International Journal of Industrial Engineering & Production Research December 2023 Vol. 34, No. 4: 1-21
DOI: 10.22068/ijiepr.34.4.11

RESEARCH PAPER

Markowitz Revisited: Addressing Ambiguity as an Important Parameter in Portfolio Optimization

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Received 20 April 2023; Revised 2 July 2023; Accepted 26 August 2023;

ABSTRACT

Since 1952, when the mean-variance model of Markowitz was introduced as a basic framework for modern portfolio theory, some researchers have been trying to add new dimensions to this model. However, most of them have neglected the nature of decision-making in such situations and have focused only on adding non-fundamental and thematic dimensions such as considering social responsibilities and green industries. Due to the nature of the stock market, the decisions made in this sector are influenced by two different parameters: (1) analyzing past trends and (2) predicting future developments. The former is derived objectively based on historical data that is available to everyone while the latter is achieved subjectively based on inside information that is only available to the investor. Naturally, due to differences in the origin of their creation, the bridge between these two types of analysis to optimize the portfolio will be a phenomenon called "ambiguity". Hence, in this paper, we revisited Markowitz's model and proposed a modification that allows the incorporation of not only return and risk but also incorporate ambiguity into the investment decision-making process. Finally, to demonstrate how the proposed model can be applied in practice, it is implemented in the Tehran Stock Exchange (TSE), and the experimental results are examined. From the experimental results, we can extract that the proposed model is more comprehensive than Markowitz's model and has a greater ability to cover the conditions of the stock market.

KEYWORDS: *Portfolio optimization; Markowitz's model; Ambiguity; Inside-information; Behavioral finance.*

1. Introduction

Portfolio optimization is one of the most important areas of financial management. Decision-making in this field, like other areas, is associated with uncertainty, so dealing with uncertainty has been a significant part of the relevant literature (for reviews of this literature see [1, 2]). In this regard, given the transparency and nature of the stock market, where historical data is easily available to everyone, some researchers argue that the uncertainty in this market is merely "stochastic" and that using other approaches in this field is not logical [3-6]. However, due to the influence of various factors such as political and social issues on the stock market; it is possible that the future behavior of a security will be different from its past behavioral patterns. Therefore, to properly interact with this observed behavior, benefiting from the investor's opinion has been considered as an appropriate approach. Indeed, due to the nature of the stock market, the decisions made in this sector are influenced by two different parameters: (1) analyzing past trends and (2) predicting future developments. The former is derived objectively based on historical data that is available to everyone while the latter is achieved subjectively based on inside information that is only available to the investor. Naturally, due to differences in the origin of their creation, the bridge between these two types of analysis to optimize the portfolio will

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be a phenomenon called "ambiguity". As we know, based on the definition provided by Knight [7] and Keynes [8], ambiguity describes a situation where the probabilities associated with the future states of nature are unknown. In this circumstance, it is necessary to mention that the use of an investor's opinion does not mean deleting or ignoring historical data in portfolio optimization process. The purpose of this work is to exploit the investor's opinion to identify the role of other factors affecting the stock market that historical data are not able to reflect or analyze.

However, despite what has been stated, the prevailing view of the relevant literature is to pay attention only to one of these two mentioned issues (historical data or investor's opinion) while the appropriate approach is to consider both of them simultaneously. In this regard, Black and Litterman [9] developed a new approach that combines the expected equilibrium between returns estimated through the Capital Asset Pricing Model (CAPM) and views to optimize the portfolio. The views in their manner represent the investor's opinion about the securities' future returns. This model yields more stable and more diversified portfolios than the standard mean-variance model of Markowitz [10]. However, Black and Litterman's original paper [9] only explains the main aspects of their idea and leaves it to others to better explain the implications of their model. An example of this claim is the works presented by [10-15], in which they explain the Black-Litterman approach in further detail. Nevertheless, neither the Black-Litterman model itself nor any of the studies that have used it paid attention to the issue of ambiguity and consequently, ambiguity aversion in portfolio optimization, which is a prominent human characteristic in the decision-making process.

By reviewing the relevant literature, we find that extensive research has been done on the use of historical data in portfolio optimization and most researchers agree on their random nature [16-19]. Indeed, it can be accepted to some extent that the use of stochastic approaches is a suitable tool for analyzing security performance in the future based on historical data but is not enough for the reasons set out above. Therefore, it is necessary to use the investor's opinion (in its most appropriate form) as a complementary tool to optimize the portfolio. But when we refer to the relevant literature in this field, we find that despite extensive studies on the use of investor opinion in portfolio optimization [20-29], few studies have paid attention to its behavioral nature. When an investor predicts the future of security, this prediction is usually made by the specific assumptions that can be changed in a short time or have not been considered in the stock market due to their emergence. Therefore, in proportion to the quantity and quality of these assumptions and also the ability of investors to predict the future behavior of security, there may be a significant difference in the investor's opinion about future return and the expected return extracted based on historical data. This difference and also the investor's belief in the correctness of his or her analysis can be considered as a suitable criterion for calculating the ambiguity in predicting the future behavior of security. Naturally, in terms of the first part, more difference indicates more ambiguity, and conversely, less difference indicates less ambiguity about the future of security. Meanwhile, in terms of the second part, more or less belief indicates less ambiguity, and, middle belief indicates more ambiguity about the future of security. Therefore, given the undeniable importance of ambiguity in human decision-making processes, this study revisited Markowitz's model and proposed a new model that also incorporates ambiguity into the investment decision-making process.

The remainder of this paper is organized as follows. In Section 2, a brief description of Markowitz's model is presented. In Section 3, the literature on measuring ambiguity in portfolio optimization is surveyed, and then based on the original definition of ambiguity provided by Ellsberg [30] and a new theoretical measure of ambiguity proposed by Blavatsky [31] our new empirical measure of ambiguity is provided for portfolio optimization. In Section 4, our new model is proposed and ambiguity aversion is incorporated in portfolio optimization. Through Section 5 and to demonstrate how the proposed model can be applied in practice, it was implemented in TSE. In Section 6, three hypothetical investment conditions are analyzed, and in addition to comparing the performance of our proposed model with Markowitz's model, the challenges ahead have been addressed. And eventually, in Section 7 the conclusions of this study are summarized and some possible future perspectives are outlined.

2. Description of the Markowitz's Model

Let us now review the standard mean-variance model of Markowitz [32] to be clear on what exactly is to be extended. In a traditional mean-variance model of Markowitz, investors maximize the expected return of the portfolio, $\mu_{PF} = \sum_{i=1}^{n} \mu_i w_i$, and minimize the portfolio risk, $\sigma_{PF} =$

$\sum_{i=1}^{n}\sum_{j=1}^{n} w_i\sigma_{ij}w_j$, while considering the constraint that the sum of all portfolio weights is equal to one ($\sum_{i=1}^{n} w_i = 1$). In this situation the mathematical formulation will be as follows:

$$Max\ \mu_{PF} = \sum_{i=1}^{n} \mu_i w_i,$$
$$Min\ \sigma_{PF} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\sigma_{ij}w_j, \qquad (1)$$
Subject to:
$$\sum_{i=1}^{n} w_i = 1$$
$$x \in F$$

where w_i, denote the portfolio weight of security i, μ_i, denoting the expected return of security i, σ_{ij}, denoting the covariance of the returns of securities i and j, and F, denoting the set of feasible solutions. The aggregation of both objectives can be done either by determining the minimum variance portfolio subject to an expected return μ^* or by maximizing the expected portfolio return subject to an affordable level of risk σ^*.

Since the introduction of the Markowitz model, many efforts have been made to incorporate other aspects of investment and bring it as close to the real-world problem as possible. Empirical evidence demonstrates that to select the best financial portfolio it is required to aggregate more than two dimensions (return and risk). As reviewed by Steuer and Na [33], there has been a long line of papers proposing methods for solving portfolio selection problems with additional criteria. But regardless of their impact and effectiveness; very few of them have mentioned the effect of ambiguity and consequently ambiguity aversion in portfolio optimization. Theoretical models of portfolio optimization that discuss ambiguity predict that investors' willingness to invest in securities is reduced when ambiguity in the stock market increases [34-38]. In this context, it is interesting to note that most of the methods used to model ambiguity have been developed based on robust optimization [39]. In this framework, Pflug and Wozabal [40] consider a maximin criterion which is based on a confidence set for probability distribution. They analyze the trade-off between return, risk, and robustness concerning ambiguity. Wozabal also does the same for the case of nonparametric ambiguity sets in [41]. Assuming that ambiguity is about the expected return vector and covariance matrix, Tütüncü and Koenig [42] treat the robust optimization problem as a saddle-point problem with some semi-defined constraints. Also, Maenhout [35] has extended the analysis of return-to-average risk premium and has obtained a solution for the optimal portfolio in the presence of ambiguity; (for reviews of this literature see [43]). However, despite the usefulness of robust optimization in ambiguity modeling; the key point in this matter that has been neglected is the role of the investor in predicting future developments. In the real world, sometimes an investor makes an investment decision based on some of the additional information he or she has at the time of the decision, which is called "insider trading" or "inside information" (for more information see [44-48]). However, this additional information is generally distorted by noise from the beginning, so the investor cannot provide a completely reliable prediction about the future based on it. Therefore, in this circumstance, ambiguity will be a phenomenon that the investor faces.

