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SEMI-RADICALS OF SUB MODULES IN MODULES

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Abstract: *Let R be a commutative ring and M be a unitary R– module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula $(s.t.s.r.f)$ and present the existence of such a module.*

***Keywords*:** *Prime sub module, semiprime sub module, radical and semi- radical of a module, modules satisfying the semi-radical formula.*

1. Introduction

ln this paper all the rings are commutative with identity and all the modules are unitary. Let R be a ring and M be an $R-$module. If N is a submodule of M we use the notation $N \leq M$. If the submodule N is generated by a subset S of M, we write $N = \langle S \rangle$.
If N and K are sub modules of M, then the set $\{r \in R \mid rK \subseteq N\}$ is denoted by $(N :_R K)$ or simply by $(N : K)$ which is clearly an ideal of *R*. If I is an ideal of the ring R, we write $I \trianglelefteq R$. In Section 2 we define prime and primary sub modules of an $R-$module M and in Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if N is a semiprime submodule of an $R-$module M, then $(N : M)$ is a semiprime ideal of R but not conversely in general. In Lemma 2.8 we prove that the converse is also true if M is a multiplication module. In Section 3 we define radical of an $R-$module M and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule of a module and also modules satisfying the semi-radical formula which is abbreviated as (s.t.s.r.f) and in Theorem 3.9 we show that such a module does exist.
Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a $P-$semiprime submodule we consider some of its properties.

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2. Some Elementary Results

We begin this section with the following definitions:

Definition 2.1. Let N be a proper submodule of an $R-$module M.
(a) N is called a prime submodule of M if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r \in (N : M)$.
(b) N is called a primary submodule of M if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r^n \in (N : M)$ for some positive integer n.
In (a) it can easily be shown that $P = (N : M)$ is a prime ideal of R and we say that N is $P-$prime.
We recall that if I is an ideal of a ring R, then the radical of I, denoted by $\sqrt{I}$, is defined as the intersection of all prime ideals containing I. Alternatively, we define the radical of I as :
$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some positive integer n}\}$.
Also if N is a primary submodule of an $R-$module M, then $(N : M)$ is a primary ideal of R and $P = \sqrt{(N : M)}$ is a prime ideal. We describe this situation by saying that N is $P-$primary.

Lemma 2.2. Let N be a proper submodule of an $R-$module M.
(i) N is a prime submodule of M if and only if $ID \subseteq N$ (with I an ideal of R and D a submodule of M) implies that $D \subseteq N$ or $I \subseteq (N : M)$.
(ii) N is a primary submodule of M if and only if for every finitely generated ideal I of R and any submodule D of M, $ID \subseteq N$ implies that $D \subseteq N$ or $I^n \subseteq (N : M)$ for some positive integer n.
(iii) Let P be a prime ideal of R, than N is a $P-$primary submodule of M if and only if (a)

$P\subseteq\sqrt{(N:M)}$, and (b) $cm\notin N$ for all $c\in R/P$, $m\in M/N$.

Proof. (i) $(\Rightarrow)$: Let $I\trianglelefteq R$ and $D\leq M$ be such that $ID\subseteq N$ and let $D\underline{\not\subseteq} N$. So there exists an element $x\in D\setminus N$. Let r be any element of I.Then $rx\in N$ and hence $r\in(N:M)$. Therefore $I\subseteq(N:M)$.

$(\Leftarrow)$: Let $r\in R$, $a\in M$ be such that $ra\in N$ and let $a\notin N$. By taking:

$I=(r)$ and $D=Ra$ we see that $ID\subseteq N$. But $D\underline{\not\subseteq} N$ and hence $I\subseteq(N:M)$,

which implies that $r\in(N:M)$. Therefore N is a prime submodule of M.

(ii) $(\Rightarrow)$: Let $D\leq M$ and I be a finitely generated ideal of R such that $ID\subseteq N$.

Then by [5, Corollary 1, P.99], $D\subseteq N$ or $I\subseteq\sqrt{(N:M)}$. Let $D\underline{\not\subseteq} N$, then $I\subseteq\sqrt{(N:M)}$ and by [5, Proposition 8. P.83], there exists a positive integer n such that $I^n\subseteq(N:M)$.

