

NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

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Abstract: We call a Cayley graph $\Gamma = \text{Cay}(G, S)$ normal for G , if the right regular representation $R(G)$ of G is normal in the full automorphism group of $\text{Aut}(\Gamma)$. In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.

Keywords: Cayley graph, normal Cayley graph, automorphism group.

1. Introduction

Let G be a finite group, and S be a subset of G not containing the identity element 1_G . The Cayley digraph $\Gamma = \text{Cay}(G, S)$ of G relative to S is defined as the graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma)$ consisting of those ordered pairs (x, y) from G for which $yx^{-1} \in S$. Immediately from the definition we find that, there are three obvious facts: (1) $\text{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of G and so Γ is vertex-transitive.

(2) Γ is connected if and only if $G = \langle S \rangle$. (3) Γ is an undirected if and only if $S^{-1} = S$.

A Cayley (di)graph $\Gamma = \text{Cay}(G, S)$ is called normal if the right regular representation $R(G)$ of G is a normal subgroup of the automorphism group of Γ .

The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see[1,2]). Given a finite group G , a natural problem is to determine all normal or non-normal Cayley (di)graphs of G . This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley's graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let $\Gamma = \text{Cay}(G, S)$ be a connected undirected Cayley graph of a finite abelian group G on S with valency 6. Then Γ is normal except when one of the following cases happens:

$$(1): G = \mathbb{Z}_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle, \\ S = \{a, b, c, abc, d, e\}.$$

$$(2): G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3), \\ S = \{a, b, c, abc, d, d^{-1}\}.$$

$$(3): G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ab, c^2, c, c^{-1}\}.$$

$$(4): G = \mathbb{Z}_2^4 \times \mathbb{Z}_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle, \\ S = \{a, b, c, d, e, e^{-1}\}.$$

$$(5): G = \mathbb{Z}_2^3 \times \mathbb{Z}_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \\ S_1 = \{a, b, c, d^2, d, d^{-1}\}, \\ S_2 = \{a, b, ab, c, d, d^{-1}\}, S_3 = \{a, b, c, ad^2, d, d^{-1}\}.$$

$$(6): G = \mathbb{Z}_2^2 \times \mathbb{Z}_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ab, c^3, c, c^{-1}\}.$$

$$(7): G = \mathbb{Z}_2^3 \times \mathbb{Z}_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle, \\ S = \{a, b, c, d^3, d, d^{-1}\}.$$

$$(8): G = \mathbb{Z}_6 \times \mathbb{Z}_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}.$$

$$(9): G = \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S = \{a, b^3, b, b^{-1}, c, c^{-1}\}.$$

$$(10): G = \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, a^{-1}, a^2, b, b^{-1}, b^m\}.$$

$$(11): G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S_1 = \{a, b, b^{-1}, b^2, c, c^{-1}\}, S_2 = \{a, b, b^{-1}, ab^2, c, c^{-1}\}.$$

$$(12): G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, b, b^{-1}, c, c^{-1}, c^m\}.$$

$$(13): G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \\ (m \geq 3), S = \{a, b, c, c^{-1}, d, d^{-1}\}.$$

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$$(14): G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3),$$

$$S = \{a, b, cd, cd^{-1}, d, d^{-1}\}.$$

$$(15): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m = 5, 10),$$

$$S = \{a, b, c, c^{-1}, c^3, c^{-3}\}.$$

$$(16): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{a, b, c, c^{-1}, c^{2m+1}, c^{2m-1}\}.$$

$$(17): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3, m \text{ is odd}),$$

$$S = \{a, a^3, b, b^{-1}, b^{m+1}, b^{m-1}\}.$$

$$(18): G = Z_4^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3),$$

$$S = \{a, a^3, b, b^3, c, c^{-1}\}.$$

$$(19): G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m \geq 2, n \geq 3),$$

$$S = \{a, a^{-1}, a^{2m+1}, a^{2m-1}, b, b^{-1}\}.$$

$$(20): G = Z_2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3, n \geq 3),$$

$$S = \{ab, a b^{-1}, b, b^{-1}, c, c^{-1}\}.$$

$$(21): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m = 5, 10, n \geq 3),$$

$$S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}.$$

$$(22): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, ab, c, abc, d\}.$$

$$(23): G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$$

$$S = \{a, b, ac^2, c, c^{-1}, c^2\}.$$

$$(24): G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, d, d^{-1}, abd^2\}.$$

$$(25): G = Z_2^2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 1),$$

$$S = \{a, b, ac^m, ac^{2m}, c, c^{-1}\}.$$

$$(26): G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle, S = \{a, b, b^3, b^5, b^7, b^9\}.$$

$$(27): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{ac, ac^{-1}, b, c^m, c, c^{-1}\}.$$

$$(28): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{a, b^2 c^m, b, b^{-1}, c, c^{-1}\}.$$

$$(29): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3),$$

$$S = \{a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}\}.$$

$$(30): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{a, b, ac, ac^{-1}, c, c^{-1}\}.$$

$$(31): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3, m \text{ is odd}),$$

$$S = \{a, b^2, b^{-2}, b^m, b^{5m}, b^{3m}\}.$$

$$(32): G = Z_2^2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{a, bc^m, bc^{3m}, bc^{5m}, c, c^{-1}\}.$$

$$(33): G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, b, c, ab, ac, abc\}.$$

$$(34): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, d, abc, abd\}.$$

$$(35): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S = \{a, b, ac^m, bc^m, c, c^{-1}\}.$$

$$(36): G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,$$

$$S_1 = \{a, b, ab, ac^2, c, c^{-1}\},$$

$$S_2 = \{a, b, ac^2, abc^2, c, c^{-1}\}.$$

$$(37): G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,$$

$$S = \{a, b, c, abcd^2, d, d^{-1}\}.$$

$$(38): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}.$$

$$(39): G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 1),$$

$$S = \{a, ab^m, ab^{2m}, ab^{3m}, b, b^{-1}\}.$$

$$(40): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S = \{a, a^{-1}, b^m, a^2 b^m, b, b^{-1}\}.$$

$$(41): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 1),$$

$$S = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}\}.$$

$$(42): G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle,$$

$$S = \{a, ab^5, b, b^9, b^3, b^7\}.$$

$$(43): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle,$$

$$S_1 = \{a, b, b^{-1}, b^m, ab, a b^{-1}\}, m \geq 2,$$

$$S_2 = \{a, ab^m, b, b^{-1}, ab, a b^{-1}\}, m \geq 2,$$

$$S_3 = \{ab^m, b^m, b, b^{-1}, ab, a b^{-1}\}, m \geq 2, S_4 = \{a, ab^m, b,$$

$$b^{-1}, b^{m+1}, b^{m-1}\}, m \geq 3, S_5 = \{a, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\},$$

$$m \geq 3, S_6 = \{a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}, m \geq 3$$

$$S_7 = \{ab^m, b, b^{-1}, b^m, ab^{m+1}, ab^{m-1}\}, m \geq 3$$

$$(44): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S_1 = \{a, b, c, c^{-1},$$

$$abc, abc^{-1}\}, m \geq 3, S_2 = \{a, b, c, c^{-1}, ac^{k+1}, ac^{k-1}\}, m = 2k,$$

$$k \geq 3, S_3 = \{a, b, c, c^{-1}, abc^{k+1}, abc^{k-1}\}, m = 2k, k \geq 3,$$

$$S_4 = \{a, bc, b c^{-1}, ack, c, c^{-1}\}, m = 2k, k \geq 2,$$

$$S_5 = \{a, bc^{k+1}, bc^{k-1}, c^k, c, c^{-1}\}, m = 2k, k \geq 3,$$

$$S_6 = \{a, bc^{k+1}, bc^{k-1}, ac^k, c, c^{-1}\}, m = 2k, k \geq 3,$$

$$S_7 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, m = 2k - 1, k \geq 2.$$

$$(45): G = Z_{4m} = \langle a \rangle \quad (m \geq 2),$$

$$S = \{a, a^{-1}, a^m, a^{-m}, a^{2m+1}, a^{2m-1}\}.$$

$$(46): G = Z_{2m} = \langle a \rangle \quad (m \geq 4),$$

$$S = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^k, a^{-k}\} \quad (2 \leq k \leq m - 2),$$

