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A Numerical Method for Backward Inverse Heat Conduction Problem With two Unknown Functions

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Abstract: This paper considers a linear one dimensional inverse heat conduction problem with non constant thermal diffusivity and two unknown terms in a heated bar with unit length. By using the WKB method, the heat flux at the end of boundary and initial temperature will be approximated, numerically. By choosing a suitable parameter in WKB method the ill-posedness of solution will be improved. Finally, a numerical example will be presented.

Keywords: Inverse heat conduction, Ill-posed problem, Finite difference method

1. Statement of the problem

This section deals with a linear heat equation

$$\frac{\partial u(x,t)}{\partial t}=a(t)\frac{\partial^2 u(x,t)}{\partial x^2}-q(t)u(x,t), \qquad (1)$$
$$D=\{(x,t)\,|\,0<x<1, 0<t<T\},$$

with boundary conditions

$$u(0,t)=g(t), \quad 0\le t\le T, \qquad (2)$$

$$u(1,t)=h(t), \quad 0\le t\le T, \qquad (3)$$

and the initial condition

$$u(x,0)=u_0(x), \quad 0\le x\le 1, \qquad (4)$$

where T is a given positive constant number, $q(t)$, $a(t)$ and $g(t)$ are known functions on $[0,T]$, and $h(t)$, $u_0(x)$ and $u(x,t)$ are unknown functions. To solve the above problem, we use the following extra conditions

$$\frac{\partial u(0,t)}{\partial t}=0, \quad 0\le t\le T, \qquad (5)$$

$$u(x,T)=f(x), \quad 0\le x\le 1, \qquad (6)$$

where $f(x)$ is an known function.
Clearly, the problem (1)-(6) is inverse heat conduction problem with two unknown terms.

In fact, these problems are of two types: backward inverse heat conduction problems (BIHCP) and determining unknown temperature histories and heat flux on a part of boundary, from known values in the body is the idea of many mathematicians.
Physically, let us consider a heated bar with unit length in one dimensional space.
A direct measurement of the heat flux or temperature in a boundary and initial time of hot body is almost impossible.
Consequently, this problem is an inverse heat conduction problem (IHCP). Some numerical and theoretical approaches for solving IHCPs when the histories of temperature at the initial time or in a boundary, not both, are unknown have been summarized in [1, 2, 3, 4, 5, 6, 7].
Beck in [1,2] has shown that, if an error is made in a known boundary condition, then there will be errors in the unknown heat flux on the other boundary.
In [4], an estimated of the solution an BIHCP, by using the regularization method, is derived.
These results are consistent with earlier observations that small values of time can produce large errors in surface flux.
In this article, we discrete the variable t and reduce the problem (1)-(6) to a system of linear, inhomogeneous secondary order differential equations.
Then, we express a relation of a parameter and increment time. By using this parameter we modify the instability of solution.

2. Time Variable Discretization

In this section, by given the following theorem we shall prove the existency and unicity for the solution of problem (1)-(6).
Theorem 1 For any final time T, let $g(t)\in \boldsymbol{H}^1([0,T])$, $a(t)\in L^\infty((0,T])$, such that, $a(t)>k>0$ where k is a constant number, $q(t)$ be an

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integrable positive function in [0, T], and f(x) be an analytical function for any 0<x<1, then there exist a unique weak solution $u \in L^2(0,T;\boldsymbol{H}^1([0,1]))$ and *Hölder* continuous function h(t), for the problem (1)-(6).

Proof In order to prove this theorem, let us consider the transformation

$$v(x,t) = u(x,t)\exp\left\{\int_0^t q(t)\,dt\right\}.$$

By using this transformation, the problem (1)-(2) and (5)-(6) becomes

$$\frac{\partial v(x,t)}{\partial t} = a(t)\frac{\partial^2 v(x,t)}{\partial x^2} - q(t)v(x,t),$$

$$D = \{(x,t)\,|\,0<x<1, 0<t<T\},$$

$$v(x,T) = u_M(x)\exp\left\{-\int_0^T q(t)\,dt\right\}$$

$$= f_1(x), \qquad 0<x<1,$$

$$v(0,t) = g(t)\exp\left\{-\int_0^t q(t)\,dt\right\}$$

$$= g_1(t), \qquad 0<t<T,$$

$$\frac{\partial v(0,t)}{\partial x} = 0, \qquad 0 \le t \le T.$$

Because, *q(t)* is a positive function and integrable in it's domain, if *g* and *f(x)* may be satisfied in the assumptions of theorem 1, then g_1 and $f_1(x)$ satisfying in these assumptions, too.