$(\Leftarrow)$:Let $r\in R, x\in M$ be such that $rx\in N$ and let $x\notin N$. By taking $I=(r)$ and $D=Rx$ we see that $ID\subseteq N$ and $D\underline{\not\subseteq} N$. So there exists a positive integer n such that $I^n\subseteq(N:M)$. This implies that $r^n\in(N:M)$ and hence N is a primary submodule of M.

(iii) $(\Rightarrow)$: If N is P–primary, then by definition $P=\sqrt{(N:M)}$. Now let $c\in R\setminus P$ and $m\in M\setminus N$. Let $cm\in N$, then there exists a positive integer n such that:

$c^n\in(N:M)$, that is, $c\in\sqrt{(N:M)}=P$ (because $m\notin N$), a contradiction. Hence $cm\notin N$.

$(\Leftarrow)$: Assume that (a), (b) hold. Let $r\in R$ and $m\in M$, $rm\in N$. Assume further that $m\notin N$, then by (b), r must belong to P and so $r\in\sqrt{(N:M)}$ by (a). Therefore N is a primary submodule of M. Next we must show that $P=\sqrt{(N:M)}$.

Let $r\in\sqrt{(N:M)}$, then $r^n\in(N:M)$ for some positive integer n, and so $r^nM\subseteq N$. Since N is proper, there exist $x\in M/N$. Now $r^nx\in N$ and $x\notin N$ so by (b) we conclude that $r^n\in P$ and, as P is prime, $r\in P$. We find that $\sqrt{(N:M)}=P$ and therefore N is $P-$primary.

The following definition is essential in the rest of the paper.

Definition 2.3. A proper submodule N of an $R-$module M is said to be semiprime in M, if for every ideal I of R and every submodule K of M, $I^2K\subseteq N$ implies that $IK\subseteq N$. Note that since the ring R is an $R-$module by itself, a proper ideal I of R is semiprime if for every ideals J and K of R, $J^2K\subseteq I$ implies that $JK\subseteq I$.

Proposition 2.4. Let M be an $R-$module.
(i) If N is a prime submodule of M, then N is semiprime.
(ii) If N is a semiprime submodule of M, then $(N:M)$ is semiprime ideal of R.

Proof. (i) Let $I\trianglelefteq R$, $K\leq M$ and $I^2K\subseteq M$. Then $I(IK)\subseteq N$ and since N is prime, $I\subseteq(N:M)$ or $IK\subseteq N$. But $(N:M)\subseteq(N:K)$ and hence $I\subseteq(N:K)$, and so $IK\subseteq N$. In any case we see that $IK\subseteq N$, and therefore N is semiprime.

(ii) Let J and K be ideals of R and $J^2K\subseteq(N:M)$. Hence $(J^2K)M\subseteq N$, and so, $J^2(KM)\subseteq N$. But $KM\leq M$, and N is semiprime, therefore $J(KM)\subseteq N$, and thus, $(JK)M\subseteq N$. Hence $JK\subseteq(N:M)$ and we conclude that $(N:M)$ is a semiprime ideal of R.

Part (i) of the above proposition implies that if P is a prime ideal of R then P is semiprime. In the next example we show that the converse of part (ii) of Proposition 2.1. is not valid in general.

Example 2.5. Let $R=Z$, $M=Z\oplus Z$ and $B=\langle(9.0)\rangle$. Then it is clear that $(B:M)=(0)$.Since Z is an integral domain, $(B:M)=(0)$ is a prime ideal and hence a semiprime ideal of Z. But B is not a semiprime submodule of M; because if we take $I=(3)$ and $K=<(2,0)>$, Then:

$$I^2K=\{(18q,0)|\ q\in Z\} \tag{1}$$

But:

$$IK=\{(6q,0)|\ q\in Z\} \tag{2}$$

is not a subset of B.

It is clear that if N is a semiprime submodule of an $R-$module M and $I\trianglelefteq R$, $K\leq M$ be such that $I^nK\subseteq N$ for some positive integer n, then $IK\subseteq N$.

Theorem 2.6. Let N be a proper submodule of an $R-$module M. Then the following statements are equivalent:
(i) N is semiprime.
(ii) Whenever $r^tm\in N$ for some $r\in R$, $m\in M$ and $t\in Z^+$, then $rm\in N$.

