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RESEARCH PAPER

A Proposal Based on Stochastic Differential Equations for Income

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ABSTRACT
Previous work has highlighted the need to apply stochastic modeling to understand the dynamics of phenomena occurring in the insurance industry. In this paper, for life insurance and applying a stochastic approach under efficient markets, we use survival probabilities and stochastic differential equations to model the actuarial reserve, changes in the constituted actuarial reserve, and estimated income over time. We present an application, sensitivity analysis, and discussion of the results using United States life tables.

KEYWORDS: *Life insurance; Actuarial science; Stochastic processes; Brownian motion.*

1. Introduction

Previous works have proposed the application of stochastic differential equations for the modeling financial phenomena [1-4]. Furthermore, another group of authors has addressed the modeling of pension systems using stochastic differential equations [5-9].

A complete review of the probabilistic treatment of life insurance can be found in [10-12]. In addition, it is included in [13] the mathematical treatment associated with the multiple causes of population decline and their respective estimation of probabilities.

Other research related to life insurance modeling focuses on using stochastic processes. These focus on multistate Markov chains to deal with disability and death. Pioneering papers in that approach also consider fundamental aspects such as actuarial value and the estimation of actuarial reserves [14-15]. Other investigations have addressed stochastic processes in continuous time through differential equations, as well as various extensions [16].

Nielsen [17] investigates the problem of optimal surplus redistribution in life insurance and pensions when the interest rate is modeled as a continuous-time stochastic process. He obtains an explicit solution leading to a characterization of the optimal strategies, indicating that some widely used redistribution schemes are suboptimal [17]. On the other hand, Noor & Mat Isa [18] evidenced that stochastic processes can be a suitable method for forecasting the purchase of life insurance. The authors used a sample of life insurance purchases between 2003 and 2006.

Kraft and Steffensen present a more economically oriented approach [19]. They model consumption and insurance problems under a time-continuous approach. The authors analyze the case where an individual makes optimal decisions in the face of the risk of death and loss of income due to disability or unemployment. Likewise, Buchardt et al. [20] propose the valuation by a semi-Markov process on finite state space. They express how to estimate the cash flow and present an approach based on the Kolmogorov equation. In addition, Buchardt et al. [20] propose using affine processes to model transition rates and interest rates. That provides alternatives for the estimate of expected present values.

Moreover, Fahrenwaldt [21] uses the Thiele differential equation to estimate the reserve of a multi-state insurance contract with functional dependence. Through an analytical approach, the author presents existence and uniqueness results and analyzes the sensitivity of reserves for

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surplus, payout rate, and transition assumptions.
Concerning forecasting in the insurance industry, Jensen [22] argues that from the perspective of the insured, traditional life insurance and pension forecasts do not adequately illustrate the financial risk and the effect of financial guarantees. The problem is addressed through the introduction of stochastic scenarios by proposing useful mathematical elements for the valuation of life insurance payouts in all types of products.
In [23] the authors investigate jump-diffusion processes for a social benefit scheme involving life insurance, unemployment and disability benefits, and retirement benefits. The authors use a four-state Markov chain with multiple decrements. The authors find that an extension of the retirement age has an indirect effect that would increase government expenditures for other social security programs. They also illustrate how worldwide life expectancy has increased in recent decades and how this situation may considerably affect the fiscal capacity of governments responsible for social security schemes. In addition, low birth rates, coupled with an aging population, imply a decline in productivity and an increase in age-related problems, unemployment, and disability.
In this article, survival probabilities are used for the case of life insurance, as presented in [24]. Also, we model the necessary actuarial reserve considering a stochastic differential equation proposed in [25] for the interest rate and a Wiener process [2] for the possible indemnity at period t. In addition, we estimate the change in the actuarial reserve constituted and the estimated income over time. The model is then presented in section 2, followed by an application and discussion of results in section 3 using the United States life tables. Finally, we conclude with the virtues of using the proposed stochastic approach for actuarial modeling.