$$(m, k) = 1, \text{ if } l > 2 \text{ or } l = 2 \text{ for } m = 4i + 2; (k = 2i, \text{ with } i$$

$$\text{odd or } k = 2i + 2, \text{ with } i \text{ even}).$$

$$(47): G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \quad (m \geq 5),$$

$$S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\} \quad (2 \leq j < \frac{m}{2}), (m, j) = p >$$

$$2; m = (t + 1)p,$$

$$S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\}, (2 \leq j < \frac{m}{2}), (m, j) = p > 2; m = (t + 1)p.$$

$$(48): G = Z_2 \times Z_8 = \langle a \rangle \times \langle b \rangle, \\ S_1 = \{ab, ab^{-1}, b, b^{-1}, b^3, b^{-3}\}, \\ S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^3, ab^{-3}\}.$$

$$(49): G = Z_{2m} \times Z_n = \langle a \rangle \times \langle b \rangle (m \geq 2, n \geq 3), \\ S = \{a, a^{-1}, a^{mb}, a^{mb^{-1}}, b, b^{-1}\}.$$

$$(50): G = Z_{2m} \times Z_{2n} = \langle a \rangle \times \langle b \rangle (m \geq 3, n \geq 2), \\ S = \{a, a^{-1}, a^{m+1}b^n, a^{m-1}b^n, b, b^{-1}\}.$$

$$(51): G = Z_{6m} = \langle a \rangle (m \geq 2), S_1 = \{a, a^{-1}, a^3, a^{-3}, a^{3m+1}, a^{3m-1}\}, \\ S_2 = \{a, a^{-1}, a^{3m+1}, a^{3m-1}, a^{3m+3}, a^{3m-3}\}.$$

$$(52): G = Z_m = \langle a \rangle (m = 7, 14), S = \{a, a^{-1}, a^3, a^{-3}, a^5, a^{-5}\}.$$

$$(53): G = Z_{3m} = \langle a \rangle (m \geq 3), \\ S = \{a, a^{-1}, a^{m-1}, a^{m+1}, a^{2m-1}, a^{2m+1}\}.$$

$$(54): G = Z_{16m-4} = \langle a \rangle (m \geq 1), \\ S = \{a, a^{-1}, a^{4m-2}, a^{12m-2}, a^{8m-3}, a^{8m-1}\}.$$

$$(55): G = Z_{16m+4} = \langle a \rangle (m \geq 1), \\ S = \{a, a^{-1}, a^{4m+2}, a^{12m+2}, a^{8m+1}, a^{8m+3}\}.$$

$$(56): G = Z_3 \times Z_3 = \langle a \rangle \times \langle b \rangle, \\ S = \{a, a^2, b, b^2, a^2b, ab^2\}.$$

$$(57): G = Z_2 \times Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, b^{-1}, c, c^{-1}, ab^2, c^2\}.$$

2. Primary Analysis

Proposition 2.1 [9, Proposition 1.5] Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of G over S , and $A = \text{Aut}(\Gamma)$. Let A_1 be the stabilizer of the identity element 1 in A .

Then Γ is normal if and only if every element of A_1 is an automorphism of G .

Proposition 2.2 [6, Theorem 1.1] Let G be a finite abelian group and S be a generating subset of $G - 1_G$. Assume S satisfies the condition that, if $s, t, u, v \in S$ with $1 \neq st = uv$, implies $\{s, t\} = \{u, v\}$. Then the Cayley graph $\text{Cay}(G, S)$ is normal.

Let X and Y be two graphs. The direct product $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$, whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product $X[Y]$ is defined as the graph vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$.

Let $V(Y) = \{y_1, y_2, \dots, y_n\}$. Then there is a natural embedding nX in $X[Y]$, where for $1 \leq i \leq n$, the i th copy of X is the subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$. Let Γ be a graph and α a permutation $V(\Gamma)$ and C_n a circuit of length n . The twisted product $\Gamma \times_{\alpha} C_n$ of Γ by C_n with respect to α is defined by;

$$V(\Gamma \times_{\alpha} C_n) = V(\Gamma) \times V(C_n) = \{(x, i) | x \in V(\Gamma), i = 0, 1, \dots, n-1\}, \\ E(\Gamma \times_{\alpha} C_n) = \{[(x, i), (x, i+1)] | x \in V(\Gamma), i = 0, 1, \dots, n-2\} \cup \{[(x, n-1), (x^{\alpha}, 0)] | x \in V(\Gamma)\} \cup \{[(x, i), (y, i)] | [x, y] \in E(\Gamma), i = 0, 1, \dots, n-1\}.$$

The graph Q_4^d denotes the graph obtained by connecting all long diagonals of 4-cube Q_4 , that is, connecting all vertices u and v in Q_4 such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is the twisted product of Q_3 by C_n such that d transposes each pair of elements on long diagonals of Q_3 . The graph $C_{2m}^d[2K_1]$ is defined by:

$$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1]),$$

$$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \cup \{[(x_i, y_j), (x_{i+m}, y_j)] | i = 0, 1, \dots, m-1, j = 1, 2\}, \text{ where } V(C_{2m}) = \{x_0, x_1, \dots, x_{2m-1}\} \text{ and } V(2K_1) = \{y_1, y_2\}.$$

Let $G = G_1 \times G_2$ be the direct product of two finite groups G_1 and G_2 , let S_1 and S_2 be subsets of G_1 and G_2 , respectively, and let $S = S_1 \cup S_2$ be the disjoint union of two subsets S_1 and S_2 . Then we have,

Lemma 2.3

- (1) $\text{Cay}(G, S) \cong \text{Cay}(G_1, S_1) \times \text{Cay}(G_2, S_2)$.
- (2) If $\text{Cay}(G, S)$ is normal, then $\text{Cay}(G_1, S_1)$ is also normal.
- (3) If both of $\text{Cay}(G_1, S_1)$ and $\text{Cay}(G_2, S_2)$ are normal and relatively prime, then $\text{Cay}(G, S)$ is normal.

3. Proof of the Main Theorem

In this section, Γ always denotes the Cayley graph $\text{Cay}(G, S)$ of an abelian group G on S with valency 6. Let $A = \text{Aut}(\Gamma)$. Then A_1 and A_1^* denote the stabilizer of 1 in A and the subgroup of A which fixes $\{1\} \cup S$, pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

Lemma 3.1 Let $G = Z_{2m} = \langle a \rangle, (m \geq 5)$, and $S = \{a^i, a^{-i}, a^{m+i}, a^{m-i}, a, a^{-1}\}, 2 \leq i < \frac{m}{2}$. Then $\Gamma = \text{Cay}(G, S)$ is normal.

Proof Let $\Gamma_2(1)$ be the subgraph of Γ with vertex set $\{1\} \cup S \cup S^2$ and edge set $\{[1, s], [s, st] \mid s, t \in S\}$. By observing the subgraph $\Gamma_2(1)$, it is easy to prove that A_1^* fixes S^2 pointwise, which implies that $A_1^* = 1$. Thus A_1 acts faithfully on S . Observing the subgraph $\Gamma_2(1)$ again, A_1 , as a permutation group on S , is generated by $(a, a^{-1})(a^{m+1}, a^{m-1})$. So $|A_1| = 2$ and $\Gamma = \text{Cay}(G, S)$ is normal.

Lemma 3.2: Let $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $m = 4k$, $k \geq 2$ and $S = \{a, b, c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma = \text{Cay}(G, S)$ is normal.