Proof. (i) $(\Rightarrow)$ (ii). Let $r^t m \in N$ where $r \in R$, $m \in M$ and $t \in Z^+$. Taking $I = (r)$ and $K = (m)$ we have $I^t K \subseteq N$ and so $IK \subseteq N$ winch implies that $rm \in N$.
(ii) $\Rightarrow$(i). Let $I \trianglelefteq R$ and $K \leq M$ be such that $I^2 K \subseteq N$. Consider the set:

$$S = \{ra |\ r \in I, a \in K\} \tag{3}$$

Then for every $r \in I, a \in K$ we have $r^2 a \in I^2 K \subseteq N$ and hence $ra \in N$. This implies that $S \subseteq N$ and since IK is the submodule of M generated by S, we must have $IK \subseteq N$. Therefore N is semiprime.

Definition 2.7. An $R-$module M is said to be a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R.
It can he easily shown that, an $R-$module M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M.

Lemma 2.8. Let M be a multiplication $R-$module. Then a submodule N of M is semiprime if and only if $(N : M)$ is a semiprime ideal of R.
Proof. $(\Rightarrow)$: This is clear from Proposition 2.4 (ii).
$(\Leftarrow)$: Let $I \trianglelefteq R$, $K \leq M$, be such that $I^2 K \subseteq N$. Hence:

$$\left(I^2 K : M\right) \subseteq (N : M). \tag{4}$$

It can be shown that:

$$I^2 (K : M) \subseteq \left(I^2 K : M\right) \tag{5}$$

and so we obtain:

$$I^2 (K : M) M \subseteq (N : M). \tag{6}$$

But $(N : M)$ is a semiprime ideal of R and hence $I(K : M) \subseteq (N : M)$. Thus we conclude that:

$$I(K : M)M \subseteq (N : M)M, \tag{7}$$

and using the fact that M is a multiplication $R-$module we have $IK \subseteq N$. Therefore N is a semiprime submodule of M.
The following lemma shows that the same situation, as above, holds for prime and primary sub modules.

Lemma 2.9. Let M be a multiplication $R-$module. Then:
(a) A submodule N of M is prime if and only if $(N : M)$ is a prime ideal of R.
(b) A submodule N of M is primary if and only if $(N : M)$ is a primary ideal of R.

Proof. (a) $(\Rightarrow)$: Clear.
$(\Leftarrow)$: Let $I \trianglelefteq R$, $D \leq M$ be such that $ID \subseteq N$, then $(ID : M) \subseteq (N : M)$. But $I(D : M) \subseteq (ID : M)$ and so $I(D : M) \subseteq (N : M)$. Since $(N : M)$ is a prime:
ideal of R we have $I \subseteq (N : M)$ or $(D : M) \subseteq (N : M)$. Suppose that $I \not\subseteq (N : M)$. Then $(D : M) \subseteq (N : M)$ and from tins we have $(D : M)M \subseteq (N : M)M$, that is, $D \subseteq N$. Hence N is a prime submodule of M by Lemma 2.2 (i).
(b)($\Rightarrow$):Clear.
($\Leftarrow$): Let $(N : M)$ be a primary ideal of R. Let I be a finitely generated ideal of R and D be a submodule of M and let $ID \subseteq N$. Suppose that for any positive integer n, $I^n \not\subseteq (N : M)$. We see that $ID \subseteq N$ implies $(ID : M) \subseteq (N : M)$ and hence $I(D : M) \subseteq (N : M)$. But $I^n \not\subseteq (N : M)$ for any positive integer n, so $(D : M) \subseteq (N : M)$, because $(N : M)$ is a primary. Hence $(D : M)M \subseteq (N : M)M$, that is, $D \subseteq N$. So N is a primary submodule of M, by Lemma 2.2 (ii). the proof is now complete.