Hereupon, this study seeks to answer the question of how an investor makes a decision when he or she is averse to ambiguity and capable of getting "inside information". In this regard, the first step is to distinguish between risky and ambiguous conditions that play an important role in the investor prediction process and the second step is to propose an empirical measure of ambiguity that is compatible with the perceptual mechanism of the investor.

3. Literature on Measuring Ambiguity in Portfolio Optimization

The concept of ambiguity was first proposed by Knight [7] and Keynes [8], and describes a situation where the probabilities associated with the future states of nature are unknown. Ellsberg [30] was the first one to speculate that people generally dislike ambiguity and try to avoid it as much as possible. Over time, this hypothesis has been confirmed by plenty of works in the field of experimental economics and psychology (for reviews of the evidence on ambiguity aversion see [49, 50]).

Due to the effect of ambiguity in the human decision-making process, some theoretical literature has addressed this concept in portfolio optimization (for reviews of this literature see [43, 51, 52]). However, despite the existence of such research, there is a lack of empirical work on this matter; this can be attributed to the lack of fair and accessible measures of ambiguity on individual securities. In recent years, few studies such as Anderson, et al. [53] have made an effort in this area, but they have generally portrayed the issue of ambiguity and consequently ambiguity aversion in the form of group decision-making and through the measuring of disagreement of investors about predicting the total portfolio profit. In this regard, you can refer to other works such as Diether, et al. [54], Qu, et al. [55], Johnson [56], Anderson, et al. [57], Ulrich [58], and Kostopoulos, et al. [59]. In particular, Qu, et al. [55], Anderson, et al. [57], and Kostopoulos, et al. [59] observe that the dispersion factors are positively related to expected returns and have explanatory power beyond traditional Fama-French and momentum factors. Conversely, Diether, et al. [54], Johnson [56], and Ulrich [58] find that higher dispersion securities have lower future returns.

But we believe that this matter before being defined as a consequence of group decision making is an individual issue and arises from the perceptual mechanism of each investor. Also, ambiguity is a phenomenon that occurs at the moment of selecting each security to participate in the portfolio and is not limited to predicting the total portfolio profit. Therefore, similar to risk, it will be necessary to define an appropriate measure of ambiguity for each security and use it to enrich the portfolio optimization process. In this case, and after much research, we found that Brenner and Izhakian [60] have a similar view and believe that a separate measure should be introduced to calculate the ambiguity of each security. They believe that investors' level of aversion to or love for ambiguity is contingent on the expected probability of favorable returns. Based on this idea, they proposed an empirical measure of ambiguity, which is independent of risk. Namely, they believed that the degree of ambiguity, denoted by $\mho^2$, be measured by the expected volatility of probabilities, across the relevant outcomes as follows:

$$\mho^2[r] = \int E[\varphi(r)] Var[\varphi(r)]\, dr, \qquad (2)$$

where $\varphi(.)$ denotes a probability density function, $E[\varphi(r)]$ denotes the expected probability of a given rate of return r, and $Var[\varphi(r)]$ denotes the variance of the probability of r. The intuition of $\mho^2$ is that, as the degree of risk can be measured by the volatility of returns, so the degree of ambiguity too can be measured by the volatility of the probabilities of returns.

To illustrate the intuition behind $\mho^2$, consider the following binomial example of security with two possible future returns: $d = -10\%$ and $u = 20\%$. Assume that the probabilities, $P(\cdot)$, of d and u are known, say $P(d) = P(u) = 0.5$. The expected return is thus 5%, and the standard deviation of the return (measuring the degree of risk) is 15%. In this case, since the probabilities are known, ambiguity is not present ($\mho^2 = 0$), and investors face only risk. Assume next that the probabilities of d and u can be either $P(d) = 0.4$ and $P(u) = 0.6$ or $P(d) = 0.6$ and $P(u) = 0.4$, where these two alternative distributions are equally likely. Investors now face not only risk but also ambiguity. The degree of ambiguity, in terms of probabilities, is $\mho = \sqrt{\sum_i E[P(i)] Var[P(i)]} =$

$$\sqrt{2 \times 0.5 \times (0.5 \times (0.4 - 0.5)^2 + 0.5 \times (0.6 - 0.5)^2)} = 0.1$$

Notice that the degree of risk, computed using the expected probabilities $E[P(d)] = E[P(u)] = 0.5$, has not changed.

In their view, a measure of ambiguity, $\mho^2$, depends only on the probabilities of outcomes, regardless of their magnitude. Therefore, they do not differentiate between different levels of utility in calculating ambiguity. However, in our view, an appropriate measure of ambiguity should have a more comprehensive definition and also take into account the differences in the level of desirability. So to achieve this goal, we first review the original definition of ambiguity provided by Ellsberg. Based on his definition [30], ambiguity is a subjective variable, but it should be possible to identify "objectively" some situations likely to present high ambiguity, by considering situations where the information contained is scarce, unreliable, or highly conflicting. Thus, like Antoniou, et al. [61] we believe that Ellsberg sees ambiguity as a negative component of what can be called the "richness" of available information to compute the expected returns. Therefore based on his attitude there are two main ways to empirically measure ambiguity: (1) quantifying the richness of the information directly, or (2) inferring this richness indirectly. Thus, in this study and based on the latter approach we exploit a theoretical measure of ambiguity proposed by Blavatsky [31] in which, in addition to the probabilities, it also contributed to the different levels of desirability as an effective parameter in calculating ambiguity. Therefore, by adapting this theoretical measure to stock market conditions, we provide an empirical measure of ambiguity for portfolio optimization.

Blavatsky [31] proposed a new theoretical approach to measure ambiguity that is analogous to axiomatic risk measurement in finance and can be viewed as a generalization of the well-known Gini [62] mean difference statistic from measuring risk to measuring ambiguity (when all events are ambiguous and not measurable in objective probabilities). It should be noted that his measure of ambiguity is derived from the following three assumptions: (1) decomposability; (2) double cancellation; and (3) elementary increase in uncertainty. Decomposability refers to the concept that ambiguity of any choice alternative (act) can be decomposed into a left-tail ambiguity (ambiguity in the realization of relatively undesirable outcomes) and a right-tail ambiguity (ambiguity in the realization of relatively desirable outcomes); also, double cancellation refers to the concept that in any choice alternative (act) ambiguity sources are independent from outcomes; and finally elementary increase in uncertainty refers to the concept that increasing a more desirable outcome in a binary choice alternative (act) necessarily increases ambiguity. The main focus of this approach is to distinguish between risky and ambiguous conditions. In this way, if events are measurable with objective probabilities, then the "uncertainty" function of the union of all such events is zero thus betting on such events is risky but not ambiguous. In opposite, if events are measurable with subjective probabilities, then the "uncertainty" function of the union of all such events is positive thus betting on such events is ambiguous and not risky. Therefore, in continuing, Blavatskyy's approach is briefly introduced and then, referring to its characteristics, we adapt this theoretical measure of ambiguity to stock market conditions and provide our empirical measure of ambiguity for portfolio optimization.