2. Materials and Methods

2.1. Survival probabilities

Following classical works in actuarial science, such as [24] and [26], consider a function s(x) that indicates the probability that the age of death X being greater than x. This function is called *survival function* and is defined as

$$s(x) = \mathbb{P}(X > x)$$

Additionally, let ${}_t p_x$ be the probability that a person of age x will survive for at least t more periods. In this case,

$$
\begin{aligned}
{}_t p_x &= \mathbb{P}(X > x + t / X > x) \\
&= \frac{\mathbb{P}(X > x + t \cap X > x)}{\mathbb{P}(X > x)} \\
&= \frac{\mathbb{P}(X > x + t)}{\mathbb{P}(X > x)} \\
&= \frac{s(x + t)}{s(x)}
\end{aligned}
$$

And let ${}_t q_x$ be the probability that a person of age x dies in less than t periods.

$$
\begin{aligned}
{}_t q_x &= 1 - {}_t p_x \\
&= 1 - \frac{s(x + t)}{s(x)}
\end{aligned}
$$

The probability that a person of age x survives t periods longer and dies in less than u periods after age $x + t$, denoted ${}_{t/u}q_x$, is

$$
\begin{aligned}
{}_{t/u}q_x &= ({}_t p_x)({}_u q_{x+t}) \\
&= \frac{s(x+t)}{s(x)} - \frac{s(x+t+u)}{s(x)}
\end{aligned}
$$

2.2. Reserve for life insurance

The actuarial reserve for life insurance can be estimated as the sum of the expected present values of each of the possible payments. In the case of insurance with coverage during n periods, with coverage beginning at this moment and compensation K payable at the end of the death period, the actuarial reserve is

$$
\begin{aligned}
R &= K[{}_{0/1}q_x](1 + i\%)^{-1} \\
&\quad + K[{}_{1/1}q_x](1 + i\%)^{-2} + \cdots \\
&\quad \cdot + K\left[{}_{(n-1)/1}q_x\right](1 + i\%)^{-n} \\
&= K \sum_{j=1}^{n} \left[{}_{(j-1)/1}q_x\right](1 + i\%)^{-j}
\end{aligned}
$$

Now suppose that the compensation in case of death is variable, denoted by $\mathcal{P}(t)$. In this case

$$R = \int_1^n \mathcal{P}(t)\left[{}_{(t-1)/1}q_x\right]e^{-r(t)t}dt \qquad (1)$$

where,
R: actuarial reserve.
$\mathcal{P}(t)$: amount of compensation to be paid if death occurs at time t.
$r(t)$: nominal continuous interest rate over time t.
x: age of the insured person.

2.3. Change in the necessary actuarial reserve

Suppose the interest rate r(t) follows the Cox-Ingersoll and Ross process

$$dr_t = (\theta - \kappa r_t)dt + \nu\sqrt{r_t}\,dBt \quad (2)$$

and suppose that the possible compensation in period t, P (t), follows the Wiener process

$$d\mathcal{P}_t = \alpha\mathcal{P}_t\,dt + \beta\mathcal{P}_t\,dU_t \quad (3)$$

with

$$Cov(dBt, dU_t) = \rho dt$$

The Itô's Lemma applied to the actuarial reserve R defines its change in relation to the variation in time t and in relation to variations both in the interest rate r(t) and in the amount of compensation $P(t)$.
In [27] it is shown that the price of a financial derivative, which depends on the price of two assets and time, can be found by a variation of the original Itô lemma using the following expression:

$$\boldsymbol{dG = \left(\frac{\partial G}{\partial S_1}a_1 + \frac{\partial G}{\partial S_2}a_2 + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S_1 \partial S_2}b_1 b_2 \rho\right)dt + \frac{\partial G}{\partial S_1}b_1 dB_t^1 + \frac{\partial G}{\partial S_1}b_2 dB_t^2}$$

with

$$dS_1 = a_1dt + b_1dB_t^1$$
$$dS_2 = a_2dt + b_2\partial B_t^2$$
$$Cov(\partial B_t^1, dB_t^2) = \rho\, dt$$