Consequently by using [1, 4, 6, 8, 9] the proof of this statement will be completed. In continuation, assume that $M \in$, $\Delta t_M = T/M$, and $t_i = i\,\Delta t_M$.

Also, we use $\hat{u}_i(x)$ instead of the approximate $u(x, i\,\Delta t_M)$, and $a_i = a(t_i)$ for any $0 \le i \le M$. Obviously, we have $u_M(x) = f(x)$. Now, apply the semi-implicit finite difference method in the form

$$\hat{u}_{i+1}(x) = \hat{u}_i(x) + \left(q\frac{\partial \hat{u}(x,t_i)}{\partial t} + q'\frac{\partial \hat{u}(x,t_{i+1})}{\partial t}\right)\Delta t_M, \qquad (7)$$

where $q>0$ and $q' = 1-q$. Then, by substituting (1)-(6) into (7) we drive the following ordinary differential equations system

$$\frac{d^2\hat{\mathbf{u}}(x)}{dx^2} = -l^2\mathbf{A}\hat{\mathbf{u}}(x) + l^2\mathbf{f}(x), \qquad (8)$$

where $l = (\Delta t_M)^{-\frac{1}{2}}$ and $\mathbf{A} = \mathbf{B}^{-1}\mathbf{C}$, such that

$$\hat{\mathbf{u}}(x) = [\hat{u}_0(x), \mathbf{K}, \hat{u}_{M-1}(x)]^T_{1\times M},$$

$$\mathbf{f}(x) = \Big[\underbrace{0, \mathbf{K}, 0}_{M-1}, f_M\Big]^T_{1\times M},$$

$$f_M = (1 + q'q\Delta t_M)u_M(x) - \Delta t_M q' a_M u''_M(x),$$

$$\mathbf{C} = [C_{ij}]_{M\times M},$$

where

$$[C_{ij}]_{M\times M} = \begin{cases} 1 - q\,q_{i-1}\Delta t_M & j=i \\ -(1-q'q_i\,\Delta t_M) & j=i+1 \\ 0 & else\ where \end{cases}$$

and

$$\mathbf{B} = \begin{pmatrix} qa_0 & q'a_1 & 0 & 0 & \mathbf{L} & 0 \\ 0 & qa_1 & q'a_2 & 0 & \mathbf{L} & 0 \\ 0 & 0 & qa_2 & q'a_3 & \mathbf{L} & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathbf{O} & \mathbf{O} & \mathrm{M} \\ 0 & 0 & \mathbf{L} & \mathbf{L} & qa_{M-2} & q'a_{M-1} \\ 0 & 0 & 0 & \mathbf{L} & 0 & qa_{M-1} \end{pmatrix}.$$

Consequently, we have

$$\hat{\mathbf{u}}(0) = [g_0, \mathbf{K}, g_{M-1}]^T$$

and

$$\hat{\mathbf{u}}'(0) = [0, \mathbf{L}, 0]^T.$$

Now, let us $f(x)=0$, then, for the solution of the equations system (8) may be in the form

$$\hat{\mathbf{u}}(x) = \cos(l\,S(x))\left(\mathbf{f}_0(x) + l^{-1}\mathbf{f}_1(x) + \mathbf{L}\right), \qquad (9)$$

or

$$\hat{\mathbf{u}}(x) = \sin(l\,S(x))\left(\mathbf{f}_0(x) + l^{-1}\mathbf{f}_1(x) + \mathbf{L}\right), \qquad (10)$$