3.1. Description of the blavatskyy's measure of ambiguity

To propose a new theoretical measure of ambiguity, Blavatskyy in [31] has considered two different elements in this matter; first, the desirability differences between various possible outcomes, and second, the type of probabilities conceivable for the occurrence of events. So to understand the concept, suppose that there is a non-empty set S (that can be finite or infinite). The elements of this set are called states of nature. Only one state of nature is true but a decision maker does not know ex ante which one. There is also a sigma-algebra Σ as the subsets of set S that are called events. Blavatsky considers $\{E_1, \dots, E_n\}$ as a partition of the state space S and divides it into $n \in \mathbb{N}$ events, where for all $i, j \in \{1, \dots, n\}$ and $i \neq j$ we have $E_1 \cup \dots \cup E_n = S$ and $E_i \cap E_j = \emptyset$. In this regard, the phrase $C \subseteq \Sigma$ denotes a set of events that are measurable with an objective probability. This set is non-empty as it contains the universal event S and the impossible event $\emptyset$ (that are measured with probabilities one and zero, respectively). Naturally, for any event $E \in C$ the complimentary event $S \backslash E$ is also contained in C. Finally, if several separate events are in C then their union is in C as well. Also, there is a continuum set $X \in \mathbb{R}$ their elements are called outcomes. In his work a choice alternative is an act $f: S \to X$ and $\{x_1, E_1; \dots; x_n, E_n\}$ denotes a step act that yields outcome $x_i \in X$ for each $E_i \in \Sigma$, where $i \in \{1, \dots, n\}$. Blavatsky assumes that outcomes are numbered in increasing order of desirability so that x_1 denotes the least desirable outcome and x_n denotes the most desirable outcome. The set of all step acts is denoted by $\mathcal{F}$. For brevity, let $\{x, E, y\} \in \mathcal{F}$ denote a binary step act that yields outcome $x \in X$ in all states that belong to event $E \in \Sigma$, and outcome $y \in X$ in all remaining states that belong to complimentary event $S \backslash E \in \Sigma$.

In this situation, Blavatskyy proposed a theoretical measure of ambiguity as a function $\beta: \mathcal{F} \to \mathbb{R}^+$ that maps acts into non-negative real numbers and considers the following equation as an ambiguity measure:

$$\beta(x_1, E_1; \dots; x_n, E_n) = \sum_{j=2}^{n} \left[v(x_j) - v(x_{j-1})\right] \varphi\left(\bigcup_{i=j}^{n} E_i\right), \quad (3)$$

where $v(.)$ denoting a desirability function which $v: X \to \mathbb{R}$, and $\varphi(.)$, denoting an "uncertainty" function which $\varphi: \Sigma \to \mathbb{R}$. Also, it should be noted that $\varphi(E) = 0$ for all events $E \in C$ and $\varphi(E) \geq 0$ for all events $E \in \Sigma$; furthermore, the desirability function $v(.)$ is monotonic.

In interpreting the outputs of the above measure, it is necessary to mention that if $E_i \in C$ for all $i \in \{1, \dots, n\}$ a step act is risky and is not ambiguous (referring to the type of probabilities conceivable for the occurrence of events); so according to the Eq. (3) ambiguity measure is zero. By the same logic, for all constant acts $x \in \mathcal{F}$ we have $\beta(x) = 0$. Also, if a step act yields outcomes, whose desirability is close to each other, then according to Eq. (3) ambiguity measure is close to zero. On the contrary, if a step act yields outcomes, in which their desirability is widely dispersed, then according to Eq. (3) ambiguity measure is relatively high (referring to the desirability differences between various possible outcomes). As it is clear from the structure of the Blavatskyy measure illustrated in Eq. (3), in the special case when $\varphi(E) = 2\pi(E)[1-\pi(E)]$, where $\pi: \Sigma \rightarrow [0,1]$ is a probability weight, Blavatskyy measure generalizes Gini [62] mean difference statistic from measuring risk to measuring ambiguity.

To better understand the computational mechanism used in the Blavatskyy ambiguity measure, we illustrate the outputs of Eq. (3) for binary acts over monetary outcomes in Figure 1. The horizontal (vertical) axis plots outcomes $x(y)$ that a decision maker receives if event $E(S\backslash E)$ is true. The Blavatskyy measure of ambiguity is always zero for all points along the 45° line which is called the "certainty line" that represents all constant and risky acts. Otherwise, if events E and $S\backslash E$ are not measurable with objective probabilities and (or) their desirability distance is relatively high, then the Blavatskyy measure of ambiguity is far from the certainty line. Naturally, the greater distance implies the greater ambiguity.

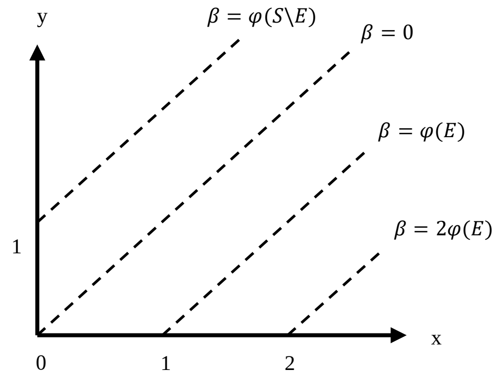

Fig. 1. Blavatskyy measure of ambiguity for binary acts over monetary outcomes (linear desirability function)

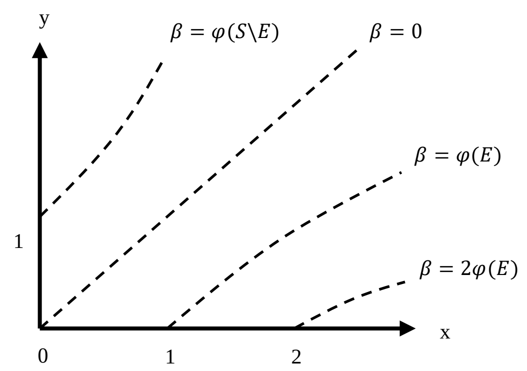

Fig. 2. Blavatskyy measure of ambiguity for binary acts over monetary outcomes (concave desirability function)

Additionally, there is another point that should be considered in describing the Blavatskyy measure and that is the type of desirability function that is used. In this regard, Figure 1 illustrates the Blavatskyy measure for a linear desirability function $v: X \rightarrow \mathbb{R}$ while Figure 2 does the same for a concave desirability function $v(.)$. For a linear desirability function iso-ambiguity graphs are parallel straight lines with a 45° slope. In other words, for a decision maker with a linear desirability function, a binary act $\{x, E, 0\}$ is as ambiguous as a binary act $\{x+y, E, y\}$ for any $y > 0$. Figure 2 illustrates the Blavatskyy measure for a concave desirability function $v(.)$. In this situation, iso-ambiguity graphs will be curved in the upper regions. In other words, for a decision maker with a concave desirability function, a binary act $\{x, E, 0\}$ is more ambiguous than an act $\{x+y, E, y\}$ for any $y > 0$. In a specific case, when outcome y is sufficiently large, a binary act $\{x+y, E, y\}$ can be viewed as a nearly constant act. At the same time, a binary act $\{x, E, 0\}$ cannot be viewed as a nearly constant act unless outcome x is quite small.

This is our main reason for including desirability differences between various possible outcomes in calculating the amount of ambiguity on each security. A phenomenon that despite its adaptation to the nature of human behavior; the proposed measure by Brenner and Izhakian [60] is unable to depict it.

3.2. Provide an empirical measure of ambiguity for portfolio optimization

As previously mentioned, due to the nature of the stock market, the decisions made in this sector are influenced by two different parameters: (1) analyzing past trends and (2) predicting future developments. The former is derived based on historical data that is available to everyone and the latter is achieved based on inside information that is only available to the investor. Both of them are generally expressed through the probability distribution function, with the difference that the former is calculated objectively while the latter is calculated subjectively. Naturally, due to differences in the origin of their creation, the bridge between these two types of analysis to optimize the portfolio will be a phenomenon called "ambiguity". In this circumstance, if these two different parameters fully confirm each other, then we will face a risky situation but not ambiguous, and in the opposite, if they are not in full compliance with each other, then we will face an ambiguous situation.

Regarding inside information, it should be noted that sometimes an investor makes an investment decision based on additional information, but this additional information is generally distorted by noise from the beginning, so the investor cannot provide a completely reliable prediction about the future based on it. Thus, in proportion to the quantity and quality of this additional information and also the ability of investors to predict the future behavior of security, there may be a significant difference in the investor's opinion about future return and the expected return extracted based on historical data. This difference and also the investor's belief in the correctness of his or her analysis can be considered as a suitable criterion for calculating the ambiguity in predicting the future behavior of a security. Naturally, in terms of the first part, more difference indicates more ambiguity, and conversely, less difference indicates less ambiguity about the future of security. Meanwhile, in terms of the second part, more or less belief indicates less ambiguity, and, middle belief indicates more ambiguity about the future of security. Therefore, if we suppose that there is a non-empty and finite set S as states of nature (maintaining the past conditions based on historical data or occurring the new conditions based on inside information), which only one state of nature is true but an investor does not know ex-ante which one. And also if we suppose Σ as the set of all available securities in the market and consider $\{x_1, \dots, x_n\}$ as a target set for investment. In this regard, the phrase $C \subseteq \Sigma$ denotes a set of securities that analyze past trends and predict future developments fully confirm each other for them or the investor can provide a completely reliable prediction about their future based on inside information. Thus, based on Blavatskyy's measure [31] and taking into account a slight but effective change, an empirical measure of ambiguity on each security can be defined as a binary step acts as follows:

$$\beta_i(\mu_{ie}, x_i, \mu_{ip}) = \Big(v(\mu_{ip}) - v(\mu_{ie})\Big)\,\varphi(x_i), \tag{4}$$

where μ_{ie}, represents the investor's opinion about the future return of security i if the new conditions occur based on his or her inside information, μ_{ip}, represents the expected return of security i if the past conditions are maintained based on the historical data, $v(.)$, representing a linear desirability function, and $\varphi(.)$, representing an "uncertainty" function. It should be noted that $\varphi(x_i) = 0$ for $x_i \in C$ and $\varphi(x_i) \geq 0$ for $x_i \in \Sigma$; furthermore, the desirability function $v(.)$ is monotonic.