Proof Set $G_1 = \langle a, b \rangle$, $G_2 = \langle c \rangle$, $S_1 = \{a, b\}$, $S_2 = \{c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma_1 = \text{Cay}(G_1, S_1) \cong K_2 \times K_2$. Note that Γ_1 and $\Gamma_2 = \text{Cay}(G_2, S_2)$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], Γ_1 and Γ_2 are normal and by Lemma 2.3, $\Gamma = \text{Cay}(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let G and S be as the following. Then the Cayley graphs $\Gamma = \text{Cay}(G, S)$ are normal.

$$(1): G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle, \\ S = \{a, b, c, d, ad, abc\}.$$

$$(2): G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \\ S = \{a, b, ac^3, c^3, c, c^{-1}\}.$$

$$(3): G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, b, abc^m, c^m, c, c^{-1}\}.$$

$$(4): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, ab^m, ab^{3m}, ab^{5m}, b, b^{-1}\}.$$

$$(5): G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2), \\ S = \{a, b^m, b^{3m}, b^{5m}, b, b^{-1}\}.$$

$$(6): G = Z_6 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3), \\ S = \{a, a^{-1}, a^3, a^3b^m, b, b^{-1}\}.$$

$$(7): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle, \\ S_1 = \{a, ab^2, ab^{-2}, b^m, b, b^{-1}\}, \quad (m \geq 4), \\ S_2 = \{a, ab^{m+2}, ab^{m-2}, ab^m, b, b^{-1}\}, \quad (m \geq 5), \\ S_3 = \{a, b^{m+2}, b^{m-2}, b^m, b, b^{-1}\}, \quad (m = 4, m \geq 6).$$

$$(8): G = Z_2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle, \\ S_1 = \{a, ab^m, ab^{3m+2}, b^{2m+1}, b, b^{-1}\}, \\ (m \geq 2), S_2 = \{a, b, b^{-1}, b^m, b^{3m+2}, b^{2m+1}\}, \quad (m \geq 2), \\ S_3 = \{a, ab^{m+1}, ab^{3m+1}, b^{2m+1}, b, b^{-1}\}, \quad (m \geq 2), \\ S_4 = \{a, b, b^{-1}, b^{m+1}, b^{3m+1}, b^{2m+1}\}, \quad (m \geq 3).$$

$$(9): G = Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \quad (m \geq 1), \\ S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}, \\ S_2 = \{a^2, a^2b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}.$$

$$(10): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, \quad (m \geq 4, m \neq 6), \\ S = \{a, b, c, c^{-1}, ac^2, ac^{-2}\}.$$

$$(11): G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2),$$

$$S_1 = \{a, ab^m, ab^{3m}, b^{2m}, b, b^{-1}\}, \\ S_2 = \{a, b, b^{-1}, b^m, b^{3m}, b^{2m}\}, \\ S_3 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}.$$

$$(12): G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 3), \\ S = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}.$$

$$(13): G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2),$$

$$S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}, S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}, \\ S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}, \\ S_4 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}.$$

$$(14): G = Z_2^3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2), \\ S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}.$$

$$(15): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m = 7, 9, m \geq 11), \\ S = \{a, b, c, c^{-1}, c^3, c^{-3}\}.$$

$$(16): G = Z_2 \times Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 1), \\ S = \{a, b^2c^{2m+1}, bc^m, b^3c^{3m+2}, c, c^{-1}\}.$$

$$(17): G = Z_2^2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 2), \\ S = \{a, c^{2m+1}, bc^m, bc^{3m+2}, c, c^{-1}\}.$$

$$(18): G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3), \\ S = \{a, ac^m, b, b^{-1}, c, c^{-1}\}.$$

$$(19): G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \quad (m \geq 6), \\ S_1 = \{a, b^m, b, b^{-1}, b^3, b^{-3}\}, \\ S_2 = \{a, ab^m, b, b^{-1}, b^3, b^{-3}\}.$$

$$(20): G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m \geq 2, n \geq 3), \\ S = \{a, a^{-1}, a^m, a^{3m}, b, b^{-1}\}.$$

$$(21): G = Z_{4m} \times Z_{4n} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m, n \neq 4), \\ S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}.$$

$$(22): G = Z_4 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m, n \neq 3), \\ S = \{a, a^3, b, b^{-1}, c, c^{-1}\}.$$

$$(23): G = Z_{2m} \quad (m \geq 5),$$

$$S = \{a, a^{-1}, a^j, a^{-j}, a^{m+j}, a^{m-j}\} \quad (2 \leq j < \frac{m}{2}).$$

$$(24): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m = 7, 9, m \geq 11, n \geq 3), \\ S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}.$$

$$(25): G = Z_{3m-1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle \quad (m \geq 2, n \geq 1), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^n, a^{2m-1}b^{2n}\}.$$

$$(26): G = Z_{3m+1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 1), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^{2n}, a^{2m+1}b^n\}.$$

$$(27): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m \geq 5, n \geq 3), \\ S = \{a, a^{-1}, b, b^{-1}, a^2b, a^{-2}b^{-1}\}.$$

$$(28): G = Z_{2m+1} \times Z_n = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 3), \\ S = \{a, a^{-1}, a^m, a^{m+1}, b, b^{-1}\}.$$

$$(29): G = Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle \quad (m, n \geq 2), \\ S = \{a, a^{-1}, b, b^{-1}, a^mb^{n+1}, a^{m+1}b^n\}.$$

(30): $G = Z_2 \times Z_{2n+1} \times Z_{2m+1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m, n \geq 1$),
 $S = \{ab^m c^{n+1}, ab^{m+1} c^n, b, b^{-1}, c, c^{-1}\}$.

(31): $G = Z_{4m} = \langle a \rangle$ ($m \geq 2$),
 $S = \{a, a^{-1}, a^k, a^{-k}, a^m, a^{-m}\}$, ($1 < k < 2m, k \neq m, 2m-1$).

(32): $G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle$ ($m \geq 3$),
 $S = \{a, a^{-1}, b, b^{-1}, ab^j, a^{-1}b^{-j}\}$, $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$,

(When $m \neq 2k$ for every j or $m = 2k, j \neq k$).

(33): $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ ($m \geq 2$),
 $S = \{a, a^{-1}, b, b^{-1}, a^2 b^j, a^2 b^{-j}\}$ $1 \leq j < m$
 (for every $j \neq 1, m-1$).

(34): $G = Z_4 \times Z_{2m-1} = \langle a \rangle \times \langle b \rangle$ ($m \geq 2$),
 $S = \{a, a^{-1}, b, b^{-1}, a^2 b^j, a^2 b^{-j}\}$ ($1 < j < \frac{2m-1}{2}$).

(35): $G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle$ ($m \geq 5$),
 $S = \{a, a^{-1}, b, b^{-1}, b^j, b^{-j}\}$ ($1 < j < \frac{m}{2}$),
 when $m \neq 2k, 5$ or $m = 2k$ ($k \geq 3, k \neq 5$), $j \neq k-1$
 or $m = 10, j \neq 3$.

(36): $G = Z_{2m} = \langle a \rangle$ ($m \geq 4$),
 $S = \{a, a^{-1}, a^j, a^{-j}, a^{m+1}, a^{m-1}\}$ ($2 \leq j \leq m-2$),
 when $(m, j) = 1$ or $(m, j) = 2, m \neq 4i+2$ ($i \geq 1$).

(37): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle$ ($m \geq 5, m \neq 8$),
 $S_1 = \{ab, ab^{-1}, b, b^{-1}, b^j, b^{-j}\}$,
 $S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^j, ab^{-j}\}$ ($2 \leq j < \frac{m}{2}$), when
 $(m, j) = p \leq 2$.

(38): $G = Z_2 \times Z_8 = \langle a \rangle \times \langle b \rangle$,
 $S_1 = \{ab, ab^7, b, b^7, b^2, b^6\}$,
 $S_2 = \{ab, ab^7, b, b^7, ab^2, ab^6\}$.

(39): $G = Z_m = \langle a \rangle$ ($m \geq 9, m \neq 14$),
 $S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ $j \neq 3, 2 \leq j < \frac{m}{2}$ when
 $m \neq 6k, \forall j$ or $m = 6k, j \neq 3k-1$.

(40): $G = Z_{14} = \langle a \rangle$,
 $S = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$ for $j = 2, 4, 6$.

(41): $G = Z_m = \langle a \rangle$ ($m \geq 7$),
 $S = \{a, a^{-1}, a^{3j}, a^{-3j}, a^j, a^{-j}\}$, ($2 \leq j < \frac{m}{2}, 3j \not\equiv 0, 1,$

$m-1, j, m-j, \frac{m}{2} \pmod{m}$), when $m \neq 7, 14, 6k$
 ($k \geq 2$) and $m = 7; j = 2$ or $m = 14; j = 2, 3, 4, 6$ or $m = 6k; j \neq 3k-1$.

