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Interval Weighted Comparison Matrices – A Review

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ABSTRACT

Nowadays, interval comparison matrices (ICM) take an important role in decision making under uncertainty. So it seems that a brief review on solution methods used in ICM should be useful. In this paper, the common methods are divided into four categories that are Goal Programming Method (GPM), Linear Programming Method (LPM), Non-Linear Programming Method (NLPM) and Statistic Analysis (SA). GPM itself is divided also into three categories. This paper is a review paper and is written to introduce the mathematical methods and the most important applications of ICM in decision making techniques.

1. Introduction

ICM is a way to compute the weights in the presence of uncertainty in decision making techniques. In traditional methods, each preference ratio in the comparison matrix (aij) is assumed to be deterministic. In real life, in most times the preference ratios are interval numbers. Two type methods are applied which one of them calculates the weights accurately while another one calculates the weights intervally.

Most real world decision problems involve multiple criteria that are often in conflict in general and it is some times necessary to conduct trade-off analysis in multiple criteria decision analysis (MCDA).As such, the estimation of the relative weights of criteria plays an important role in a MCDA process. Among many frameworks developed for weight estimation, pair wise comparison matrices provide a natural frame work to elicit preferences from decision makers and have been used in several weight generation methods. However, due to the complexity and uncertainty involved in real world decision problems and the inherent subjective nature of human judgments, it is sometimes unrealistic and infeasible to acquire exact judgments. It is more natural or easier to provide fuzzy or interval judgments for parts or all of the judgments in a pair wise comparison matrix. A number of techniques have been developed to use such a fuzzy or interval comparison matrix to generate weights.

Differences of each method according to input data type (interval comparison matrix) and output data type (achieved weights) are exhibited in the table 1 in appendix1.

2. Definitions

2.1. Comparison Matrices

In the conventional AHP, a judge estimates (by filling out a questionnaire, say) ratios of priorities, which are arranged in the upper triangle of a pair wise comparison (Saaty) matrix:

$$A = (a_{ij}) = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{2n} & \dots & 1 \end{bmatrix} \quad (1)$$

Each element a_{ij} of the upper triangle in Eq. (1) represents an estimate of the ratio of preferences α_i and α_j of the *i*th and the *j*th objects. That is,

$$a_{ij} \equiv \frac{\alpha_i}{\alpha_j}, \quad i = 1,...,n \ ; \quad j = 1,...,n. \quad (2)$$

The elements in the lower triangle of matrix (1) are taken as follow:

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$$a_{ji} = \frac{1}{a_{ij}}, \quad j = 1,...,n; \quad i = 1,...,n. \tag{3}$$

The priority vector α is estimated as the right eigenvector for the maximal eigenvalue k in the following eigen problem (Saaty, 1980):

$$A\alpha = \lambda\alpha \tag{4}$$

2-2. Interval Comparison Matrices

Suppose the decision maker provides interval judgments instead of precise judgments for a pair wise comparison. For example, it could be judged that criterion i is between l_{ij} and u_{ij} times as important as criterion j with l_{ij} and u_{ij} being non-negative real numbers and $l_{ij} \le u_{ij}$. Then, an interval comparison matrix can be represented by:

$$A = (a_{ij})_{n\times n} = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \dots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \dots & 1 \end{bmatrix} \tag{5}$$

Where $l_{ij} = 1/u_{ij}$ and $u_{ij} = 1/l_{ij}$ and $l_{ij} \le a_{ij} \le u_{ij}$. About the above interval comparison matrix, we give the following definition and theorem:

Let $A = (a_{ij})_{n\times n}$ is an interval comparison matrix defined by (5) with $l_{ij} \le a_{ij} \le u_{ij}$ and $l_{ii} = a_{ii} = u_{ii} = 1$ for $i, j = 1,\dots,n.$ If the convex feasible region $S_w = \left\{ w = (w_1,..., w_n) \mid l_{ij} \le w_i / w_j \le u_{ij}, \sum_{i=1}^{n} w_i = 1, w_j > 0, j = 1,..., n \right\}$ is nonempty, and then A is said to be a consistent interval comparison matrix.