In interpreting the outputs of the above measure, it should be noted that, unlike the Blavatskyy measure, we do not assume that potential outcomes are numbered in an increasing order of desirability. Therefore, our empirical measure of ambiguity as a function $\beta: \mathcal{F} \to \mathbb{R}$ maps acts into real numbers. In this situation, $\beta_i > 0$ means that the inside information indicates a decrease in the desirability of security i. Thus, due to the nature of loss aversion which is hidden in an investor's decisions, he or she will avoid placing security i in his or her optimal portfolio. Conversely, $\beta_i < 0$ means that the inside information indicates an increase in the desirability of security i. Thus, by the same logic, he or she will place security i in his or her optimal portfolio.

In this circumstance, for a risk-free asset as a constant act $x \in \mathcal{F}$, we have $\beta(x) = 0$. Also, if for a security i the investor predicts a future return μ_{ie}, which its desirability is close to the expected return μ_{ip}, and (or) the investor's belief in the correctness of his or her analysis is very high or very low then according to the Eq. (4) ambiguity measure is close to zero. On the contrary, if for a security i the investor predicts a future return μ_{ie}, whose desirability is far from the expected return

μ_{ip}, and (or) the investor's belief in the correctness of his or her analysis is moderate then according to the Eq. (4) ambiguity measure is relatively high. This approach of measuring ambiguity, which has been proposed in this study, despite its simplicity, corresponds closely to the original definitions provided by Ellsberg [30]. According to his definition, when investors arrive at conflicting views (when the difference between prediction based on historical data and inside information is large), the actual distribution can be described as more ambiguous. In addition, this approach is consistent with the theoretical literature in portfolio optimization that considers ambiguity as uncertainty in generating returns [63]. Since each investor relies more on his or her inside information to make a prediction, a large difference in these two different parameters (analyzing past trends and predicting future developments), signals a situation where different models are possible and therefore, ambiguity is increased.

4. Description of the Proposed Model

In this section and based on what was discussed in the previous section, we will develop the standard mean-variance model of Markowitz. Therefore, in the proposed model, Eq. (1) is expanded to allow investors to incorporate the ambiguity of securities into their decision-making process. In this regard, the ambiguity score of the portfolio, β_{PF} is given by Eq. (5):

$$\beta_{PF} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \beta_{ij} \gamma_{ij} w_j, \tag{5}$$

where w_i , denoting the portfolio weight of security i, β_{ij}, denoting the ambiguity rating of security i which $\beta_{ii} = \beta_i$ and $\beta_{ij} = 0$, and γ_{ij}, denoting the score value of correlation between the ambiguity rating of securities i and j. To get closer to the real world, it is assumed that the ambiguity ratings are correlated with each other and these relationships are also determined by the investor based on his or her inside information.

In this regard, due to the powerful representation capability of the Intuitionistic Fuzzy Sets (IFSs) proposed by Atanassov [64], we construct $\tilde{\zeta}_{ij}$ as a pairwise comparison matrix to determine the correlation between the ambiguity rating of securities i and j based on the investor's opinion. Afterward, to obtain γ_{ij}, we use a novel score function proposed by Zeng, et al. [65] to compare Intuitionistic Fuzzy Values (IFVs) that are produced by the investor. Thus, in the following, we briefly review the definitions of IFSs, IFVs, and novel score functions for ranking IFVs.

Definition 4.1. An IFS $\tilde{A}$ in the universe of discourse $X = \{u_1, \dots, u_n\}$ can be represented as follows [64]:

$$\tilde{A} = \{\langle u_i, \mu_{\tilde{A}}(u_i), \nu_{\tilde{A}}(u_i) | u_i \in X \rangle\}, \tag{6}$$

where $\mu_{\tilde{A}}$ and $\nu_{\tilde{A}}$ are the membership function and the non-membership function of the IFS $\tilde{A}$, respectively, $\mu_{\tilde{A}}: X \to [0,1]$ and $\nu_{\tilde{A}}: X \to [0,1]$. $\mu_{\tilde{A}}(u_i)$ and $\nu_{\tilde{A}}(u_i)$ are the membership degree and the non-membership degree of element u_i belonging to the IFS $\tilde{A}$, respectively, $0 \le \mu_{\tilde{A}}(u_i) + \nu_{\tilde{A}}(u_i) \le 1$ and $1 \le i \le n$. The hesitation degree of element u_i belonging to the IFS $\tilde{A}$ is denoted by $\pi_{\tilde{A}}(u_i) = 1 - \mu_{\tilde{A}}(u_i) - \nu_{\tilde{A}}(u_i)$, where $\pi_{\tilde{A}}(u_i) \in [0,1]$ and $1 \le i \le n$.

Definition 4.2. Let $d = \langle a, b \rangle$ be an IFV, where $a \in [0,1]$, $b \in [0,1]$ and $0 \le a + b \le 1$. Let π_d be the hesitant degree of the $d = \langle a, b \rangle$, where $\pi_d \in [0,1]$. The score value $\gamma(d)$ of the IFV d is defined as follows [65]:

$$\gamma(d) = a - b - \left(\pi_d \times \frac{\log_2(1+\pi_d)}{100}\right) \tag{7}$$

where $\gamma(d) \in [0,1]$. The larger the value of $\gamma(d)$, the larger the IFV d.

In Eq. (7), the score value $\gamma(d)$ of the IFV $d = \langle a, b \rangle$ consists of two parts, i.e., $(a - b)$ and $\left(\pi_d \times \frac{\log_2(1+\pi_d)}{100}\right)$. The former is for general evaluation, in which the larger the value of $(a - b)$, the larger the score value $\gamma(d)$; while the latter is used to distinguish between two IFVs that have the same value of $(a - b)$, the smaller the value of $\left(\pi_d \times \frac{\log_2(1+\pi_d)}{100}\right)$, the larger the score value $\gamma(d)$ of the IFV d.

Thus, the correlation between the ambiguity rating of securities based on IFVs can be concisely expressed in a pairwise comparison matrix. Suppose that there is a set of securities $X = \{x_1, \dots, x_n\}$, so an investor has to state the correlation between the ambiguity rating of securities i and j with IFVs.

Definition 4.3. If an intuitionistic fuzzy pairwise comparison matrix $\tilde{\zeta}$ on the set X is defined as $\tilde{\zeta} = (\tilde{\zeta}_{ij})_{n \times n} \subset X \times X$, then:

$$\tilde{\zeta} = \begin{bmatrix} \tilde{\zeta}_{11} & \cdots & \tilde{\zeta}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{\zeta}_{n1} & \cdots & \tilde{\zeta}_{nn} \end{bmatrix} \tag{8}$$