Proposition 2.10. Let $\{P_i\}_{i \in I}$ be a non-empty family of semiprime sub modules of an $R-$module M. Then $P = \bigcap P_i$ is a semiprime submodule of M. Further if $\{P_i\}_{i \in I}$ is totally ordered (by inclusion), then $T = \bigcap P_i$ is also a semiprime submodule whenever $T \neq M$.
Proof. Let $I \trianglelefteq R$ and $K \leq M$ be such that $I^2 k \subseteq P = \bigcap P_i$. Then $I^2 k \subseteq P_i$ for every $i \in I$, and since P_i is semiprime we have $Ik \subseteq P_i$. Hence $IK \subseteq \bigcap P_i = P$ and P is semiprime. Next we let $T = \bigcap P_i \neq M$. The fact that $\{P_i\}_{i \in I}$ is totally ordered by inclusion makes it clear that T is a submodule of M. Let $I \trianglelefteq R$ and $K \leq M$ be such that $I^2 K \subseteq T$. Consider the set:

$$S = \{rk |\ r \in R, k \in K\} \tag{8}$$

Then S is a generating set for the submodule IK. If $r \in I$, $k \in K$ then $r^2 k \in I^2 K \subseteq T$ and so for some $i \in I, r^2 k \in P_i$. Since P_i is semiprime this implies that $rk \in P_i$. It follows that $S \subseteq T$ and hence $IK = \langle S \rangle \subseteq T$. Therefore T is also a semiprime submodule of *M*.

Remark. Some authors define a semiprime submodule as an intersection of prime sub modules. But by our definition of a semiprime submodule of a module we can find a semiprime submodule of a module which is not an intersection of prime sub modules. for example look at .J. Jenkins and P.F. Smith in the proof of [1].

3. Radicals and Semi-Radicals

Let M be an $R-$module and N a submodule of M. If there exists a prime submodule of M which contain N, then the intersection of all prime sub modules containing N, is called the $M-$radical of M and is denoted by $rad_M N$, or simply by $radN$. If there is no prime submodule containing N, then we define $rad_M N = M$; in particular $rad_M M = M$. An ideal I of a ring R is called a radical ideal if $\sqrt{I} = I$. Similarly, we say that a submodule B of an $R-$module M is a radical submodule if $radB = B$. It is easy to see that an ideal I of a ring R is semiprime if and only if it is radical. Because, let I be semiprime, and let $x \in \sqrt{I}$. Then $x^k \in I$ for some positive integer k. So $x^k.1 \in I$, and since I is semiprime we have $x.1 = x \in I$. Therefore $I = \sqrt{I}$.

On the other hand, if $I = \sqrt{I}$ then by definition of $\sqrt{I}$ and Propositions 2.4 (i) and 2.10, I is semiprime. Finally by Propositions 2.4 (i)and 2.10 we see that for any submodule B of an $R-$module M, $radB$ is a semiprime submodule whenever $radB \neq M$.

Theorem 3.1. Let M be a finitely generated multiplication $R-$module and let N be a proper submodule of M. Then N is semiprime if and only if it is radical.

Proof. Since $ann_R(M) \subseteq (N:M)$, by [2, Theorem 3, P.216],

$$\sqrt{(N:M)}M = rad(N:M)M. \qquad (9)$$

As M is a multiplication module we have $(N:M)M = M$, and if N is semiprime, $(N:M)$ is a radical ideal. Therefore $\sqrt{(N:M)}M = rad(N:M)M$ iff $(N:M)M = rad(N:M)M$. If $N = radN$ this implies that N is a radical submodule of M, that is, $N = radN = \bigcap P(P$ is a prime submodule of M containing N). Hence by Propositions 2.4 (1) and 2.10 N is a semiprime submodule of M. The proof is now complete.

After Remark 2.11 we may ask under what condition a semiprime submodule is the intersection of prime submoclules containing it. The following corollary can be considered as an answer.

Corollary 3.2. Let M be a finitely generated multiplication $R-$module and let N be a proper submodule of M. Then N is semiprime if and only if $N = \bigcap P$ (P_i a prime submodule of M containing N).

Proof. $(\Rightarrow)$: If N is semiprime then by Theorem 3.1, it is radical, that is, $N = \bigcap P$ (P_i a prime submodule of M containing N).

$(\Leftarrow)$:By Propositions 2.4 (i) and 2.10, N is semiprime.

Proposition 3.3. If M is a finitely generated $R-$module, then every proper submodule of M is contained in a semiprime sub module.

Proof. By Corollary of [3, Proposition 4, P.63], every proper submodule of M is contained in a prime submodule . So by Proposition 2.4 (i), we have the result.

