal lemon period is inversely proportional to the expected warranty cost rate, ii) The higher product reliability will reduce the warranty cost significantly, and ease the manufacturer to cope with the Lemon Laws regulation. This study suggests that the manufacturer improves the product reliability to make the warranty strategy offered more effective and efficient in fulfilling the Lemon laws regulation. Finally, this research can be extended in the following ways - (i) the warranty cost model allowing the consumer to choose either refund or replacement options when a lemon is declared and (ii) the lemon invoked by the product's downtime (out-of-service time). These topics are currently ongoing.

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