Where $\tilde{\zeta}_{ij} = \langle a_{ij}, b_{ij} \rangle$, is an IFVs. a_{ij} indicates the extent to which the investor aligns the role of his or her inside information in predicting the future of securities i and j. Also, b_{ij} indicates the extent to which the investor does not align the role of his or her inside information in predicting the future of securities i and j. In addition, $a_{ij} \subset [0,1]$, $b_{ij} \subset [0,1]$, $0 \leq a_{ij} + b_{ij} \leq 1$.
With this structure, the mathematical formulation of the proposed model will be as follows:

$$\begin{aligned}
&Max\ \mu_{PF} = \sum_{i=1}^{n} (\omega_{ie}\mu_{ie} + \omega_{ip}\mu_{ip}) w_i, \\
&Min\ \sigma_{PF} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \sigma_{ij} w_j, \\
&Min\ \beta_{PF} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \beta_{ij}\gamma_{ij} w_j, \\
&\text{Subject to:} \\
&\sum_{i=1}^{n} w_i = 1 \\
&x \in F
\end{aligned} \tag{9}$$

where ω_{ie} and ω_{ip} indicate the weight (the degree of participation) of the investor's opinion and historical data in the estimation of expected return for security i in the first objective function, respectively. Therefore, if the investor is quite confident of achieving μ_{ie}, then ω_{ip} will be zero.
Regarding the description of the above objectives, it should be noted that from the past to the present, the expected return of the portfolio, μ_{PF}, is the most practical objective which is usually used in portfolio optimization. In this study, for considering analyzing past trends and predicting future developments simultaneously, with a slight change, the expected return of security i is characterized by a weighted combination of expected return based on the investor's opinion and historical data. Regarding the second objective, which is the minimization of portfolio risk, σ_{PF}, assuming that the variance obtained from historical data and the investor's opinion are equal ($\sigma_p = \sigma_e$), we have done this matter by the standard mean-variance model of Markowitz. In addition, regarding the third objective, which is the minimization of the ambiguity score of the portfolio, β_{PF}, it is necessary to mention that due to the existence of inside information in the real market situation, ambiguity and consequently ambiguity aversion is a main concern of investors which in this study we tried to properly deal with this phenomenon and enrich the portfolio optimization process.
The portfolio optimization as given by Eq. (9), can easily be solved by applying the Weighted Goal Programming (WGP) approach as given by Eq. (10):

$$\begin{aligned}
&Min\ D = \frac{W_1(\alpha_1^- d_1^-)}{\theta_1} + \frac{W_2(\alpha_2^+ d_2^+)}{\theta_2} + \frac{W_3(\alpha_3^+ d_3^+)}{\theta_3}, \\
&\text{Subject to:} \\
&\sum_{i=1}^{n} (\omega_{ie}\mu_{ie} + \omega_{ip}\mu_{ip}) w_i + d_1^- - d_1^+ = b_1 \\
&\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \sigma_{ij} w_j + d_2^- - d_2^+ = b_2 \\
&\sum_{i=1}^{n}\sum_{j=1}^{n} w_i \beta_{ij}\gamma_{ij} w_j + d_3^- - d_3^+ = b_3 \\
&\sum_{i=1}^{n} w_i = 1 \\
&x \in F
\end{aligned} \tag{10}$$

where d_k^+ and d_k^- denote the kth positive and negative deviational variables, respectively, α_k^+ and α_k^- denoting the weighting factors for positive and negative deviational variables k, respectively, W_k denoting the weighting factors for k th objective function, θ_k denoting the normalization constant for deviational variable k, b_k denoting the kth target value, and F, denoting the set of feasible solutions.
In general, the investor's preferences can be incorporated into the WGP models by assigning weights to the unwanted deviational variables, and objectives and also setting the desired target values for each objective. In WGP, a normalization technique is used to measure the deviations from each objective with the same unit of measurement. Thus, in this study according to the data conditions, we used Euclidean normalization which is particularly useful when the target values or objective functions are very small or close to zero. The normalization constant, θ_k, for the following objective function k:

$$\sum_{j=1}^{n} a_{kj} x_j + d_k^- - d_k^+ = b_k, \tag{11}$$

is calculated as:

$$\theta_k = \sqrt{a_{k1}^2 + a_{k2}^2 + \cdots + a_{kn}^2}, \qquad (12)$$

Now it should be noted that if a specific set of preference parameters is given, it is possible to calculate the single optimal portfolio. However, there are cases, where the preference parameters are unknown for any reason. According to the recent literature, the former is referred to as applying the model in a priori fashion while the latter is referred to as applying the model in a posteriori fashion [66]. Therefore, according to the situation or the goals that we pursue from our analysis, we will choose one of the above approaches to solve the problem.

Finally, at the end of this section and based on what was discussed in this study, we can illustrate the framework of our proposed model in Figure 3 and recognize different types of priority as summarized in Table 1. Security A is strictly preferred to security B if one of the following rules is true.

Tab. 1. The priority rules in portfolio optimization

	μ/σ priority	β/σ priority	$\mu/\sigma/\beta$ priority
$A \succ B$	$\mu_A > \mu_B \cup \sigma_A = \sigma_B$	$\beta_A < \beta_B \cup \sigma_A = \sigma_B$	$\mu_A > \mu_B \cup \sigma_A = \sigma_B \cup \beta_A = \beta_B$
	$\mu_A = \mu_B \cup \sigma_A < \sigma_B$	$\beta_A = \beta_B \cup \sigma_A < \sigma_B$	$\mu_A > \mu_B \cup \sigma_A < \sigma_B \cup \beta_A = \beta_B$
	$\mu_A > \mu_B \cup \sigma_A < \sigma_B$	$\beta_A < \beta_B \cup \sigma_A < \sigma_B$	$\mu_A > \mu_B \cup \sigma_A = \sigma_B \cup \beta_A < \beta_B$
			$\mu_A > \mu_B \cup \sigma_A < \sigma_B \cup \beta_A < \beta_B$
			$\mu_A = \mu_B \cup \sigma_A < \sigma_B \cup \beta_A = \beta_B$
			$\mu_A = \mu_B \cup \sigma_A = \sigma_B \cup \beta_A < \beta_B$
			$\mu_A = \mu_B \cup \sigma_A < \sigma_B \cup \beta_A < \beta_B$

5. Empirical Results

A representative data set is required for the empirical analysis in this study. To construct such a data set of both conventional and ambiguous ratings, we use the annual bulletin of TSE published in 2020. It should be noted that in the first step, 10 securities were identified as the top assets of the market in the desired period based on fundamental analysis and the use of historical data and then in the second step based on an experienced investor's opinion and his inside-information we obtained ambiguity ratings (β_i), ambiguity correlation matrix ($\tilde{\zeta}_{ij}$) and ambiguity score matrix (γ_{ij}) for all of these securities. The daily closing prices of all 10 securities are obtained from 1 January 2010 to 31 December 2020 and based on them, the expected return, variance, and covariance matrix are determined. Tables 2-4 provide all of the relevant descriptive statistics about these securities.

Tab. 2. Data set descriptive statistics

Securities	μ_{ip}	σ_i^2	μ_{ie}	$\pi(x_i)$	$\varphi(x_i)$	β_i
x_1	0.0091	0.0295	0.0062	0.7000	0.4200	0.0012
x_2	0.0025	0.0206	0.0038	0.3500	0.4550	-0.0006
x_3	0.0032	0.0212	0.0032	0.0000	0.0000	0.0000
x_4	0.0044	0.0066	0.0035	0.6000	0.4800	0.0004
x_5	0.0010	0.0187	0.0013	0.9000	0.1800	-0.0001
x_6	0.0046	0.0196	0.0035	0.8000	0.3200	0.0004
x_7	0.0039	0.0127	0.0035	0.5000	0.5000	0.0002
x_8	0.0149	0.1318	0.0176	0.7500	0.3750	-0.0010
x_9	0.0026	0.0044	0.0033	0.6500	0.4550	-0.0003
x_{10}	0.0047	0.0063	0.0064	1.0000	0.0000	0.0000