According to the above, since R depends on r(t) and (t), which follow the stochastic differential equations (2) and (3), and t, then

$$dR = \left(\frac{\partial R}{\partial r}(\theta - kr_t) + \frac{\partial R}{\partial \mathcal{P}}\alpha\mathcal{P}_t + \frac{\partial R}{\partial t} + \frac{1}{2}\frac{\partial^2 R}{\partial r \partial \mathcal{P}}\beta\mathcal{P}_t\nu\sqrt{r_t}\rho\right)dt + \frac{\partial R}{\partial r}\nu\sqrt{r_t}\,d\beta_t + \frac{\partial R}{\partial \mathcal{P}}n\mathcal{P}_t dU_t \quad (4)$$

2.4. Change in the constituted actuarial reserve

Suppose the insurance company has an actuarial reserve of R_{C_t} . This reserve is invested in a portfolio whose stochastic differential equation has a trend parameter of μ and volatility σ. Furthermore, in each period the surpluses of ψ_t are withdrawn, generated by the change in the probability of transition from alive to dead as the insured survives one more period. These surpluses constitute income for the insurance company.

$$dR_{C_t} = \mu R_{C_t}dt + \sigma R_{C_t}dK_t - \psi_t dt = (\mu R_{C_t} - \psi_t)dt + \sigma R_{C_t}dK_t \quad (5)$$

2.5. Estimated income in time t

To maintain actuarial equilibrium it is necessary that dR = dRCt , then equating the equations (4) and (5), we have:

$$\mu R_{C_t} - \psi_t = \frac{\partial R}{\partial r}(\theta - kr_t) + \frac{\partial R}{\partial \mathcal{P}}\alpha\mathcal{P}_t + \frac{\partial R}{\partial t} + \frac{1}{2}\frac{\partial^2 R}{\partial r \partial \mathcal{P}}\beta\mathcal{P}_t\nu\sqrt{r_t}\rho$$

Then

$$\psi_t = \mu R_{C_t} - \frac{\partial R}{\partial r}(\theta - kr_t) - \frac{\partial R}{\partial \mathcal{P}}\alpha\mathcal{P}_t - \frac{\partial R}{\partial t} - \frac{1}{2}\frac{\partial^2 R}{\partial r \partial \mathcal{P}}\beta\mathcal{P}_t\nu\sqrt{r_t}\rho \quad (6)$$

where the partial derivatives are obtained from (1). Then,

$$\frac{\partial R}{\partial r} = -\int_1^n t\mathcal{P}(t)\left[{}_{(t-1)/1}q_x\right]e^{-r(t)t}dt$$
$$\frac{\partial R}{\partial t} = \mathcal{P}(n)\left[{}_{(n-1)/1}q_x\right]e^{-r(n)n} - \mathcal{P}(1)\left[{}_1q_x\right]e^{-r(1)}$$
$$\frac{\partial R}{\partial \mathcal{P}} = \int_1^n \left[{}_{(t-1)/1}q_x\right]e^{-r(t)t}dt$$
$$\frac{\partial^2 R}{\partial r \partial \mathcal{P}} = -\int_1^n t\left[{}_{(t-1)/1}q_x\right]e^{-r(t)t}dt$$

3. Results and Discussion

To analyze the results obtained with the modeling presented in the previous section, consider an insurance company that invests its reserves in a portfolio. It follows the behavior of the index *iShares Core U.S. Aggregate Bond ETF*, one of the most popular *exchange-traded funds* (ETF) for investing actuarial reserves in the United States. Figure 1 shows the value of the ETF for the last five years.