Definition 3.4. (1) A semiprime submodule P of an $R-$module M is called a minimal semiprime of a proper submodule N if $N \subseteq P$ and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of $0 = <0_M>$ is called a minimal semiprime submodule of M.

Theorem 3.5. Let M be an $R-$module. If a submodule N of M is contained in a semiprime submodule P, then P contains a minimal semiprime submodule of N.

Proof. It is similar to the proof of [5, Theorem 4. P.84].

Proposition 3.6. Every proper submodule of a finitely generated $R-$module M possesses at least one minimal semiprime submodule of M.

Proof. Let N be a proper submodule of M, then by Proposition 3.3, N is contained in a semiprime submodule of M.

Corollary 3.7. Every semiprime submodule of an $R-$module M contains at least one minimal semiprime submodule of M.

Proof. Let P be a semiprime submodule of M and take $N = <0>$ in the Theorem 3.5. Then P contains a minimal semiprime submodule of $<0>$, and so a minimal semiprime submodule of M.

Definition 3.8. Let M be an $R-$module and $N \leq M$. If there exists a semiprime submodule of M which contains N, then the intersection of all semiprime sub modules containing N is called the semi−radical of N and is denoted by $S-rad_M N$, or simply by $S-radN$. If there is no semiprime submodule containing N. then we define

$S-radN=M$, in particular $S-radM=M$. We call $S-rad\langle 0\rangle$ the semiprime radical of M.

If $N\leq M$, then the envelope of N, denoted by $E(N)$, is defined as:

$$E\langle N\rangle=\left\{\begin{array}{l} x\in M \mid x=ra \text{ for some } r\in R, a\in M \\ \text{and } r^n a\in N \text{ for some } n\in\mathbb{Z}^+ \end{array}\right\} \quad (10)$$

We say that M satisfies the semi-radical formula, M (s.t.s.r.f) if for any $N\leq M$, the semi-radical of N is equal to the submodule generated by its envelope, that is, $S-radN=\langle E(N)\rangle$. We already know that $\langle E(N)\rangle\subseteq radN$, by [4, P.1815]. Now let $x\in E(N)$ and P be a semiprime submodule of M containing N. Then $x=ra$ for some $r\in R, a\in M$ and for positive integer $n, r^n a\in N$. But $r^n a\in P$ and since P is semiprime we have $ra\in P$. Hence $E(N)\subseteq P$. We conclude that $E(N)\subseteq\bigcap P$ (P is a semiprime submodule containing N). So $E(N)\subseteq S-radN$. On the other hand, since every prime submodule of M is clearly semiprime, we have $S-radN\subseteq radN$. We see that:

$$\langle E(N)\rangle\subseteq S-radN\subseteq radN \quad (11)$$

Now we present an $R-$module which satisfies the semi-radical formula.

Theorem 3.9. Let M be a finitely generated multiplication $R-$module. Then M satisfied the semi-radical formula.

Proof. Let $N\leq M$, then by [4. Theorem 4.4], we have $(\langle E(N)\rangle : M)=(radN : M)$.

Hence $(\langle E(N)\rangle : M)M=(radN : M)M$ and since M is a multiplication $R-$module, $\langle E(N)\rangle=radN$.

Next from (*) we have:

$$(\langle E(N)\rangle : M)M\subseteq(S-radN : M)M\subseteq (radN : M)M \quad (12)$$

that is,

$$(\langle E(N)\rangle\subseteq S-radN\subseteq radN. \quad (13)$$

Thus we find that $S-radN=\langle E(N)\rangle$.

Remark. Under the conditions of Theorem 3.9, we see that for any submodule $N\neq M$ of M we always have $RadN=S-RadN$.

Proposition 3.10. Let M be a finitely generated $R-$module. Then the semi-radical of a proper submodule N of M is the intersection of its minimal semiprime sub modules.

Proof. This is clear by using Theorem 3.5 and Proposition 3.6.

For the rest of this section we state and prove some properties of semi-radical of sub modules.

Theorem 3.11. Let B and C be sub modules of an $R-$module M. Then,

(1) $B\subseteq S-radB$.