2-3. Consistency of Interval Comparison Matrix

$A = (a_{ij})_{n\times n}$ Is a consistent interval comparison matrix if and only if it satisfies the following inequality constraints:

$$\max_k (l_{ik} l_{kj}) \le \min_k (u_{ik} u_{kj}), \quad \textit{for all } i, j, k = 1,\dots,n. \tag{6}$$

Proof . If A is a consistent interval comparison matrix, then the convex feasible region S_w is nonempty, which means that there is no contradiction among the following inequality constraints:

$$l_{ik} \le w_i / w_k \le u_{ik}, \quad i, k = 1,\dots,n \tag{7}$$

$$l_{kj} \le w_k / w_j \le u_{kj}, \quad i, j = 1,\dots,n \tag{8}$$

Multiplying (7) by (8) leads to the following implied indirect inequalities:

$$l_{ik}.l_{kj} \le w_i / w_j \le u_{ik}.u_{kj}, \quad i, j, k = 1,\dots,n \tag{9}$$

Since (9) holds for any $k = 1,...,n$, it follows that $\max_k (l_{ik} l_{kj}) \le \min_k (u_{ik} u_{kj})$ holds for all $i, j, k = 1,\dots,n$. Conversely, if (6) holds for $\forall i, j, k$ then $l_{ij} \le w_i / w_j \le u_{ij}$ holds for any $i, j = 1,\dots,n$. So, S_w cannot be empty. By definition, A is a consistent interval comparison matrix.

2-4. The Degree of Preference

The degree of preference of a over b (or $a > b$) is defined as:

$$P(a > b) = \frac{\max(0, a_2 - b_1) - \max(0, a_1 - b_2)}{(a_2 - a_1) + (b_2 - b_1)} \tag{10}$$

The degree of preference of b over a (or $b > a$) can be defined in the same way. That is:

$$P(b > a) = \frac{\max(0, b_2 - a_1) - \max(0, b_1 - a_2)}{(a_2 - a_1) + (b_2 - b_1)} \tag{11}$$

Let $a = [a_1, a_2]$ and $b = [b_1, b_2]$ be two interval weights, whose possible relationships are as shown in Fig.1. We refer to the degree of one interval weight being greater than another one as the degree of preference. Accordingly, we have the following definitions and properties.

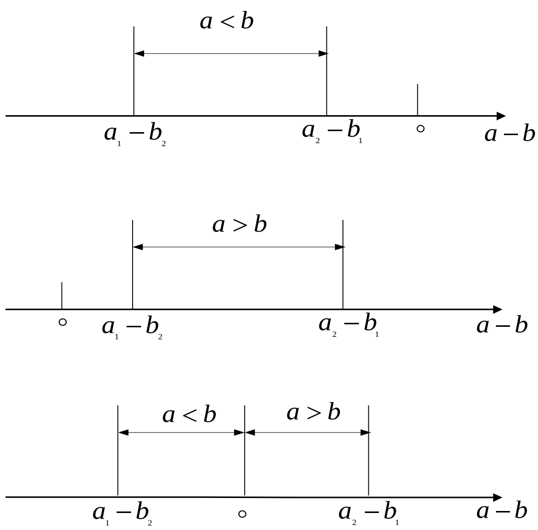

Fig. 1. Relationships between two interval weights a and b

It is obvious that $P(a > b) + P(b > a) = 1$ and $P(a > b) = P(b > a) \equiv 0.5$ when $a = b$, i.e. $a_1 = b_1$ and $a_2 = b_2$.

- If $P(a>b) > P(b>a)$, then a is said to be superior to b to the degree of $P(a>b)$, denoted by $a \overset{P(a>b)}{\succ} b$;
- If $P(a>b) = P(b>a) = 0.5$, then a is said to be indifferent to b, denoted by $a \sim b$;
- If $P(b>a) > P(a>b)$, then a is said to be inferior to b to the degree of $P(b>a)$, denoted by $a \overset{P(a>b)}{\prec} b$.

Property1. $P(a>b)=1$ if and only if $a \ge b$.

Property2. If $a_1 \ge b_1$ and $a_2 \ge b_2$, then $P(a>b) \ge 0.5$ and $P(b>a) \le 0.5$.