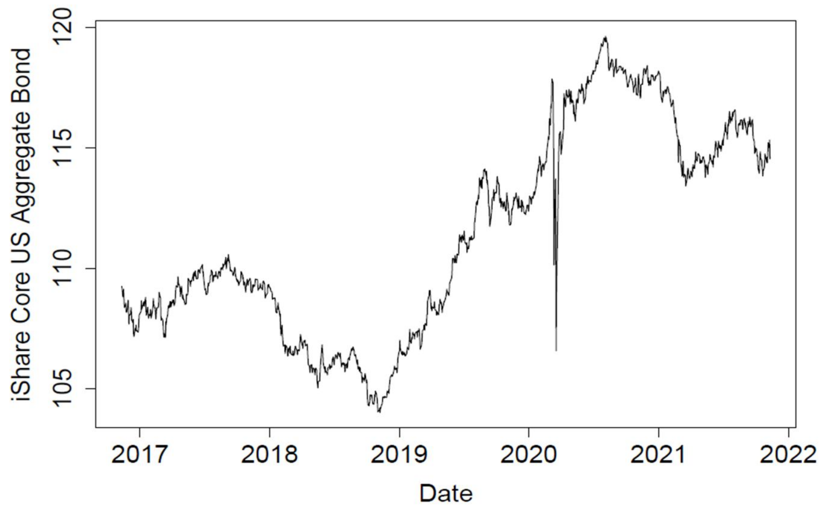

Fig. 1. Price per unit of the ishare core US aggregate bond from 2016-11-11 to 2021-11-10.

The portfolio rate of return (α) was estimated at 0.004209292% and volatility (β) at 0.2959636%, both estimates in daily terms. The stochastic differential equation followed by the process is then

$d\mathcal{P}_t = 0.00004209292\mathcal{P}_t \; dt + 0.002959636\mathcal{P}_t dU_t$

Moreover, using the est.cir function of the R SMFI5 library, the parameters of the stochastic differential equation governing the United States 10Y bond were estimated. The rates for the last five years are presented in Figure 2.

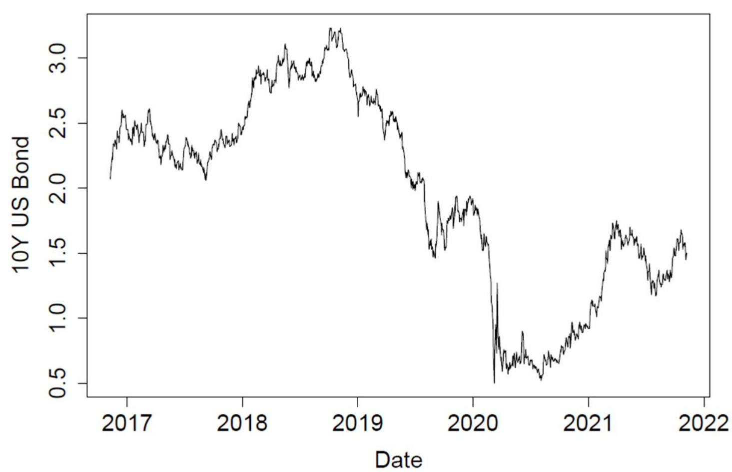

Fig. 2. 10-year US treasury bond interest rate from 2016-11-9 to 2021-11-8.

The stochastic differential equation for the US 10-year bond interest rate is:

$dr_t = (1.278589 - 0.6511135r_t)dt + 0.6168297\sqrt{r_t}dBt$

then θ = 1.278589, κ = 0.6511135 and ν = 0.6168297.

For illustrative purposes, we assume a 40-year-old U.S. citizen. A company insured this person's life for one year, with an initial indemnity in the event of death of USD 100,000. The indemnity will vary each day by the same percentage as the iShares Core U.S. Aggregate Bond ETF index, followed by the actuarial reserve portfolio.

Regarding the life tables, we used those corresponding to the entire population of the United States and interpolated them using the linear method. Figure 3 shows the survival function plot s(x) for that population. According to the above, we present the simulation of a possible trajectory for the indemnity value (Figure 4) and the interest rate (Figure 5).

In this scenario, the insurer's average daily income is $38.12. This income increases in variability over time, reflecting the increase in risk as the time horizon gets longer. Results are shown in Figure 6.

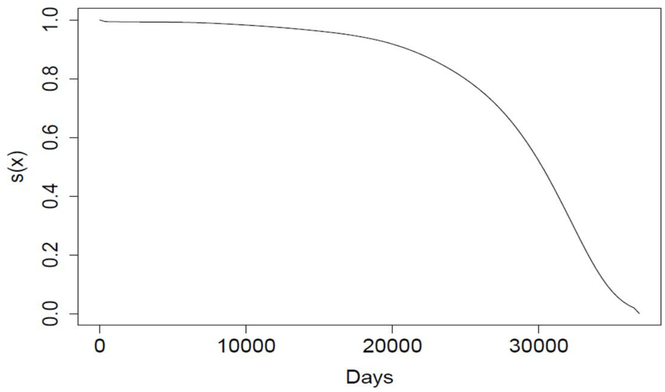

Fig. 3. Survival function *s*(*x*) interpolated from age 0 to age 100.