(2) $S-rad(S-radB)=S-radB$,

(3) $S-rad(B\bigcap C)\subseteq S-radB\bigcap S-radC$, and we have the equality when for every semiprime submodule P, $B\cap C\subseteq P$ implies that $B\subseteq P$ or $C\subseteq P$,

(4) $S-rad(B+C)=S-rad(S-radB+S-radC)$,

(5) $\sqrt{(B:M)}\subseteq(S-radB:M)$,

(6) If M is finitely generated, then $S-radB=M$ if and only if $B=M$,

(7) If M is finitely generated, then $B+C=M$ if and only if $S-RadB+S-RadC=M$,

(8) $S-radI\,M=S-rad\sqrt{I}M$ for every ideal I of R.

Proof. (1) clear.

(2) Since $S-RadB$ is semiprime by Proposition 2.10, we have:

$$S-Rad(S-RadB)=S-RadB. \quad (14)$$

(3) Let P be a semiprime submodule of M such that $B\subseteq P$, so $B\bigcap C\subseteq P$ and hence $S-rad(B\bigcap C)\subseteq P$. But P is arbitrary, therefore $S-rad(B\bigcap C)\subseteq S-radB$. By a similar argument we have $S-rad(B\bigcap C)\subseteq S-radC$. Now let P be a semiprime submodule of M such that $B\bigcap C\subseteq P$ and assume that $B\subseteq P$. Then $S-radB\subseteq P$ and so $S-radB\bigcap S-radC\subseteq P$. Since P is arbitrary this implies that $S-radB\bigcap S-radC\subseteq S-rad(B\bigcap C)$ and hence we have the equality.

(4) Let P be a semiprime submodule of M such that $(S-radB+S-radC)\subseteq P$. So $S-radB\subseteq P$ and $S-radC\subseteq P$. Hence $B\subseteq C$ and $C\subseteq P$ which implies $B+C\subseteq P$. Therefore $S-rad(B+C)\subseteq P$. But P is chosen arbitrary, so:

$$S-rad(B+C)\subseteq S-rad(S-radB+S-radC). \quad (15)$$

Now suppose that P be a semiprime submodule such that $B+C\subseteq P$. So $B\subseteq P$, and $C\subseteq P$. Hence $S-radB\subseteq P$ and $S-radC\subseteq P$ and therefore $S-radB+S-radC\subseteq P$.

But $S-rad(S-radB+S-radC)\subseteq P$ and we conclude that:

$$S-rad(S-radB+S-radC)\subseteq S-rad(B+C). \quad (16)$$

(5) If $S-radB=M$, then we have the result. So let P be a semiprime submodule of M such that $B\subseteq P$. So $(B:M)\subseteq(P:M)$. We know that $(P:M)$ is a semiprime ideal of R and we have shown that $\sqrt{(P:M)}=(P:M)$. Hence $\sqrt{(B:M)}\subseteq\sqrt{(P:M)}=(P:M)$ implies that:
$\sqrt{(B:M)}M\subseteq(P:M)P\subseteq P$,
and since P can be any semiprime submodule of M containing B, we have $\sqrt{(B:M)}M\subseteq S-radB$, that is, $\sqrt{(B:M)}M\subseteq(S-radM:M)$.
(6) If $B=M$, then $S-radB=S-radM=M$. Conversely, let $S-radB=M$, but $B\neq M$. Since M is finitely generated. it contains a prime and so a semiprime submodule P containing B, by Corollary after Proposition 4 of [3]. Hence $S-radB\neq M$, a contradiction.
(7) Using parts (4) and (6) we have:
$B+C=M$ iff $S-rad(B+C)=M$
iff $S-rad(S-radB+S-radC)=M$
iff $S-radB+S-radC=M$.
(8) If M has no semiprime submodule containing IM, then $S-radIM=M$ and we have:

$$
\begin{aligned}
&I\subseteq\sqrt{I}\Rightarrow IM\subseteq\sqrt{I}M\Rightarrow S-radIM\subseteq S-rad\sqrt{I}M\\
&\Rightarrow M\subseteq S-rad\sqrt{I}M\Rightarrow M=S-rad\sqrt{I}M:\\
&=S-radIM.
\end{aligned}
\tag{17}
$$

Now let P he a semiprime submodule of M such that $IM\subseteq P$,so $I\subseteq(IM:M)\subseteq(P:M)$ and since $(P:M)$ is semiprime $\sqrt{I}\subseteq\sqrt{(P:M)}=(P:M)$.
So $\sqrt{I}M\subseteq P$ and hence $S-rad\sqrt{I}M\subseteq P$. Since P is arbitrary we have:
$S-rad\sqrt{I}M\subseteq S-radIM$.
Therefore $S-radIM=S-rad\sqrt{I}M$. The proof is now complete.