Property3. If b is nested in a, i.e. $a_1 \le b_1$ and $a_2 \ge b_2$, then $P(a>b) \ge 0.5$ if and only if $\dfrac{a_1+a_2}{2} \ge \dfrac{b_1+b_2}{2}$.

Property4. If $P(a>b) \ge 0.5$ and $P(b>c) \ge 0.5$, then $P(a>c) \ge 0.5$.

2.5. Multiplicative Constraint

The multiplicative constraint, i.e. $\prod_{i=1}^{n} \ln w_i = 1$ which is equivalent to $\sum_{i=1}^{n} \ln w_i = 0$. Such multiplicative constraint is widely used in multiplicative AHP.

3.Solution Methods

3.1. Goal Programming Based Methods

3.1.1. Goal Programming

a. Model (1)

Ying-Ming Wang and Taha M.S.Elhag 0 developed a method for deriving interval weight based on goal programming. Suppose a decision maker (DM) provides an interval judgment instead of precise judgment for a pair wise comparison matrix. For example, the importance of criterion i in respect to criterion j, lies between l_{ij} and u_{ij}, with l_{ij} and u_{ij} being non-negative real numbers and $l_{ij} \le u_{ij}$. An interval comparison matrix can be expressed as the matrix (1). Where $l_{ij} = 1/u_{ij}$ and $u_{ij} = 1/l_{ij}$. For all $i, j = 1,\dots,n; i \ne j$. The above interval comparison matrix can be split into two crisp nonnegative matrices:

$$A_L = \begin{bmatrix} 1 & l_{12} & \cdots & l_{1n} \\ l_{21} & 1 & \cdots & l_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \qquad A_U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ u_{21} & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & u_{n2} & \cdots & 1 \end{bmatrix} \quad (12)$$

Where $A_L \le A \le A_U$. Note that A_L, A_U are no longer the reciprocal matrices.

For the interval comparison matrix A, there should exist a normalized interval weight vector, $W = ([w_1^L, w_1^U],\dots,[w_n^L, w_n^U])^T$ which is close to A in the sense that $a_{ij} = [l_{ij}, u_{ij}] \approx [w_i^L, w_i^U] \big/ [w_j^L, w_j^U]$ for all $i, j = 1,\dots,n; i \ne j$.

According to 0, the interval weight vector W is said to be normalized if and only if:

$$\sum_i w_i^U - \max_j (w_j^U - w_j^L) \ge 1, \tag{13}$$

$$\sum_i w_i^L + \max_j (w_j^U - w_j^L) \le 1, \tag{14}$$

Which can be equivalently rewritten as:

$$w_i^L + \sum_{j=1, j\ne i}^{n} w_j^U \ge 1, \qquad i = 1,\cdots,n, \tag{15}$$

$$w_i^U + \sum_{j=1, j\ne i}^{n} w_j^L \le 1, \qquad i = 1,\cdots,n. \tag{16}$$

As is known, if the interval comparison matrix A is the precise comparison about the interval weight vector W, namely, $a_{ij} = [l_{ij}, u_{ij}] \equiv [w_i^L, w_i^U] \big/ [w_j^L, w_j^U]$ and then A can be written as follows:

$$A = \begin{bmatrix} 1 & \dfrac{[w_1^L, w_1^U]}{[w_2^L, w_2^U]} & \cdots & \dfrac{[w_1^L, w_1^U]}{[w_n^L, w_n^U]} \\ \dfrac{[w_2^L, w_2^U]}{[w_1^L, w_1^U]} & 1 & \cdots & \dfrac{[w_2^L, w_2^U]}{[w_n^L, w_n^U]} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{[w_n^L, w_n^U]}{[w_1^L, w_1^U]} & \dfrac{[w_n^L, w_n^U]}{[w_2^L, w_2^U]} & \cdots & 1 \end{bmatrix} \tag{17}$$