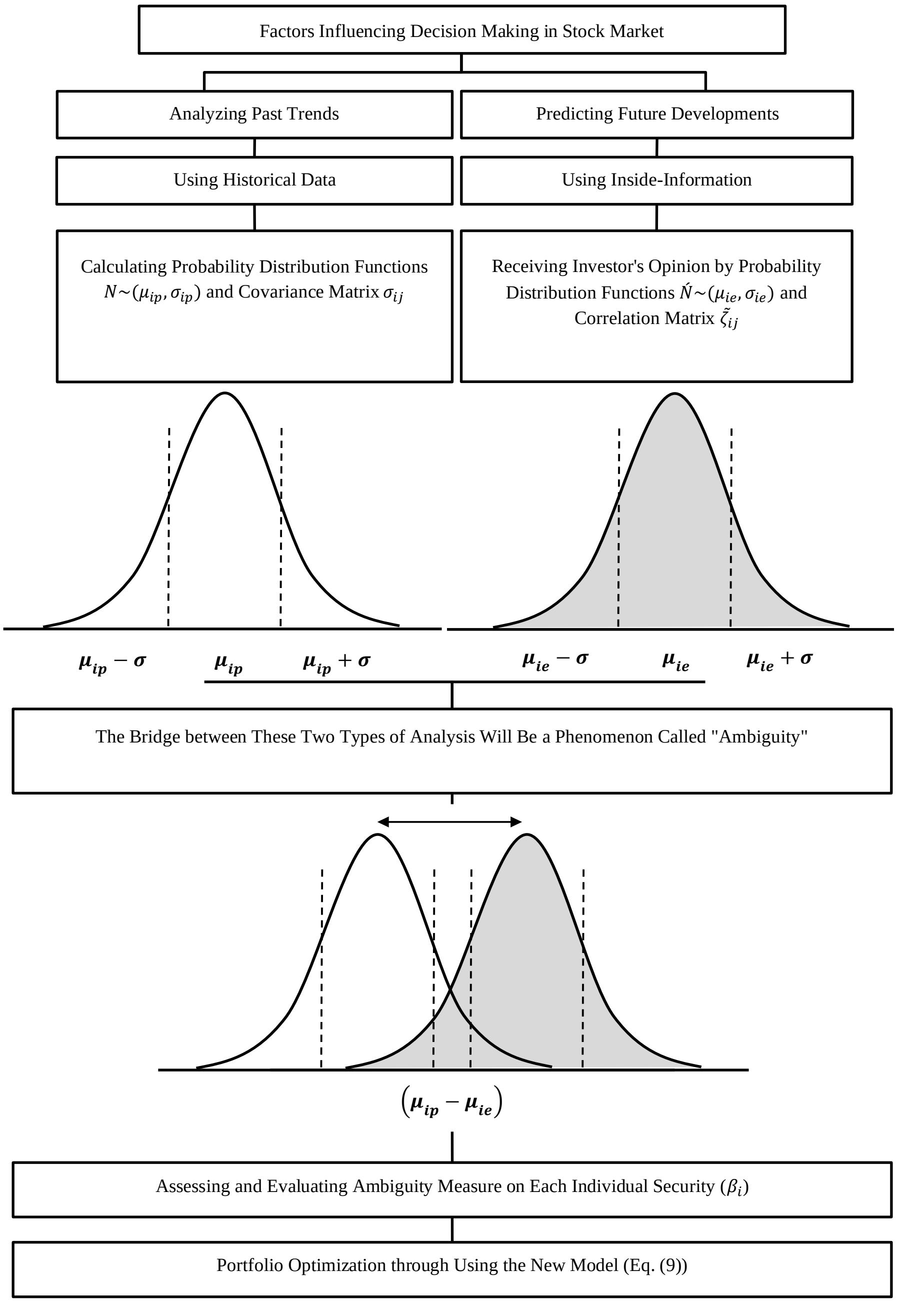

Fig. 3. The origin of ambiguity and how to measure and incorporate it in portfolio optimization

Tab. 3. Correlation between the ambiguity rating of securities ($\tilde{\zeta}_{ij}$)

Securities	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	⟨1.00,0.00⟩	⟨0.20,0.60⟩	⟨0.50,0.50⟩	⟨0.70,0.20⟩	⟨0.15,0.80⟩	⟨0.75,0.20⟩	⟨0.65,0.20⟩	⟨0.20,0.75⟩	⟨0.15,0.70⟩	⟨0.10,0.80⟩
x_2	⟨0.20,0.60⟩	⟨1.00,0.00⟩	⟨0.50,0.50⟩	⟨0.20,0.60⟩	⟨0.70,0.20⟩	⟨0.20,0.70⟩	⟨0.20,0.60⟩	⟨0.70,0.20⟩	⟨0.65,0.20⟩	⟨0.70,0.20⟩
x_3	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨1.00,0.00⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩	⟨0.50,0.50⟩
x_4	⟨0.70,0.20⟩	⟨0.20,0.60⟩	⟨0.50,0.50⟩	⟨1.00,0.00⟩	⟨0.20,0.70⟩	⟨0.70,0.20⟩	⟨0.65,0.25⟩	⟨0.20,0.70⟩	⟨0.15,0.65⟩	⟨0.15,0.75⟩
x_5	⟨0.15,0.80⟩	⟨0.70,0.20⟩	⟨0.50,0.50⟩	⟨0.20,0.70⟩	⟨1.00,0.00⟩	⟨0.15,0.80⟩	⟨0.20,0.70⟩	⟨0.75,0.20⟩	⟨0.75,0.20⟩	⟨0.90,0.05⟩
x_6	⟨0.75,0.20⟩	⟨0.20,0.70⟩	⟨0.50,0.50⟩	⟨0.70,0.20⟩	⟨0.15,0.80⟩	⟨1.00,0.00⟩	⟨0.70,0.20⟩	⟨0.20,0.75⟩	⟨0.20,0.70⟩	⟨0.10,0.85⟩
x_7	⟨0.65,0.20⟩	⟨0.20,0.60⟩	⟨0.50,0.50⟩	⟨0.65,0.20⟩	⟨0.20,0.70⟩	⟨0.70,0.20⟩	⟨1.00,0.00⟩	⟨0.20,0.65⟩	⟨0.20,0.60⟩	⟨0.20,0.70⟩
x_8	⟨0.20,0.75⟩	⟨0.70,0.20⟩	⟨0.50,0.50⟩	⟨0.20,0.70⟩	⟨0.75,0.20⟩	⟨0.20,0.75⟩	⟨0.20,0.65⟩	⟨1.00,0.00⟩	⟨0.70,0.15⟩	⟨0.80,0.10⟩
x_9	⟨0.15,0.70⟩	⟨0.65,0.20⟩	⟨0.50,0.50⟩	⟨0.15,0.65⟩	⟨0.75,0.20⟩	⟨0.20,0.70⟩	⟨0.20,0.60⟩	⟨0.70,0.15⟩	⟨1.00,0.00⟩	⟨0.75,0.15⟩
x_{10}	⟨0.10,0.80⟩	⟨0.70,0.20⟩	⟨0.50,0.50⟩	⟨0.15,0.75⟩	⟨0.90,0.05⟩	⟨0.10,0.85⟩	⟨0.20,0.70⟩	⟨0.80,0.10⟩	⟨0.75,0.15⟩	⟨1.00,0.00⟩

Tab. 4. The score value of correlation between the ambiguity rating of securities (γ_{ij})

Securities	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	1.0000	-0.4005	0.0000	0.4999	-0.6500	0.5500	0.4497	-0.5500	-0.5503	-0.7001
x_2	-0.4005	1.0000	0.0000	-0.4005	0.4999	-0.5001	-0.4005	0.4999	0.4497	0.4999
x_3	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
x_4	0.4999	-0.4005	0.0000	1.0000	-0.5001	0.4999	0.4497	-0.5001	-0.5005	-0.6001
x_5	-0.6500	0.4999	0.0000	-0.5001	1.0000	-0.6500	-0.5001	0.5500	0.5500	0.8500
x_6	0.5500	-0.5001	0.0000	0.4999	-0.6500	1.0000	0.4999	-0.5500	-0.5001	-0.7500
x_7	0.4497	-0.4005	0.0000	0.4497	-0.5001	0.4999	1.0000	-0.4503	-0.4005	-0.5001
x_8	-0.5500	0.4999	0.0000	-0.5001	0.5500	-0.5500	-0.4503	1.0000	0.5497	0.6999
x_9	-0.5503	0.4497	0.0000	-0.5005	0.5500	-0.5001	-0.4005	0.5497	1.0000	0.5999
x_{10}	-0.7001	0.4999	0.0000	-0.6001	0.8500	-0.7500	-0.5001	0.6999	0.5999	1.0000

Through the optimization process of WGP, firstly we solved each of the objective functions with linear and nonlinear programming methods and obtained their target values for the data set and secondly substituted these values in Eq. (10) for optimizing the portfolio by various combinations of objective functions. These values are provided in Table 5.

Tab. 5. The target values for each objective

Portfolio models	Return (b_1)	Risk (b_2)	Ambiguity (b_3)
EWM	0.0051	0.0511	-
MVM	0.0149	0.0348	-
MVAM	0.0162	0.0348	-0.0006

In this point, due to the difference in dimensions of the studied models, we defined specific combinations of W_k to construct portfolio models and obtain objective functions. To this end, we considered the weight of the first objective function as the most important goal, 50%, and divided the other 50% equally among the other objective functions. In this regard, the Equally Weighted Model (EWM) was used as a benchmark test for the portfolio models. In addition, to evaluate the performance of the portfolio models, we have used three relative indexes (σ/μ, β/μ and β/σ). Therefore, the values of objective functions and relative indexes are provided in Table 6.