The simulation for a possible trajectory for the value of the compensation and for the interest rate is presented in Figure 4 and 5.

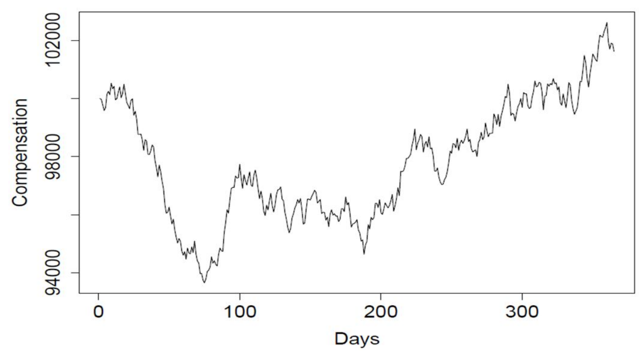

Fig. 4. Simulated compensation values.

To analyze the distribution of income, 1,000 trajectories were simulated, both for compensation and interest rates. Results indicate that the insurer's income, discounted at an inflation rate of 1% per year, has a symmetrical distribution, with a minimum of USD 12,829, a mean of USD 14,656, and a maximum value of USD 16,643. Figure 7 shows the income histogram with a normal curve for comparison. Tukey's test yields one lower outlier and five upper outliers. According to the same test, none of the data can be classified as extreme outliers.

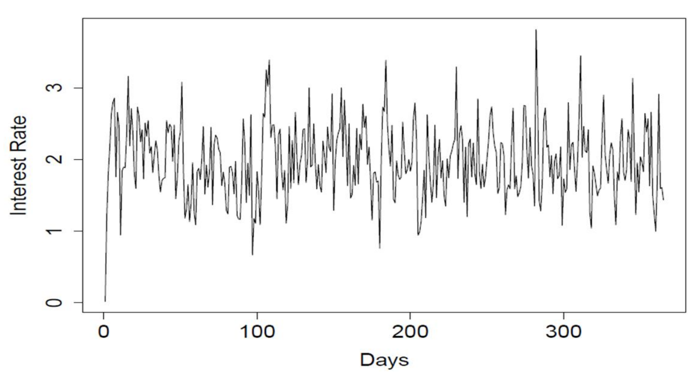

Fig. 5. Simulated interest rate values

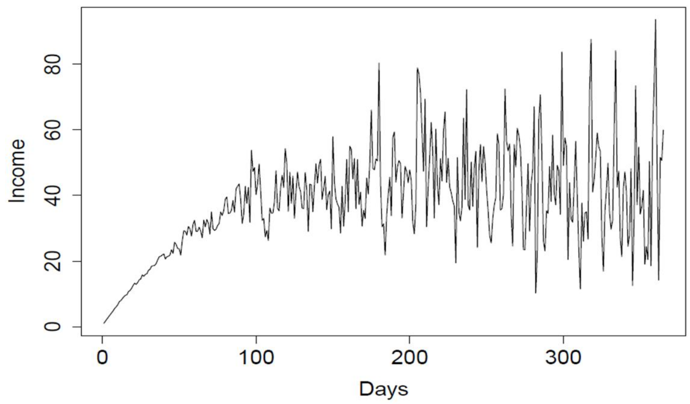

Fig. 6. Simulated income values.