Where $S(x)$ is an unknown function and $\mathbf{f}_0(x)$, $\mathbf{f}_1(x)$, $\mathbf{L}$, are unknown vector-functions. By substituting (9) and (10) into the ordinary differential equations system (8), cancel the cosine or sine term and simplifying the produced results, then we obtain a recurrent system of equations

$$\left(\mathbf{A} - S'^2(x)\mathbf{I}\right)\mathbf{f}_0(x) = \mathbf{0}, \qquad (11)$$

$$\left(\mathbf{A} - S'^2(x)\mathbf{I}\right)\mathbf{f}_1(x) = \mathbf{f}_0(x)S''(x) + 2\mathbf{f}'_0(x)S'(x), \qquad (12)$$

$$\left(\mathbf{A} - S'^2(x)\mathbf{I}\right)\mathbf{f}_k(x) = \mathbf{f}_{k-1}(x)S''(x)$$
$$+2\mathbf{f}'_{k-1}(x)S'(x) + \mathbf{f}''_{k-2}(x), \quad k \ge 2. \qquad (13)$$

If $a(t)$ is a monotone function, then the characteristic equation (8), has not turning points for any $x \in [0,1]$ ([5]).

Then $\mathbf{A}$ has M unequal eigenvalues and M linear independent eigenvectors corresponding to eigenvalues of matrix $\mathbf{A}$.

By using (11)-(13) we derive $2M$ independent solutions for (8). It follows from (11) that $S'^2(x)$ is an eigenvalue, and $\mathbf{f}_0(x)$ is an eigenvector of $\mathbf{A}$. Let $\{\mathbf{e}_0(x), \mathbf{K}, \mathbf{e}_{M-1}(x)\}$ be a base of eigenvectors. Then, we derive

$$S_j(x) = \frac{x\sqrt{1-q\,q_j\,\Delta t_M}}{\sqrt{q\,a_j}}, \quad 0 \le j \le M-1,$$

$$\mathbf{f}_i^{(j)}(x) = a_{i,j}(x)\,\mathbf{e}_j(x), \quad i \ge 0, 0 \le j \le M-1,$$

where

$$a_{0,j}(x) = \sqrt[4]{a_j(x)}, \quad 0 \le j \le M-1,$$

and

$$a_{i,j}(x) = -\frac{a_{0,j}(x)}{2}\int_0^x \frac{a''_{(i-1),j}(s)}{a_{0,j}(s)}\,ds = 0,$$
$$0 \le j \le M-1, \quad i > 1.$$

Then, for finding $\hat{u}_i(x)$ for any $i = 0, 1, \mathbf{K}, M-1$, setting

$$\hat{\mathbf{u}}(x) = \sum_{i=0}^{M-1} C_i^{(1)}\hat{\mathbf{u}}_i^{(1)}(x) + \sum_{i=0}^{M-1} C_i^{(2)}\hat{\mathbf{u}}_i^{(2)}(x), \tag{14}$$

where

$$\hat{\mathbf{u}}_i^{(1)}(x) = \sin\left(\frac{x\sqrt{1-q\,q_i\,\Delta t_M}}{\sqrt{q a_i\,\Delta t_M}}\right)\mathbf{f}_0^{(i)}(x), \tag{15}$$

$$\hat{\mathbf{u}}_i^{(1)}(x) = \cos\left(\frac{x\sqrt{1-q\,q_i\,\Delta t_M}}{\sqrt{q a_i\,\Delta t_M}}\right)\mathbf{f}_0^{(i)}(x), \tag{16}$$
$$i = 0, \mathbf{K}, M-1,$$

such that, $C_i^{(j)}$ for any $i = 0, \mathbf{K}, M-1$, and $j = 1, 2$ are unknown constants and will be found from initial conditions

$$\hat{\mathbf{u}}(0) = [g_0, \mathbf{K}, g_{M-1}]^T$$

and

$$\hat{\mathbf{u}}'(0) = [0, \mathbf{L}, 0]^T.$$

Now, if $f(x) \ne 0$, then, a particular solution $\mathbf{u}^{(p)}(x)$ of the inhomogeneous system can be found by the method of variation of parameters in the form