Tab. 6. The values of objective functions and relative indexes

Combinations of α_k^+, α_k^-	Combinations of W_k	Portfolio models	Values of objective functions: Return	Risk	Ambiguity	Values of relative indexes: σ/μ	β/μ	β/σ
$\alpha_k^+ = 0.5, \alpha_k^- = 0.5$	Does not matter	EWM	0.0051	0.0511	0.0000	10.0196	0.0000	0.0000
	$W_1 = 0.5, W_2 = 0.5$	MVM	0.0100	0.1403	0.0002	14.0300	0.0200	0.0014
	$W_1 = 0.5, W_2 = 0.25, W_3 = 0.25$	MVAM	0.0132	0.2500	-0.0006	18.9394	-0.0455	-0.0024

* EWM: Equally Weighted Model/ MVM: Mean-Variance Model/ MVAM: Mean-Variance Ambiguity Model

As is shown in Table 6, the best values for return and ambiguity are obtained by MVAM, while the best value for risk is obtained by EWM. Also, these models were tested by relative indexes of σ/μ, β/μ, and β/σ, respectively, which indicate better performance for portfolios when they are smaller. However, to comprehensively evaluate the mentioned models and determine their overall ranking, we have used the combined method based on GRA and TOPSIS presented by Makui, et al. [67], Which is shown as follows:

Step 1: We determined the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS).

$$r^+ = ({a_1}^+, {a_2}^+, \dots, {a_6}^+) \tag{13}$$

$$r^{-} = (a_1{}^{-}, a_2{}^{-}, \dots, a_6{}^{-}) \quad (14)$$

where

$$r_j{}^{+} = (a_j{}^{+}) = \left(\max_i a_{ij}\right), j \in 1,2,\dots,6 \quad (15)$$

$$r_j{}^{-} = (a_j{}^{-}) = \left(\min_i a_{ij}\right), j \in 1,2,\dots,6 \quad (16)$$

therefore:

$$r^{+} = (0.0132, 0.0511, -0.0006, 10.0196, -0.0455, -0.0024)$$
$$r^{-} = (0.0051, 0.2500, 0.0002, 18.9394, 0.0200, 0.0014)$$

Step2: We calculated the gray relational coefficients of each model from PIS and NIS using the following equations, respectively:

$$\xi_{ij}^{+} = \frac{\min_{1\le i\le 3}\min_{1\le j\le 6} d(r_{ij}, r_{ij}^{+}) + \rho \max_{1\le i\le 3}\max_{1\le j\le 6} d(r_{ij}, r_{ij}^{+})}{d(r_{ij}, r_{ij}^{+}) + \rho \max_{1\le i\le 3}\max_{1\le j\le 6} d(r_{ij}, r_{ij}^{+})}, \quad (17)$$

$$\xi_{ij}^{-} = \frac{\min_{1\le i\le 3}\min_{1\le j\le 6} d(r_{ij}, r_{ij}^{-}) + \rho \max_{1\le i\le 3}\max_{1\le j\le 6} d(r_{ij}, r_{ij}^{-})}{d(r_{ij}, r_{ij}^{-}) + \rho \max_{1\le i\le 3}\max_{1\le j\le 6} d(r_{ij}, r_{ij}^{-})}, \quad (18)$$

where the identification coefficient, ρ, is equal to 0.5 and using the normalized Hamming distance. Therefore:

$$\xi_{ij}^{+} = \begin{bmatrix} 0.3333 & 1.0000 & 0.4000 & 1.0000 & 0.4186 & 0.4435 \\ 0.5586 & 0.5272 & 0.3333 & 0.5265 & 0.3333 & 0.3333 \\ 1.0000 & 0.3333 & 1.0000 & 0.3333 & 1.0000 & 1.0000 \end{bmatrix}$$
$$\xi_{ij}^{-} = \begin{bmatrix} 1.0000 & 0.3333 & 0.6667 & 0.3333 & 0.6207 & 0.5730 \\ 0.4525 & 0.4755 & 1.0000 & 0.4760 & 1.0000 & 1.0000 \\ 0.3333 & 1.0000 & 0.3333 & 1.0000 & 0.3333 & 0.3333 \end{bmatrix}$$

Step3: We calculated the degree of gray relational coefficients of each model from PIS and NIS using the following equations, respectively:

$$\xi_i^{+} = \sum_{j=1}^{6} w_j \xi_{ij}^{+}, \qquad i = (1,2,3) \quad (19)$$

$$\xi_i^{-} = \sum_{j=1}^{6} w_j \xi_{ij}^{-}, \qquad i = (1,2,3) \quad (20)$$

therefore:

$$\xi_1^{+} = 0.5992, \qquad \xi_2^{+} = 0.4354, \qquad \xi_3^{+} = 0.7778,$$
$$\xi_1^{-} = 0.5878, \qquad \xi_2^{-} = 0.7340, \qquad \xi_3^{-} = 0.5556,$$

Step4: We calculated the relative grey relational degree of each model from the PIS using the following equation:

$$\xi_i = \frac{\xi_i^{+}}{\xi_i^{+} + \xi_i^{-}}, i = 1,2,3. \quad (21)$$

therefore:

$$\xi_1 = 0.5048, \quad \xi_2 = 0.3723, \quad \xi_3 = 0.5833,$$

Step 5: We ranked all the models and selected the best one(s) by ξ_i. If any model has the highest ξ_i value, then, it is the best one.

$$\xi_3 \succ \xi_1 \succ \xi_2$$
$$\text{MVAM} \succ \text{EWM} \succ \text{MVM}$$

Thus, according to the calculations made in previous stages, it is clear that the proposed model in this study (MVAM) has a better performance than the other two models (EWM and MVM). Also, the following figures (Figure 4 and Figure 5) show that although there is an overall ranking of the models, the investor(s) can observe the ranking of the models in every single criterion and according to different situations can select each of them.

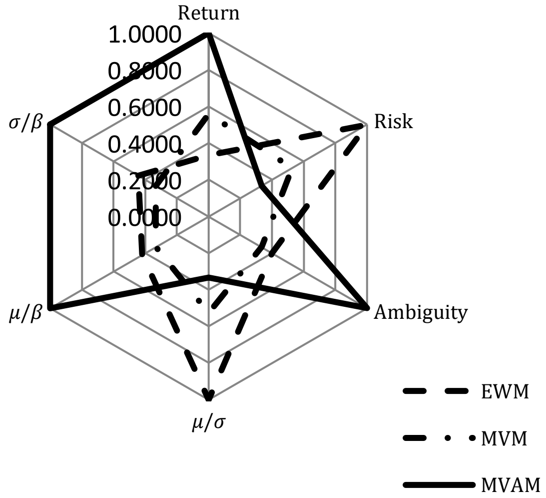

Fig. 4. The shape similarity between PIS and the models

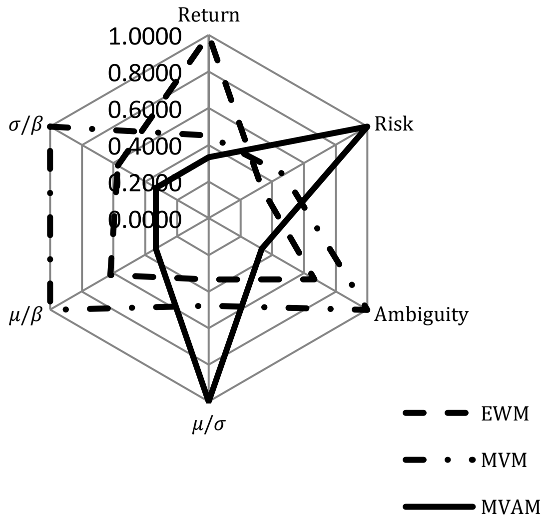

Fig. 5. The shape similarity between NIS and the models

Interestingly, according to the results, it can be seen that contrary to the claims made in many types of research (for reviews of this literature see [68]), the EWM is not the best approach to cover ambiguity in portfolio optimization problems and in this respect, it is significantly lower than the proposed model in this study. Another interesting point is that the proposed model has acceptable performance not only in terms of covering ambiguity but also in other indexes presented in Table 6 and has the best ranking in most of them. Therefore, it can be concluded that the proposed model has the necessary ability to meet the needs of investors and can be used as a suitable tool to optimize the portfolio of securities.

6. Discussions

In this study, we claimed that the proposed model is more comprehensive than Markowitz's model and has a greater ability to cover the conditions of the stock market. In this section, we briefly describe how to achieve this result. In this regard, three hypothetical investment conditions by different market movements are examined separately.

In the first hypothetical investment conditions, assume that based on inside information that is only available to the investor; there is no significant fluctuation in the market and there is no specific information about any of the securities. In such a situation, the market and the securities are expected to show the same behavior as in previous periods. In this case, the investor's opinion about the future return of security i , μ_{ie} , and the expected return of security i, μ_{ip} will be the same. Thus, due to there being no difference between its desirability based on investor prediction and historical estimation, according to Eq. (4) ambiguity measure β_i, is zero. Therefore, given the structure of the objective functions presented in Eq. (9) to calculate the expected return of the portfolio, μ_{PF} , the portfolio risk, σ_{PF} , and the ambiguity score of the portfolio, β_{PF}, the expected results will be as follows:

$$(Max\ \mu_{PF})_{\text{MVAM}} = (Max\ \mu_{PF})_{\text{MVM}}, \quad (Min\ \sigma_{PF})_{\text{MVAM}} = (Min\ \sigma_{PF})_{\text{MVM}}, \quad (Min\ \beta_{PF})_{\text{MVAM}} = (Min\ \beta_{PF})_{\text{MVM}}, \tag{22}$$

From the analysis result, it can be seen that if there is no additional information at the time of the decision, which is called "inside information", there is no difference between our proposed model in this study and the standard mean-variance model of Markowitz [32] and both of them can get equal positive returns at the same level of risk and ambiguity.