(42): $G = Z_m = \langle a \rangle$ ($m \geq 8, m \neq 14$),
 $S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ (if $m = 2k$ then $2 \leq j \leq \frac{m}{2}-3$ and if $m = 2k+1$ then $2 \leq j \leq \frac{m}{2}-1$). When $m \neq 3k$ for every j and when $m = 3k$, for k odd ; $j \neq k-1$
 and for k even ; $j \neq k-1, 3\frac{k}{2}-3$.

(43): $G = Z_{14} = \langle a \rangle$,
 $S = \{a, a^{-1}, a^{2+j}, a^{-2-j}, a^j, a^{-j}\}$ for $j = 2, 4$.

(44): $G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$),
 $S = \{a, ab^2 c^m, b, b^{-1}, c, c^{-1}\}$.

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and [6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that Γ is not normal. In view of Proposition 2.2, we have the following assumption: $\exists s, t, u, v \in S$ such that $st = ub \neq 1$ but $\{s, t\} \neq \{u, v\}$. (*).

We divide S into four cases:

Case 1: $S = \{a, b, c, d, e, f\}$, where a, b, c, d, e, f are involutions. In this case G is an elementary abelian 2-group and a, b, c, d, e, f are not independent by the assumption (*). Consequently $G = Z_2^3$ or $G = Z_2^4$ or $G = Z_2^5$. If $G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ by the assumption (*) we can let $S = \{a, b, c, ab, ac, abc\}$. We have $\sigma = (a, abc) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; and by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (33) of Theorem 1.1. If $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ by the assumption (*) we see that S is one of the following cases:

- (i) $S_1 = \{a, b, c, d, abc, abd\}$,
- (ii) $S_2 = \{a, b, c, d, ab, abc\}$,
- (iii) $S_3 = \{a, b, c, d, abc, abc\}$.

When $S = S_1, \sigma = (a, b) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (34) of Theorem 1.1. When $S = S_2$, we have the Case (22) of the main theorem. Also when $S = S_3, \Gamma$ is normal by Lemma 3.3. If $G = Z_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$ we can let $S = \{a, b, c, d, e, abc\}$ and hence $\Gamma = \text{Cay}(G, S)$ is non-normal, the Case (1) of Theorem 1.1.

Case 2: $S = \{a, b, c, d, e, e^{-1}\}$, where a, b, c, d are involutions but e is not. In this case, $S^2 - 1 = \{ab, ac, ad, ae, ae^{-1}, bc, bd, be, be^{-1}, cd, ce, ce^{-1}, de, de^{-1}, e^2, e^{-2}\}$. By the assumption (*) $d = abc, o(e) = 4$ or $d = e^3$. Suppose $d = abc$. Then $G = Z_2^2 \times Z_{2m}, (m \geq 2)$ or

$G = Z_2^3 \times Z_m, (m \geq 3)$.

If $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, ($m \geq 2$), we can let $S = \{a, b, ac^m, bc^m, c, c^{-1}\}$ or $S = \{a, b, c^m, abc^m, c, c^{-1}\}$. When $S = \{a, b, ac^m, bc^m, c, c^{-1}\}$, $\sigma = (ab, abc^m)(abc, abc^{m+1}) \dots (abc^{m-1}, abc^{2m-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (35) of the main theorem. When $S = \{a, b, c^m, abc^m, c, c^{-1}\}$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(3). If $G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, ($m \geq 3$), $S = \{a, b, c, abc, d, d^{-1}\}$, the Case (2) of Theorem 1.1. Suppose $o(e) = 4$. Then $G =$

$Z_2^2 \times Z_4$, $Z_2^3 \times Z_4$ or $Z_2^4 \times Z_4$. If $G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, we have S is one of the following cases:
 $S_1 = \{a, b, ab, ac^2, c, c^{-1}\}$, $S_2 = \{a, b, ae^2, bc^2, c, c^{-1}\}$,
 $S_3 = \{a, b, ac^2, abc^2, c, c^{-1}\}$,
 $S_4 = \{a, b, ab, c^2, c, c^{-1}\}$,
 $S_5 = \{a, b, ac^2, c^2, c, c^{-1}\}$,
 $S_6 = \{a, b, abc^2, c^2, c, c^{-1}\}$.

When $S = S_1$, $\sigma = (ac^2, c)(ac, c^2)(bc, abc^2)(abc, bc^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (36 – S_1) of Theorem 1.1. When $S = S_2$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (35, $m = 2$) of Theorem 1.1. When $S = S_3$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.4, $\Gamma = \text{Cay}(G, S)$ is not normal the Case (36 – S_2) of Theorem 1.1. When $S = S_4$, we have the Case (3) of Theorem 1.1. When $S = S_5$, we have the Case (23) of Theorem 1.1. When $S = S_6$, Γ is normal by Lemma 3.3 (3, $m=2$). If $G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, we have $S = \{a, b, c, d, d^{-1}, u\}$, where $u = d^2, ab, ad^2, abc, abd^2$ or $abcd^2$. When $u = d^2$, we have the Case (5– S_1) of Theorem 1.1. When $u = ab$, we have the Case (5 – S_2) of Theorem 1.1. When $u = ad^2$, we have the Case (5 – S_3) of Theorem 1.1. When $u = abc$, we have the Case (2) of Theorem 1.1. When $u = abd^2$, we have the Case (24) of Theorem 1.1. When $u = abcd^2$, $\sigma = (abcd^2, d)(bcd^2, ad)(acd^2, bd)(abd^2, cd)(abcd, d^2)(cd^2, abd)(bd^2, acd)$ and $(bcd, ad^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (37) of Theorem 1.1. If $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \{a, b, c, d, e, e^{-1}\}$, we have the Case (4) of Theorem 1.1. Now suppose $d = e^3$. Then $G = Z_2^2 \times Z_6$ or $G = Z_2^3 \times Z_6$. If $G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, we see that S is one of the following cases: $S_1 = \{a, b, ab, c^3, c, c^{-1}\}$, $S_2 = \{a, b, ac^3, c^3, c, c^{-1}\}$, $S_3 = \{a, b, abc^3, c^3, c, c^{-1}\}$. When $S = S_1$, we have the Case (6) of Theorem 1.1. For S_2 and S_3 , we have the Cases (2) and (3, $m = 3$) of Lemma 3.3 respectively. If $G = Z_2^3 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, then $S = \{a, b, c, d^3, d, d^{-1}\}$, the Case (7) of Theorem 1.1.

Case 3: $S = \{a, b, c, c^{-1}, d, d^{-1}\}$, where a, b are involutions but c, d are not. By the assumption (*) and the symmetry of c, c^{-1}, d and d^{-1} , we have five sub cases (I) $a = c^3$, (II) $a = c^2d$, (III) $o(c) = 4$, (IV) $c^3 = d$ and (V) $c^2 = d^2$. Suppose $a = c^3$, then G is isomorphic to one of the following: $Z_2 \times Z_{6m}$ ($m \geq 2$), $Z_2 \times Z_6$, $Z_6 \times Z_{2m}$ ($m \geq 2$), $Z_2^2 \times Z_{3m}$ ($m \geq 1$), $Z_2 \times Z_6 \times Z_m$ ($m \geq 3$). If $Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle$, ($m \geq 2$), we see that S is one of the following cases:
 $S_1 = \{a, b^{3m}, ab^{2m}, ab^{4m}, b, b^{-1}\}$, $S_2 = \{a, ab^{3m}, ab^m, ab^{5m}, b, b^{-1}\}$, $S_3 = \{a, b^{3m}, b^m, b^{5m}, b, b^{-1}\}$. When $S = S_1$, $\sigma = (a, ab^{2m}, ab^{4m})(ab, ab^{2m+1}, ab^{4m+1}) \dots (ab^{2m-1}, ab^{4m-1}, ab^{6m-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by

Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (38) of the main theorem. For the Cases $S = S_2$ and $S = S_3$, we have the Cases (4) and (5) of Lemma 3.3. If $G = Z_2 \times Z_6 = \langle a \rangle \times \langle b \rangle$, we see that S is one of the following cases:

$S_1 = \{a, b^3, ab^2, ab^4, b, b^{-1}\}$, $S_2 = \{a, b^3, b, b^{-1}, b^2, b^4\}$,
 $S_3 = \{a, b^3, b, b^{-1}, ab, ab^{-1}\}$.
 When $S = S_1$, $\sigma = (a, ab^2, ab^4)(ab, ab^3, ab^5) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 – S_5) of Theorem 1.1. When $S = S_2$, we have the Case (29, $m=3$) of Theorem 1.1. When $S = S_3$, $\sigma = (b^5, ab^3)(b^2, ab^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 – S_1) of Theorem 1.1. If $G = Z_6 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$, we see that S is one of the following cases:

$S_1 = \{a^3, b^m, a, a^{-1}, b, b^{-1}\}$, $S_2 = \{a^3, a^3b^m, a, a^{-1}, b, b^{-1}\}$.
 When $S = S_1$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (8) of Theorem 1.1. For $S = S_2$, when $m = 2$, $\sigma = (b^2, a^3b)(ab^2, a^4b)(a^2b^2, a^5b)(a^3b^2, b)(a^4b^2, ab)(a^5b^2, a^2b) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (40, $m=3$) of Theorem 1.1, and when $m \geq 3$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(6). If $G = Z_2^2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$), $S = \{a, b, ac^m, ac^{2m}, c, c^{-1}\}$. Then we obtain the Case (25) of Theorem 1.1. If $G = Z_2 \times Z_6 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$), $S = \{b^3, a, b, b^{-1}, c, c^{-1}\}$. Then we obtain the Case (9) of Theorem 1.1. Suppose $a = c^2d$. Then we have one of the following cases:
 (1): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ ($m \geq 3$),
 $S = \{a, b^m, b, b^{-1}, ab^{-2}, ab^2\}$.

(2): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$,
 $S_1 = \{ab^m, a, b, b^{-1}, ab^{m-2}, ab^{m+2}\}$ ($m \geq 3$),
 $S_2 = \{b^m, a, b, b^{-1}, b^{m-2}, b^{m+2}\}$, $m \geq 4$,

(3): $G = Z_2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle$,
 $S_1 = \{a, b, b^{-1}, b^{2m+1}, ab^m, ab^{3m+2}\}$ ($m \geq 1$),
 $S_2 = \{a, b, b^{-1}, b^{2m+1}, b^m, b^{3m+2}\}$, $m \geq 2$
 $S_3 = \{a, b, b^{-1}, b^{2m+1}, b^{3m+1}, b^{m+1}\}$ ($m \geq 1$),
 $S_4 = \{a, b^{2m+1}, ab^{3m+1}, ab^{m+1}, b, b^{-1}\}$, $m \geq 1$,

(4): $G = Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle$,
 $S_1 = \{a^2b^{2m+1}, b^{2m+1}, ab^m, a^3b^{3m+2}, b, b^{-1}\}$, $m \geq 1$
 $S_2 = \{a^2b^{2m+1}, a^2, ab^m, a^3b^{3m+2}, b, b^{-1}\}$, $m \geq 1$.

(5): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$),
 $S = \{a, b, c, c^{-1}, ac^{-2}, ac^2\}$.

(6): $G = Z_2 \times Z_4 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$),
 $S = \{a, b^2c^{2m+1}, bc^m, b^{-1}c^{-m}, c, c^{-1}\}$.

(7): $G = Z_2^2 \times Z_{4m+2} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$),
 $S = \{a, c^{2m+1}, bc^m, bc^{-m}, c, c^{-1}\}$.

In the Case (1), when $m = 3$, $\sigma = (b^2, b^4) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not

normal, the Case (43– S_5 , $m = 3$) of Theorem 1.1. When $m \geq 4$, Γ is normal by Lemma 3.3(7– S_1).

In the Case (2), $S = S_1$ when $m = 3$, $\sigma = (b^2, ab^2)(b^5, ab^5) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43– S_2) of Theorem 1.1.

When $m = 4$, $\sigma = (b, b^7)(b^2, b^6)(b^3, b^7) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case 39 ($m = 2$) of Theorem 1.1. When $m \geq 5$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (7– S_2). In the Case (2), $S = S_2$, when $m = 5$, we have the Case (26) of Theorem 1.1. When $m \geq 6$, Γ is normal by Lemma 3.3 (7– S_3).

In the Case (3), $S = S_1$, when $m = 1$, we have the Case (43 – S_1) of Theorem 1.1. When $m \geq 2$, Γ is normal by Lemma 3.3 (8 – S_1). In the Case (3), $S = S_2$, Γ is normal by Lemma 3.3 (8 – S_2). In the Case (3), $S = S_3$, when $m = 1, 2$, we have the Cases (29, $m = 3, 5$) of Theorem 1.1 respectively. When $m \geq 3$, Γ is normal by Lemma 3.3(8 – S_4). In the Case (3), $S = S_4$, when $m = 1$, $\sigma = (ab, ab^5) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (29, $m = 3$) of Theorem 1.1. When $m \geq 2$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(8 – S_3). In the Case (4), $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(9). In the Case (5), when $m = 3, 6$, by Proposition 2.1, Γ is not normal, the Case (25, $m = 1, 2$) of Theorem 1.1. Otherwise Γ is normal by Lemma 3.3(10). In the Case (6), Γ is normal by Lemma 3.3(16). In the Case (7), when $m = 1$, by Proposition 2.1, Γ is not normal, the Case 27 ($m = 1$) of Theorem 1.1. When $m \geq 2$, Γ is normal by Lemma 3.3 (17). Suppose $o(c) = 4$. Then we have one of the following cases:

(I) $G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle$, $S_1 = \{a, b^2, b, b^{-1}, ab, ab^{-1}\}$,

(II) $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle$, $S_1 = \{a, b^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$, ($m \geq 2$), $S_2 = \{a, ab^{2m}, ab^m, ab^{3m}, b, b^{-1}\}$, ($m \geq 1$), $S_3 = \{a, b^{2m}, b^m, b^{3m}, b, b^{-1}\}$, ($m \geq 2$), $S_4 = \{a, ab^{2m}, b^m, b^{3m}, b, b^{-1}\}$, ($m \geq 2$).

(III) $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ ($m \geq 2$), $S_1 = \{a^2, b^m, a, a^{-1}, b, b^{-1}\}$, $S_2 = \{a^2, a^2b^m, a, a^{-1}, b, b^{-1}\}$, $S_3 = \{a^2b^m, b^m, a, a^{-1}, b, b^{-1}\}$.

(IV): $G = Z_2^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$, $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$.

(V): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S_1 = \{a, b, abc^m, abc^{3m}, c, c^{-1}\}$, $S_2 = \{a, b, ac^m, ac^{3m}, c, c^{-1}\}$, $S_3 = \{a, b, c^m, c^{3m}, c, c^{-1}\}$.

(VI): $G = Z_2 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$), $S_1 = \{a, b^2, b, b^{-1}, c, c^{-1}\}$, $S_2 = \{a, ab^2, b, b^{-1}, c, c^{-1}\}$.

(VII): $G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S_1 = \{a, c^m, b, b^{-1}, c, c^{-1}\}$, $S_2 = \{a, ac^m, b, b^{-1}, c, c^{-1}\}$, $S_3 = \{a, b^2c^m, b, b^{-1}, c, c^{-1}\}$, $S_4 = \{a, ab^2c^m, b, b^{-1}, c, c^{-1}\}$.

(VIII): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$),

$S_1 = \{a, c^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$, $S_2 = \{a, ac^{2m}, bc^m, bc^{3m}, c, c^{-1}\}$.

(IX): $G = Z_2^2 \times Z_4 \times \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ ($m \geq 3$), $S = \{a, b, c, c^{-1}, d, d^{-1}\}$.

(X): $G = Z_2^3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ ($m \geq 1$), $S = \{a, b, cd^m, cd^{3m}, d, d^{-1}\}$.