According to the division operation rule on interval numbers, i.e. $[b_L, b_U]/[d_L, d_U] = [b_L/d_U, b_U/d_L]$, where $[b_L, b_U]$ and $[d_L, d_U]$ are two positive interval numbers, the interval comparison matrix A defined by (17) can be further rewritten as:

$$A = \begin{bmatrix} 1 & \left[\dfrac{w_1^L}{w_2^U}, \dfrac{w_1^U}{w_2^L}\right] & \cdots & \left[\dfrac{w_1^L}{w_n^U}, \dfrac{w_1^U}{w_n^L}\right] \\ \left[\dfrac{w_2^L}{w_1^U}, \dfrac{w_2^U}{w_1^L}\right] & 1 & \cdots & \left[\dfrac{w_2^L}{w_n^U}, \dfrac{w_2^U}{w_n^L}\right] \\ \vdots & \vdots & \vdots & \vdots \\ \left[\dfrac{w_n^L}{w_1^U}, \dfrac{w_n^U}{w_1^L}\right] & \left[\dfrac{w_n^L}{w_2^U}, \dfrac{w_n^U}{w_2^L}\right] & \cdots & 1 \end{bmatrix} \tag{18}$$

This can be split into the following two crisp nonnegative matrices:

$$A_L=\begin{bmatrix} 1 & \frac{w_1^L}{w_2^U} & \cdots & \frac{w_1^L}{w_n^U} \\ \frac{w_2^L}{w_1^U} & 1 & \cdots & \frac{w_2^L}{w_n^U} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{w_n^L}{w_1^U} & \frac{w_n^L}{w_2^U} & \cdots & 1 \end{bmatrix} \qquad A_U=\begin{bmatrix} 1 & \frac{w_1^U}{w_2^L} & \cdots & \frac{w_1^U}{w_n^L} \\ \frac{w_2^U}{w_1^L} & 1 & \cdots & \frac{w_2^U}{w_n^L} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{w_n^U}{w_1^L} & \frac{w_n^U}{w_2^L} & \cdots & 1 \end{bmatrix} \tag{19}$$

It is easy to prove that

$$A_L W_U = W_U + (n-1)W_L, \tag{20}$$

$$A_U W_L = W_L + (n-1)W_U, \tag{21}$$

Where $W_L=(w_1^L,\dots,w_n^L)$ and $W_U=(w_1^U,\dots,w_n^U)$. Relations (20) and (21) are important links between the lower and the upper bounds of the interval weight vector *W*.

Due to the presence of subjectivity and uncertainty, the DM's subjective judgments cannot be 100% exact. Therefore, Relations (20) and (21) may not hold precisely. Based on such an analysis, consider the following deviation vectors:

$$E=(A_L-I)W_U-(n-1)W_L, \tag{22}$$

$$\Gamma=(A_U-I)W_L-(n-1)W_U, \tag{23}$$

Where $E=(\varepsilon_1,\dots,\varepsilon_n)^T$, $\Gamma=(\gamma_1,\dots,\gamma_n)^T$ and I is an $n \times n$ unit matrix whose elements on the leading diagonal are 1, and all the other elements are 0.

It is most desirable that the absolute values of deviation variables should be kept as small as possible, which leads to the following optimization model to be constructed:

$$\text{Minimiz} \quad J=\sum_{i=1}^{n}(|\varepsilon_i|+|\gamma_i|)$$

$$\text{s.t}\quad \begin{cases} (A_L-I)W_U-(n-1)W_L-E=0, \\ (A_U-I)W_L-(n-1)W_U-\Gamma=0, \\ w_i^L+\sum_{j=1,j\neq i}^{n} w_j^U \geq 1, \quad i=1,\cdots,n, \\ w_i^U+\sum_{j=1,j\neq i}^{n} w_j^L \leq 1, \quad i=1,\cdots,n, \\ W_U-W_L\geq 0, \\ W_U,W_L\geq 0, \end{cases} \tag{24}$$

Where the first two constraints are relations (22) and (23), the middle two constraints are the normalization constraints on the interval weight vector *W*, and the last two constraints are those on the lower and upper bounds of *W*.