In the second hypothetical investment condition, assume that based on inside information that is only available to the investor; the market will have a growing movement that has never been seen before in historical data. In such a situation, the investor's opinion about the future return of security i, μ_{ie} is larger or ultimately equal to what has been determined based on historical data μ_{ip}. Thus, due to there is a difference between its desirability based on investor prediction and historical estimation regardless of the relevant degree of investor's belief in the correctness of his or her analysis, according to the Eq. (4) ambiguity measure β_i, is non-positive (which confirms that the predicted return is higher than the expected return). Naturally in this circumstance, the ambiguity score matrix,γ_{ij} for all securities will be non-negative due to the alignment created in the ambiguity correlation matrix, $\tilde{\zeta}_{ij}$. Therefore, given the structure of the objective functions presented in Eq. (9) to calculate the expected return of the portfolio, μ_{PF} , the portfolio risk, σ_{PF} , and the ambiguity score of the portfolio, β_{PF}, the expected results will be as follows:

$$(Max\ \mu_{PF})_{\text{MVAM}} \geq (Max\ \mu_{PF})_{\text{MVM}}, \quad (Min\ \sigma_{PF})_{\text{MVAM}} \geq (Min\ \sigma_{PF})_{\text{MVM}}, \quad (Min\ \beta_{PF})_{\text{MVAM}} \leq (Min\ \beta_{PF})_{\text{MVM}}, \tag{23}$$

From the analysis result, it can be seen that if the investor could access more reliable and accurate inside information, which indicates that there is a growing movement in the market and any of the securities, our proposed model in this study in addition to considering the historical rate of return, will select those securities that have the highest value growth in new conditions, so it performs better than the standard mean-variance model of Markowitz [32] and it is possible to estimate more appropriate portfolios.

In the third hypothetical investment condition, assume that based on inside information that is only available to the investor; the market will have a falling movement that has never been seen before in historical data. In such a situation, the investor's opinion about the future return of security i, μ_{ie} is smaller or ultimately equal to what has been determined based on historical data μ_{ip}. Thus, due to there is a difference between its desirability based on investor prediction and historical estimation and regardless of the relevant degree of investor's belief in the correctness of his or her analysis, according to the Eq. (4) ambiguity measure β_i, is non-negative (which confirms that the predicted return is lower than the expected return). Naturally in this circumstance, the ambiguity score matrix,γ_{ij} for all securities will be non-negative due to the alignment created in the ambiguity correlation matrix, $\tilde{\zeta}_{ij}$. Therefore, given the structure of the objective functions presented in Eq. (9) to calculate the expected return of the portfolio, μ_{PF} , the portfolio risk, σ_{PF} , and the ambiguity score of the portfolio, β_{PF}, the expected results will be as follows:

$$(Max\ \mu_{PF})_{\text{MVAM}} \leq (Max\ \mu_{PF})_{\text{MVM}}, \quad (Min\ \sigma_{PF})_{\text{MVAM}} \geq (Min\ \sigma_{PF})_{\text{MVM}}, \quad (Min\ \beta_{PF})_{\text{MVAM}} \leq (Min\ \beta_{PF})_{\text{MVM}}, \tag{24}$$

From the analysis result, it can be seen that if the investor could access more reliable and accurate inside information, which indicates that there is a falling movement in the market and any of the securities, our proposed model in this study in addition to considering the historical rate of return, will select those securities that have the lowest value loss in new conditions, so it performs better than the standard mean-variance model of Markowitz [32] and it is possible to estimate more appropriate portfolios.

However, it should be noted that due to the nature of our proposed model in this study which simultaneously used two different parameters in portfolio optimization: analyzing past trends and predicting future developments; a true and more accurate assessment is possible only in the form of a posterior approach. , after passing the desired period, the objective results obtained from the use of both models are compared and conclusions are made. In this situation, if the inside information that is only available to the investor was sufficiently accurate, our proposed model in a growing market has certainly achieved greater returns and also in a falling market has certainly suffered fewer losses compared to the standard mean-variance model of Markowitz [32].

Another interesting point that emerged in this study was the inadequacy of the variance measure to estimate the level of risk in our proposed model. As mentioned earlier, given the nature of our proposed model; the variance measure due to its structural limitations cannot include future-based analysis in its calculations which has led to a low performance of our proposed model in the context of risk. Thus, we suggest that the best way to avoid the risk in our proposed model is to use of "safety first" approach introduced by Roy [69]. Investors who use this theory first determine the safety limit by the method indicated in Roy [69] and then move on to the other goals. Their purpose is to secure their capital in most cases of nature. Therefore, in this condition, the portfolio optimization problem can be shown in the following form:

$$
\begin{aligned}
&Max\ \mu_{PF} = \sum_{i=1}^{n} (\omega_{ie}\mu_{ie} + \omega_{ip}\mu_{ip}) w_i,\\
&Min\ \beta_{PF} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i \beta_{ij} \gamma_{ij} w_j,\\
&\text{Subject to:}\\
&P(Max\ \mu_{PF} < A) < \varepsilon_r\\
&P(Min\ \beta_{PF} > B) < \varepsilon_a\\
&\sum_{i=1}^{n} w_i = 1\\
&x \in F
\end{aligned}
\tag{25}
$$

Where A and B are the desired level of expected return and ambiguity score for the portfolio, respectively; and ε_r and ε_a are the maximum probability of decreasing the level of expected return and ambiguity score for the portfolio, respectively.

Finally, it can be inferred that the proposed model in this study can be modified for investment problems with different types of investor information about the past and future of the stock market. However, according to analysis results, the standard mean-variance model of Markowitz [32] could not able to take into account these two different types of investor information simultaneously. Therefore, in a volatile and inefficient stock market such as TSE, the portfolios obtained from this configuration could not give reasonable returns compared to our proposed model. This situation is a common shortcoming of the conventional portfolio selection models and their derivatives. As a result, the proposed model in this study can fulfill this shortcoming because it handles all types of available information to investors in the desired period simultaneously.

7. Conclusions and Future Researches

Empirical evidence demonstrates that to select the best portfolio it is required to aggregate more than two conventional dimensions (maximize the expected return of the portfolio and minimize the portfolio risk). Thus, in this study, considering the undeniable importance of ambiguity in human decision-making, especially in the field of investment, we revisited Markowitz's model and proposed a modification that allows incorporating not only return and risk but also incorporate ambiguity into the investment decision-making process. In this way, our contributions can be summarized as follows:

(1) We construct an investment model in portfolio optimization that simultaneously uses two different parameters: analyzing past trends and predicting future developments. The former is derived based on historical data that is available to everyone and the latter is achieved based on inside information that is only available to the investor.

(2) We proposed an empirical measure of ambiguity which is based on the difference between the views of investors and the results extracted from the historical data. This approach to measuring ambiguity, despite its simplicity, corresponds closely to the original definitions of ambiguity provided by Ellsberg [30].

(3) We explored portfolio optimization from a new perspective and improved upon existing methods that fail to consider the

views of investors about the effectiveness of decision information. We propose a comprehensive model that expands the investment climate from a purely past-oriented to an information-driven mode, which makes the decision-making process more realistic.

In the end, it should be noted, that an interesting line for future research can be considered as applying this approach for multi-period portfolio optimization given that the credibility of decision information decreases with time, so it is significant to capture the dynamic credibility of decision information in the temporal dimension. Finally, due to the differences in the origin of ambiguity and risk, the development of more appropriate methods can be put on the agenda to simultaneously contribute these two effective components in the portfolio optimization process in the form of individual and group decision-making.

Declarations

Availability of data and material: The datasets generated and analyzed during the current study are available from the corresponding author upon reasonable request.

Conflicts of interest/Competing interests: The authors whose names are listed in the current study declare that they have no conflict of interest.

Funding: The current study did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Authors' contributions: All authors contributed to the study's conception and design. Material preparation, data collection, analysis, and writing of the first draft of the manuscript were performed by Seyed Erfan Mohammadi. All authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Acknowledgments: Not applicable.

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