In the Case (I), $\sigma = (ab, b^3) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 – S_1) of Theorem 1.1. In the Case (II), $S = S_1$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(11 – S_1). In the Case (II), $S = S_2$, $\sigma = (b, b^{-1})(b^2, b^{-2}) \dots (b^{2m-1}, b^{2m+1})(a, ab^m) \dots (ab^{2m+1}, ab^{-(m+1)}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (39) of Theorem 1.1. In the Case (II), $S = S_3$, and $S = S_4$, Γ is normal by Lemma 3.3, the Case (11 – S_2, S_3). In the Case (III), when $S = S_1$, we have the Case (10) of Theorem 1.1. When $S = S_2$, $m = 2$, $\sigma = (a^2b^2, b)(a^3b^2, ab)(ab^2, a^3b)(b^2, a^2b) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (40, $m = 2$) of Theorem 1.1. When $S = S_2$, $m \geq 3$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(12). When $S = S_3$, $\sigma = (a^2, ab^m)(a^2b, ab^{m+1}) \dots (a^2b^{2m-1}, ab^{m+(2m-1)}) \in A_1$ but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (40) of Theorem 1.1.

In the Case (IV), when $S = S_1$, $\sigma = (c^2, ac^2)(bc^2, abc^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44– S_2) of Theorem 1.1. When $S = S_2$, $\sigma = (ac^2, bc^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44– S_3) of Theorem 1.1. In the Case (V), $S = S_1$, when $m = 1$, with an argument similar to the Case (IV – S_2) we obtain the same result. When $m \geq 2$, Γ is normal by Lemma 3.3 (13– S_1). In the Case (V), $S = S_2$, when $m = 1$, with an argument similar to the Case (IV– S_1), we obtain the same result.

When $m \geq 2$, Γ is normal by Lemma 3.3 (13 – S_2). In the Case (V), $S = S_3$, Γ is normal by Lemma 3.3(13– S_3). In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), $S = S_1$, $S = S_3$ and $S = S_2$ ($m = 2$), we have the Cases (12), (28) and (11 – S_2 , $m = 4$) of Theorem 1.1 respectively. In the Case (VII), $S = S_2$, $m \geq 3$, Γ is normal by Lemma 3.3(18). In the Case (VII), $S = S_4$, for $m = 2$, $\sigma = (b^3, c)(ab^3, ac)(abc^2, ab^2c^3)(b^2, bc)(b^3c^3, c^2)(b^2c, b^2c^3)(ab^2, abc)(ab^3c^3, ac^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (57) of Theorem 1.1, and for $m \geq 3$, Γ is normal by Lemma 3.3(44). In the Case (VIII), $S = S_1$ when $m = 1$, we have the Case (21, $m = 2$) of Theorem 1.1. If $m \geq 2$, Γ is normal by Lemma 3.3 (13 – S_4). In the Case (VIII), $S = S_2$, $\sigma = (ab, abc^{2m})(abc, abc^{2m+1}) \dots (abc^{2m-1}, abc^{4m-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (41) of Theorem 1.1. In the Case

(IX), we have the Case (13) of Theorem 1.1. In the Case (X), $m = 1$, we have the Case (14) of Theorem 1.1, and for $m \geq 2$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(14). Suppose $c^3 = d$, then $G = Z_2^2 \times Z_{2m}$, ($m \geq 4$) or $G = Z_2^2 \times Z_m$ ($m \geq 5$, $m \neq 6$). If $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ ($m \geq 4$), we can let S to be $S_1 = \{a, b^m, b, b^{-1}, b^3, b^{-3}\}$ or $S_2 = \{a, ab^m, b, b^{-1}, b^3, b^{-3}\}$. Let $S = S_1$, for $m = 4, 5$ we have the Cases (29), (26) of Theorem 1.1 respectively, and for $m \geq 6$, Γ is normal by Lemma 3.3(19 - S_1). Let $S = S_2$. When $m = 4$, $\sigma = (ab^2, ab^6) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_4), $m = 4$) of Theorem 1.1. When $m = 5$, $\sigma = (b^3, b^7)(ab^3, ab^7)(b^2, b^8)(ab^2, ab^8) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (42) of Theorem 1.1. When $m \geq 6$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(19 - S_2). If $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 5$, $m \neq 6$), $S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$. When $m = 5, 10$ and $m = 8$ we have the Cases (15), and (16) of Theorem 1.1 respectively. When $m = 7, 9, m \geq 11$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(15). Suppose $c^2 = d^2$, then $G = Z_2 \times Z_{2m}$, $G = Z_2^2 \times Z_{2m}$ ($m \geq 3$) $G = Z_2^2 \times Z_{2m-1}$ ($m \geq 2$) or $G = Z_2^2 \times Z_m$ ($m \geq 3$). If $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ we see that S is one of the following cases:

- 1) $S_1 = \{a, b^m, b, b^{-1}, ab, ab^{-1}\}$, $m \geq 2$,
- 2) $S_2 = \{a, ab^m, b, b^{-1}, ab, ab^{-1}\}$, $m \geq 2$,
- 3) $S_3 = \{a, b^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$, $m \geq 3$,
- 4) $S_4 = \{a, ab^m, b, b^{-1}, b^{m+1}, b^{m-1}\}$, $m \geq 3$,
- 5) $S_5 = \{a, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$, $m \geq 3$,
- 6) $S_6 = \{a, ab^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$, $m \geq 3$,
- 7) $S_7 = \{ab^m, b^m, b, b^{-1}, ab, ab^{-1}\}$, $m \geq 2$,
- 8) $S_8 = \{ab^m, b^m, b, b^{-1}, ab^{m+1}, ab^{m-1}\}$, $m \geq 2$.

In the Case (1), $m \geq 2$, when $m = 2i$, $\sigma = (b^i, ab^i)(b^{3i}, ab^{3i}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and when $m = 2i + 1$, $\sigma = (b^{i+1}, ab^{i+1})(b^{3i+2}, ab^{3i+2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_1) of Theorem 1.1. In the Case (2), similarly Case (1), $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_2) of Theorem 1.1. In the Case (3), we have the Case (29) of Theorem 1.1. In the Case (4), when $m = 2i$, $\sigma = (ab^i, ab^{3i}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and when $m = 2i + 1$, $\sigma = (ab^{i+1}, ab^{3i+2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_4) of Theorem 1.1. In the Case (5), when $m = 2i$, $\sigma = (b^{3i}, ab^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and when $m = 2i + 1$, $\sigma = (b^{i+1}, ab^{3i+2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_5) of Theorem 1.1. In the Case (6), when $m = 2i$, $\sigma = (b^i, ab^{3i})(b^{3i}, ab^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and when $m = 2i + 1$, $\sigma = (b^{i+1}, ab^{3i+2})(b^{3i+2}, ab^{i+1}) \in A_1$,

but $\sigma \notin \text{Aut}(G, S)$. Hence by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_6) of Theorem 1.1.

In the Case (7), for $m = 2i$ and $m = 2i + 1$, $\sigma = (b^{i+1}, ab^{i+1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_3) of Theorem 1.1. In the Case (8), for $m = 2i$ and $m = 2i - 1$, $\sigma = (b^i, ab^{i+m})(b^{m+i}, ab^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 - S_1) of Theorem 1.1. If $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, we can let S to be one of the following cases:

- (1): $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$, $m \geq 2$,
- (2): $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$, $m \geq 2$,
- (3): $S_3 = \{a, b, c, c^{-1}, c^{m+1}, c^{m-1}\}$, $m \geq 3$,
- (4): $S_4 = \{a, b, c, c^{-1}, ac^{m+1}, ac^{m-1}\}$, $m \geq 2$,
- (5): $S_5 = \{a, b, c, c^{-1}, abc^{m+1}, abc^{m-1}\}$, $m \geq 2$,
- (6): $S_6 = \{a, cm, c, c^{-1}, bc, b c^{-1}\}$, $m \geq 2$,
- (7): $S_7 = \{a, ac^m, c, c^{-1}, bc, bc^{-1}\}$, $m \geq 2$,
- (8): $S_8 = \{a, c^m, c, c^{-1}, bc^{m+1}, bc^{m-1}\}$, $m \geq 2$,
- (9): $S_9 = \{a, ac^m, c, c^{-1}, bc^{m+1}, bc^{m-1}\}$, $m \geq 2$.

In the Case (1), Γ is not normal, the Case (30) of Theorem 1.1. In the Case (2), $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_1) of Theorem 1.1. In the Case (3), when $m = 2i$, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (16) of Theorem 1.1.