$$\hat{\mathbf{u}}^{(p)}(x) = \sum_{i=0}^{M-1} u_i^{(1)}(x)\,\hat{\mathbf{u}}_i^{(1)}(x) + \sum_{i=0}^{M-1} u_i^{(2)}(x)\,\hat{\mathbf{u}}_i^{(2)}(x),$$

where, $u_i^{(j)}(x)$ for any $i = 0, \mathbf{K}, M-1$ and $j = 1, 2$ are unknown functions and will be found from (8) and (14)-(16). Now, for each $n \in$ and $0 \le i \le M-1$, if $\frac{1-q\,q_i\,\Delta t_M}{a_i\,q\,\Delta t_M} \ne (np)^2$, then the solution (11) is unique ([10]). The above result may be summarized in the following statement.

Theorem 2 *If $f(x)$ be the analytical function, and for each $n \in$ and $0 \le i \le M-1$, if $\frac{1-q\,q_i\,\Delta t_M}{a_i\,q\,\Delta t_M} \ne (np)^2$, then the equations system (8) has a unique solution.*

Proof See the analysis preceding the above theorem statement.

In the next section we consider the one example, and show that, choosing an appropriate q produce convergent solution for problem (1)-(4).

3. Numerical Example

This section will present a simulated case to evaluate the capability of the proposed robust input estimation scheme.

Example Assume that

$$T = 1, \quad q(t) = 2t,$$
$$f(x) = e^{-1}\cosh(2)\cos(x), \quad 0 \le x \le 1,$$
$$a(t) = 3 - 2t, \quad 0 \le t \le 1,$$
$$g(t) = e^{-t^2}\cosh(t^2 - 3t), \quad 0 \le t \le 1.$$

Clearly, $f(x)$ and $g(t)$ satisfy in assumptions of theorems 1 and 2.

Therefore, there is a unique solution for this sample problem. Obviously, $u(x,t) = \cosh(t^2 - 3t)\cos x$ for any $0 \le x \le 1$, $0 \le t \le T$ and the above assumptions, satisfies in problem (1)-(6). Now, we use the above numerical method to this problem.

For $x = 1, \Delta t_M = 0.1$, $q = 10$, the result are given in the table 1.

One can see from the data in the table 1 the relation errors generated through the computation show that the approximate and the exact solutions are vanished.

In the fifth column, the produced errors of area, between u and $\hat{u}$ in the interval [0,1], no more than five percentage, although, the relative errors in $\hat{u}_i(1)$, for some of $0 \le i \le M$ may be *23%*, but the maximum error in area region of between u and $\hat{u}$ in their domain no more than *0.03* (*3.7%* relative error).

Consequently this technique can be applied for the similar inverse problems.

Table. 1. Exact and Estimate of the Temperature in x=1 with $\Delta t_M = 0.1, q = 10.$

t	$u(1,t)$	$\hat{u}(1,t)$	***relative error***	$\|u(x,t_i)\|_{L[0,1]}$	$\|u(x,t_i)-\hat{u}(x,t_i)\|_{L[0,1]}$
0	0.540302	0.438671	**18.8 %**	0.806089	0.035381
0.1	0.563181	0.428510	**23.9 %**	0.830280	0.04682
0.2	0.627258	0.482832	**23.0 %**	0.926704	0.05019
0.3	0.727453	0.591342	**18.7 %**	1.085646	0.04729
0.4	0.859802	0.745534	**13.2 %**	1.299360	0.03970
0.5	1.020319	0.936711	**8.1 %**	1.560007	0.02904
0.6	1.204232	1.154731	**4.1 %**	1.858290	0.01718
0.7	1.405515	1.387395	**1.2 %**	2.182683	0.00627
0.8	1.616714	1.620597	**0.24 %**	2.519261	0.00137
0.9	1.829042	1.839611	**0.57 %**	2.852268	0.00370

4. Conclusion

In this paper we shown that, if we choose the appropriate of parameter q such that, the estimated solution of this problem well-posed, then we can to tend Δt_M to zero and we derive the convergency and stability of this problem.
In order to, reduce of effect measurements error in the final time and boundary, we use the source term $q(t)\ u(x,t)$ in the problem (1)-(6).

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