Corollary 3.12. Let M be an $R-$module and I an ideal of R. Then $S-radI^nM=S-radM$ for every positive integer n.

Proof. We know that $\sqrt{I^n}=\sqrt{I}$. so by part (8) of Theorem 3.11:

$$
\begin{aligned}
&S-radI^nM=S-rad\sqrt{I^n}M=\\
&S-rad\sqrt{I}M=S-radIM.
\end{aligned}
\tag{18}
$$

Proposition 3.13. Let Q be a $P-$primary submodule of an $R-$module A. Then $S-radQ=S-rad(Q+PA)$.

Proof. We have $Q\subseteq Q+PA$, so $S-radQ\subseteq S-rad(Q+PA)$.Let $S-radQ=\bigcap_{i\in I}P_i$, where any P_i is a semiprime submodule of A containing Q. We see that

$$P=\sqrt{(Q:A)}\subseteq\sqrt{(P_i:A)}=(P_i:A) \tag{19}$$

implies $PA\subseteq P_i$. So $(Q+PA)\subseteq P_i$, for every $i\in I$ and hence $S-rad(Q+PA)\subseteq P_i$. Therefore $S-rad(Q+PA)\subseteq\bigcap P_i$ and so $S-radQ=S-rad(Q+PA)$.

Definition 3.14. Let N be a semiprime submodule of an $R-$module M, and let $P=\sqrt{(N:M)}=(N:M)$. We call N a P semiprime submodule of M, if P is prime ideal of R.

Lemma 3.15. Let M be a finitely generated $R-$module and let K be a maximal ideal of R. If Q is a $K-$primary submodule of M, then $S-radQ$ is a $K-$semiprime sub module.
Proof. By Theorem 3.11, part (5), we have $K=\sqrt{(Q:M)}\subseteq(S-radQ:M)$.
But K is a maximal ideal of R, so $(S-radQ:M)=R$ or $(S-radQ:M)=K$. If $(S-radQ:M)=R$ then $S-radQ=M$ and by Theorem 3.11, part (6) we have $Q=M$ which is a contradiction since Q is primary. Hence $(S-radQ:M)=K$ and since $S-radQ$ is an intersection of semiprime sub modules containing Q it is semiprime and in fact $K-$semiprime.

Proposition 3.16. Let $N_1,N_2,....,N_t$, be $P-$semiprime sub modules of an $R-$module M. Then $N=N_1\bigcap N_2\bigcap\dots\bigcap N_t$ is also $P-$semiprime.
Proof. By Proposition 2.10, N is semiprime and we have:

$$
\begin{aligned}
&(N:M)=(N_1\cap N_2\cap\dots\cap N_t:M)=\\
&(N_1:M)\cap(N_2:M)\cap\dots\cap(N_t:M)
\end{aligned}
\tag{20}
$$

$=P\bigcap P\bigcap\dots\bigcap P=P.$ Therefore N is $P-$semiprime.

Lemma 3.17. Let M be a multiplication $R-$module and L, N be sub modules of M. Also let K be a prime ideal of R and P be a $K-$semiprime submodule of M such that $N\bigcap L\subseteq P$. If $(N:M)\not\subset K$ then $L\subseteq P$.
Proof. We have $N\bigcup L\subseteq P\Rightarrow(N\bigcap L:M)\subseteq(P:M)=K\Rightarrow$
$(N:M)\bigcap(L:M)\subseteq K.$

and since K is a prime ideal of R, $(N:M)\subseteq K$ or $(L:M)\subseteq K$. Since $(N:M)\not\subset K$, we find that $(L:M)\subseteq K$. From this we conclude that $(L:M)M\subseteq KM$, that is, $L\subseteq KM$. But $(P:M)=K$ implies that $KM\subseteq P$. Therefore $L\subseteq KM\subseteq P$.

4. Conclusion

In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

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