When $m = 2i + 1$, $\Gamma = \text{Cay}(G, S)$ is not normal, we have the Case 14 (with m odd) of Theorem 1.1. In the Case (4), when $m = 2i$, $i \geq 2$, $\sigma = (c^i, ac^{3i})(ac^i, c^{3i})(bc^i, abc^{3i})(abc^i, bc^{3i}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and when $m = 2i + 1$, $\sigma = (c^{i+1}, ac^{3i+2})(ac^{i+1}, c^{3i+2})(bc^{i+1}, abc^{3i+2})(abc^{i+1}, bc^{3i+2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_2) of Theorem 1.1. In the Case (5), when $m = 2i$, $i \geq 2$, $\sigma = (c^{3i}, abc^i)(ac^{3i}, bc^i)(bc^{3i}, ac^i)(abc^{3i}, c^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and when $m = 2i + 1$, $\sigma = (c^{3i+2}, abc^{i+1})(ac^{3i+2}, bc^{i+1})(bc^{3i+2}, ac^{i+1})(abc^{3i+2}, c^{i+1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_3) of Theorem 1.1. In the Case (6), $m \geq 2$, Γ is not normal, we have the Case (27) of Theorem 1.1. In the Case (7), if $m \geq 3$, for $m = 2i$ and $m = 2i - 1$, $\sigma = (ci, bci)(aci, abc^i)(ci+m, bci+m)(aci+m, abc^i+m) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and if $m = 2$, $\sigma = (b, bc^2)(ab, abc^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$. Then by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_4) of Theorem 1.1. In the Case (8), for $m = 2i$ and $m = 2i - 1$, $\sigma = (c^i, bc^{i+m})(ac^i, abc^{i+m})(c^{i+m}, bc^i)(ac^{i+m}, abc^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_5) of Theorem 1.1. In the Case (9), similarly Case (8), $\Gamma = \text{Cay}(G, S)$ is not normal. We have the Case (44 - S_6) of Theorem 1.1.

If $G = Z_2^2 \times Z_{2m-1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, ($m \geq 2$), then S is $S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}$ or $S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}$. When $S = S_1$, $\sigma = (cm, acm)(bcm, abcm) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44 - S_7) of the main theorem.

When $S = S_2$, $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (44- S_1) of Theorem 1.1. If $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, we can consider $m \geq 3$, $S = \{a, b, d, d^{-1}, cd, cd^{-1}\}$. In this case for $m = 2i$ and $m = 2i-1$, ($i \geq 2$) $\sigma = (d^i, cd^i)(ad^i, acd^i)(bd^i bcd^i)(abd^i, abcd^i) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$ and by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal the Case (14) of Theorem 1.1.

Case 4: $S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$, where the elements of the set S are not involution. By the assumption (*), $o(a) = 4$, $a^2 = b^2$, $a^3 = b$ or $c = a^2b$. Suppose $o(a) = 4$, then G is isomorphic to one of the following: Z_{4m} ($m \geq 2$), $Z_4 \times Z_m$, $Z_{4m} \times Z_n$ ($m \geq 2, n \geq 3$), $Z_{4m} \times Z_{4n}$ ($m \geq 1, n \geq 1$), $Z_4 \times Z_m \times Z_n$ ($m, n \geq 3$). If $G = Z_{4m} = \langle a \rangle$ ($m \geq 2$), we can let $S = \{a^m, a^{-m}, a, a^{-1}, a^j, a^{-j}\}$, where $1 < j < 2m$, $j \neq m$. When $j = 2m - 1$, $\sigma = (a^m, a^{-m}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (45) of Theorem 1.1. When $j \neq 2m - 1$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(31). If $G = Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle$, we can let S to be one of the following cases:

- (1): $S_1 = \{a, a^3, b, b^{-1}, ab^j, a^3b^{-j}\}$, $m \geq 3, 1 \leq j \leq \lfloor m/2 \rfloor$,
- (2): $S_2 = \{a, a^3, b, b^{-1}, a^2b^j, a^2b^{-j}\}$, $m \geq 2, 1 \leq j \leq (m/2)$,
- (3): $S_3 = \{a, a^3, b, b^{-1}, b^j, b^{-j}\}$, $m \geq 5, 1 < j < (m/2)$.

When $S = S_1$, for $m = 2j$, $\sigma = (a^2, a^2b^j)(a^2b, a^2b^{j+1}) \dots (a^2b^{j-1}, a^2b^{2j-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (49) of the main theorem. Otherwise, Γ is normal by Lemma 3.3(32). When $S = S_2$, $j = 1$ for $m = 2k$ and $m = 2k - 1$, $k \geq 2$, $\sigma = (ab^k, a^3b^k) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and when $j = k - 1$, $m = 2k$ ($k \geq 3$), $\sigma = (b^{k-1}, a^2b^{-1})(ab^{k-1}, a^3b^{-1})(a^2b^{k-1}, b^{-1})(a^3b^{k-1}, ab^{-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, then these graphs are non-normal and we have the Cases (49, 50) of Theorem 1.1. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (33, 34). When $S = S_3$, for $j = k - 1$, $m = 2k$, if k is odd we have the Case (17) of Theorem 1.1 and if k is even we have the Case 19 ($m = 4$) of the main theorem. For $m = 5$; $j = 2$ and $m = 10$; $j = 3$ we have the Case 21 ($m = 4$) of the main theorem.

Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (35). If $G = Z_{4m} \times Z_n = \langle a \rangle \times \langle b \rangle$ ($m \geq 2, n \geq 3$), $S = \{a^m, a^{-m}, a, a^{-1}, b, b^{-1}\}$, then $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(20). If $G = Z_{4m} \times Z_{4n} = \langle a \rangle \times \langle b \rangle$ ($m \geq 1, n \geq 1$), $S = \{a^m b^n, a^{-m} b^{-n}, a, a^{-1}, b, b^{-1}\}$, then $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(21). If $G = Z_4 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m, n \geq 3$), we can consider $S = \{a, a^3, b, b^{-1}, c, c^{-1}\}$. In this case, for $m = 4$, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (18) of Theorem 1.1, and for $m, n \neq 4$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(22). Suppose $a^2 = b^2$. Then G is isomorphic to one of the following: $Z_{2m}, Z_2 \times Z_m$ ($m \geq 5$), $Z_{2m} \times Z_{2n+1}, Z_{2m} \times Z_{2n}$ ($m \geq 3, n \geq 2$), $Z_2 \times Z_n$ ($m \geq 3, n \geq 3$). If $G = Z_{2m} = \langle a \rangle$, we can let S to be $S_1 = \{a^j, a^{-j}, a^{m+j}, a^{m-j}, a, a^{-1}\}$,

$2 \leq j \leq m/2, m \geq 5$, or $S_2 = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^j, a^{-j}\}$, $2 \leq j \leq m - 2, m \geq 4$. When $S = S_1$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(23). When $S = S_2$, (m, j) = 2, for $m = 4i + 2, j = 2i$ (with i odd) and $j = 2i + 2$ (with i even), $\sigma = (a^2, a^{2+m/2})(a^6, a^{6+m/2}) \dots (a^{2m-2}, a^{m/2-2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and when (m, j) = 1 > 2, then $\sigma = (a^2, a^{m+2})(a^{2+1}, a^{m+2+1}) \dots (a^{m+2-1}, a^{2-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, then by Proposition 2.1 these graphs are non-normal, and we have the Case (46) of the main theorem. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (36). If $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle$ $m \geq 5$, we can let S to be $S_1 = \{b, b^{-1}, ab, ab^{-1}, b^j, b^{-j}\}$, $2 \leq j > m/2$ or $S_2 = \{b, b^{-1}, ab, ab^{-1}, ab^j, ab^{-j}\}$, $2 \leq j > m/2$. Let $S = S_1$. When (m, j) = $p > 2$; $m = (t + 1)p$, $\sigma = (b, ab)(b^{p+1}, ab^{p+1}) \dots (b^{tp+1}, ab^{tp+1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (47- S_1) of the main theorem.