On the other hand, from the equation (6), it follows that

$$\frac{\partial \psi_t}{\partial k} = -\frac{\partial R}{\partial r} r(t)$$
$$\frac{\partial \psi_t}{\partial \bar{r}_t} = \frac{\partial R}{\partial r} k, \text{ con } \theta = \bar{r}_t \kappa$$
$$\frac{\partial \psi_t}{\partial \nu} = \frac{1}{2}\frac{\partial^2 R}{\partial r \partial \mathcal{P}} \beta \mathcal{P}(t)\sqrt{r(t)}\rho$$
$$\frac{\partial \psi t}{\partial \alpha} = \frac{\partial R}{\partial \mathcal{P}} \mathcal{P}(t)$$
$$\frac{\partial \psi_t}{\partial \beta} = \frac{\partial^2 R}{\partial r \partial \mathcal{P}} \nu \mathcal{P}(t)\sqrt{r(t)}\rho$$

Observe that as the actuarial reserve and interest rate have inverse behaviors, while the reversion speed (κ) increases, the insurer's income ψ_t increases by r(t). Also, as the economy's average interest rate increases, the insurer's income decreases by κ. Regarding ν, as income increases, it also increases. However, at a lower rate than in the presence of a variation in κ. Figure 8 illustrates this behavior for the trajectories depicted in Figures 4 and 5.

For variations in the stochastic differential equation parameters for indemnity, as α and β increase, so does income. However, the increase in income as α grows fades as the end of the coverage period approaches. Figure 9 shows the results for the trajectories depicted in Figures 4 and 5.

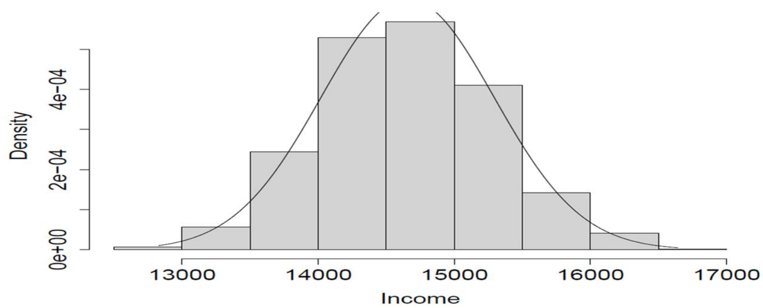

Fig. 7. Histogram of simulated earnings

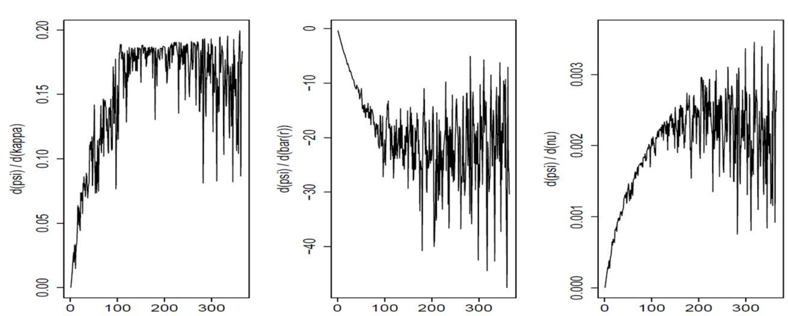

Fig. 8. Sensitivity of income to changes in the parameters of the stochastic differential equation for interest rates.

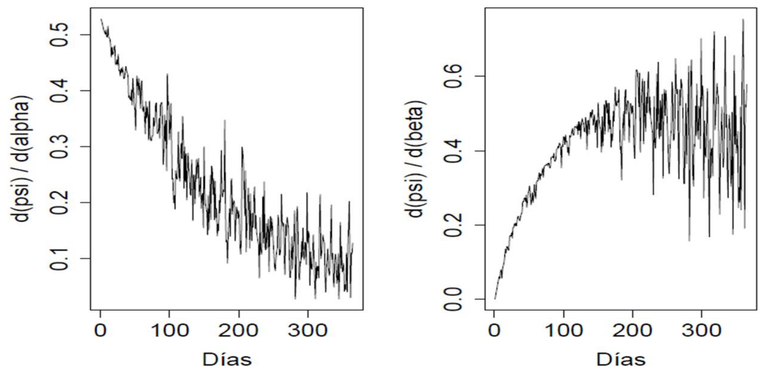

Fig. 9. Sensitivity of income to changes in the parameters of the stochastic differential equation for compensation.

The analysis of the above results must also consider the possibility of facing inefficient markets. Using Brownian motion and especially the Martingale property, this paper implicitly assumes the efficiency of asset and interest rate markets. Future work should focus on the use of fractional Brownian motion, which appears in [28], to cover inefficient financial markets as well.

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