When $m = 8, j = 3$, $\sigma = (b^2, b^6)(ab, a b^7)(a b^3, a b^5) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (48- S_1) of Theorem 1.1. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(37, 38- S_1). Let $S = S_2$. When (m, j) = $p > 2$; $m = (t + 1)p$, $\sigma = (b, ab)(b^{p+1}, ab^{p+1}) \dots (b^{tp+1}, ab^{tp+1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (47- S_2) of Theorem 1.1. When $m = 8, j = 3$, $\sigma = (b^2, b^6)(b^3, b^5)(b, b^7) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case(48- S_2) of main theorem. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(37, 38- S_2). If $G = Z_{2m} \times Z_n = \langle a \rangle \times \langle b \rangle$, we can let S to be one of the following cases:

- (1): $S_1 = \{a, a^{-1}, a^{m-1}, a^{m-1}, b, b^{-1}\}$, $m \geq 3$,
- (2): $S_2 = \{b, b^{-1}, a^m b, a^m b^{-1}, a, a^{-1}\}$, $m \geq 2$,
- (3): $S_3 = \{b, b^{-1}, a^{m+1} b^l, a^{m-1} b^l, a, a^{-1}\}$, $n = 2l, l \geq 2$.

Let $S = S_1$. When $m = 2i$, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (19) of Theorem 1.1. When $m = 2i + 1$, $\sigma = (a^{m-1}, a^{2m-1})(a^{m-1}b, a^{2m-1}b) \dots (a^{m-1}b^{n-1}, a^{2m-1}b^{n-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.4, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case 20 (with m odd) of Theorem 1.1. Let $S = S_2$. When $n = 2j$, $2j - 1$ ($j \geq 2$), $\sigma = (b^j, a^m b^j)(ab^j, a^{m+1} b^j) \dots (a^{m-1} b^j, a^{2m-1} b^j) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (49) of Theorem 1.1. When $S = S_3$, $\sigma = (a^{m-1}, a^{-1} b^l)(a^{m-1} b, a^{-1} b^{l+1}) \dots (a^{m-1} b^{2l-1}, a^{-1} b^{l-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (50) of Theorem 1.1. If $G = Z_2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $m \geq 3, n \geq 3$, $S = \{b, b^{-1}, ab, ab^{-1}, c, c^{-1}\}$, we have the Case (20) of the main theorem. Suppose $a^3 = b$, then we have one of the following cases :

- (1): $G = Z_m = \langle a \rangle$, $m \geq 7$, $S_1 = \{a, a^{-1}, a^3, a^{-3}, a^j, a^{-j}\}$, ($j \neq 3, 2 \leq j \leq m/2$), $S_2 = \{a^j, a^{-j}, a^{3j}, a^{-3j}, a, a^{-1}\}$, ($2 \leq j \leq m/2, 3j \neq 0, 1, m - 1, j, m - j, m/2 \pmod{m}$).
- (2): $G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle$, ($n \geq 3, m \geq 5, m \neq 6$), $S = \{a, a^{-1}, a^3, a^{-3}, b, b^{-1}\}$.
- (3): $G = Z_{3m-1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle$, ($m \geq 2, n \geq 1$),

$$S = \{a^m b^n, a^{2m-1} b^{2n}, a^3, a, a^{-1}, b, b^{-1}\}.$$

$$(4): G = Z_{3m+1} \times Z_{3n} = \langle a \rangle \times \langle b \rangle, (m, n \geq 1), S = \{a^{2m+1} b^n, a^m b^{2n}, a, a^{-1}, b, b^{-1}\}.$$

In the Case (1), when $m = 6k, j = 3k-1, k \geq 2, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (51) of Theorem 1.1. In this case for S_1 , when $m = 7, j = 2, \sigma = (a^2, a^5) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of Theorem 1.1. When $m = 8, j = 2, \sigma = (a^2, a^6) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (45) of the main theorem.

When $m = 14; j = 5, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of Theorem 1.1. Also for S_2 , when $m = 7; j = 3, \sigma = (a^3, a^4) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of Theorem 1.1. When $m = 14; j = 3, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of Theorem 1.1. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(39, 40, 41). In the Case (2), when $m = 5, 10$ and 8 we have the Cases (21) and (19, $m = 2$) of Theorem 1.1 respectively. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (24). In the Cases (3) and (4), $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (25, 26). Suppose $c = a^2 b$. Then we have one of the following cases:

$$(1): G = Z_m = \langle a \rangle (m \geq 7), S = \{a, a^{-1}, a^j, a^{-j}, a^{2j}, a^{-2j}\}, \text{ if } m = 2k, 2 \leq j \leq (m/2) - 3 \text{ and if } m = 2k + 1, 2 \leq j \leq (m/2) - 1.$$

$$(2): G = Z_m = \langle a \rangle (m \geq 7), S_1 = \{a^j, a^{-j}, a, a^{-1}, a^{2j+1}, a^{-2j-1}\}, 2 \leq j \leq m - 2, j \neq m/2 \text{ and } 2j + 1 \neq m/2, 0, 1, m - 1, j, m - j \pmod{m}$$

$$(3): G = Z_m \times Z_n = \langle a \rangle \times \langle b \rangle (m, n \geq 3), S = \{a, a^{-1}, b, b^{-1}, a^2 b, a^{-2} b^{-1}\}.$$

$$(4): G = Z_{2m+1} \times Z_n = \langle a \rangle \times \langle b \rangle (m \geq 2, n \geq 3), S = \{a^m, a^{m+1}, a, a^{-1}, b, b^{-1}\}.$$

$$(5): G = Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle (m, n \geq 1), S = \{a^m b^{n+1}, a^m b^n, a, a^{-1}, b, b^{-1}\}.$$

$$(6): G = Z_2 \times Z_{2m+1} \times Z_{2n+1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle (m, n \geq 1), S = \{ab^m c^{n+1}, ab^{m+1} c^n, b, b^{-1}, c, c^{-1}\}.$$

In the Case (1), if $m = 3k, k \geq 3, j = k - 1, \sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (53) of Theorem 1.1. If $m = 6k, k \geq 3, j = 3k - 3, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (51 - $S_2, m \geq 3$) of Theorem 1.1. If $m = 7; j = 2, \sigma = (a^3, a^4) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and if $m = 14, j = 2, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of the main theorem.

Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(42, 43). In the Case (2), if $m = 7, j = 4, \sigma = (a^5, a^9) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, and if $m = 14, j = 5, \sigma = (a^2, a^{12})(a^5, a^9) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (52) of Theorem 1.1. If $m = 3k, j = k - 1, k \geq 3, \sigma = (a^k, a^{2k}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (53) of Theorem 1.1. If $m = 4j, j \geq 2, \sigma = (a^j, a^{3j}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (45) of Theorem 1.1. If $m = 6k, j = 3k+1, k \geq 3, \sigma = (a, a^{3k+1})(a^4, a^{3k+4}) \dots (a^{3k-2}, a^{6k-2}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (51 - S_1) of Theorem 1.1. If $m = 8k + 4, k \geq 1, \text{ for } k = 2i - 1, j = 4i - 2, i \geq 1, \sigma = (a^2, a^{12i-1})(a^6, a^{12i+3}) \dots (a^{m-2}, a^{12i-5}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (54) of Theorem 1.1, and for $k = 2i, j = 12i + 2, i \geq 1, \sigma = (a^2, a^{4i+3})(a^6, a^{4i+7}) \dots (a^{m-2}, a^{4i-1}) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (55) of Theorem 1.1. In the Case (3), if $m = n = 3, \sigma = (ab, a^2 b^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (56) of the main theorem. If $m = 4, \sigma = (ab^2, a^3 b^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (50) of Theorem 1.1. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(27).

In the Case (4), if $m = 2$, we have the Case (21) of Theorem 1.1. if $m \geq 3, \Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(28). In the Case (5), if $m = n = 1, \sigma = (ab, a^2 b^2) \in A_1$, but $\sigma \notin \text{Aut}(G, S)$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (56) of Theorem 1.1. Otherwise, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(29). In the Case (6), $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(30).

4. Conclusion

Let $\Gamma = \text{Cay}(G, S)$ be a connected Cayley graph of an abelian group G on S . In this paper we have shown all non-normal Cayley graph Γ with valency 6.

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