Let

$$\varepsilon_i^+=\frac{\varepsilon_i+|\varepsilon_i|}{2} \quad \text{and} \quad \varepsilon_i^-=\frac{-\varepsilon_i+|\varepsilon_i|}{2}, \quad i=1,\cdots,n, \tag{25}$$

$$\gamma_i^+=\frac{\gamma_i+|\gamma_i|}{2} \quad \text{and} \quad \gamma_i^-=\frac{-\gamma_i+|\gamma_i|}{2}, \quad i=1,\cdots,n. \tag{26}$$

Then

$$E^+=(\varepsilon_1^+,\cdots,\varepsilon_n^+)^T\geq 0$$

$$E^-=(\varepsilon_1^-,\cdots,\varepsilon_n^-)^T\geq 0$$

$$\Gamma^+=(\gamma_1^+,\cdots,\gamma_n^+)^T\geq 0$$

$$\Gamma^-=(\gamma_1^-,\cdots,\gamma_n^-)^T\geq 0$$

Based on and ε_i^+ and ε_i^-, ε_i and $|\varepsilon_i|$ can be expressed as

$$|\varepsilon_i|=\varepsilon_i^++\varepsilon_i^-, \quad i=1,\dots,n$$

$$|\varepsilon_i|=\varepsilon_i^+-\varepsilon_i^-, \quad i=1,\dots,n$$

Where $\varepsilon_i^+.\varepsilon_i^-=0$ for $i=1,\dots,n$. γ_i and $|\gamma_i|$ can be expressed as

$$|\gamma_i|=\gamma_i^++\gamma_i^-, \quad i=1,\dots,n$$

$$|\gamma_i|=\gamma_i^+-\gamma_i^-, \quad i=1,\dots,n$$

Where $\gamma_i^+.\gamma_i^-=0$ for $i=1,\dots,n$. Accordingly, the optimization model (24) can be rewritten as

$$\text{Minimiz } J=\sum_{i=1}^{n}(\varepsilon_i^++\varepsilon_i^-+\gamma_i^++\gamma_i^-)=e^T(E^++E^-+\Gamma^++\Gamma^-)$$

$$\text{s.t.}\quad \begin{cases} (A_L-I)W_U-(n-1)W_L-E^++E^-=0, \\ (A_U-I)W_L-(n-1)W_U-\Gamma^++\Gamma^-=0, \\ w_i^L+\sum_{j=1,j\neq i}^{n} w_j^U\geq 1, \quad i=1,\cdots,n, \\ w_i^U+\sum_{j=1,j\neq i}^{n} w_j^L\leq 1, \quad i=1,\cdots,n, \\ W_U-W_L\geq 0, \\ W_U,W_L,E^+,E^-,\Gamma^+,\Gamma^-\geq 0, \end{cases} \tag{27}$$

Where $e^T=(1,\dots,1)$, ε_i^+ and ε_i^- as well as γ_i^+ and γ_i^- cannot be simultaneously the basic variables in the simplex method. This method for obtaining interval weights from an interval comparison matrix is referred as the GP method (GPM). Since crisp comparison matrices are a special case of interval comparison matrices, the above GP model (27) is also applicable to crisp comparison matrices.

3.1.2. Numerical Example

Consider the following comparison matrix:

$$\begin{bmatrix} 1 & [3,5] & [1/2,4] \\ [1/5,1/3] & 1 & [2,4] \\ [1/4,2] & [1/4,1/2] & 1 \end{bmatrix}$$

We have:

$$A_L = \begin{pmatrix} 1 & 3 & 1/2 \\ 1/5 & 1 & 2 \\ 1/4 & 1/4 & 1 \end{pmatrix} \qquad A_U = \begin{pmatrix} 1 & 5 & 4 \\ 1/3 & 1 & 4 \\ 2 & 1/2 & 1 \end{pmatrix}$$

$$W_L = \left(w_1{}^L, w_2{}^L, w_3{}^L\right)$$

$$W_U = \left(w_1{}^U, w_2{}^U, w_3{}^U\right)$$

$$n = 3$$

$$E = \left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right)$$

$$\Gamma = \left(\gamma_1, \gamma_2, \gamma_3\right)$$

By using model (27) the following linear programming is obtained:

$$\min \varepsilon_1{}^+ + \varepsilon_2{}^+ + \varepsilon_3{}^+ + \varepsilon_1{}^- + \varepsilon_2{}^- + \varepsilon_3{}^-$$
$$+\gamma_1{}^+ + \gamma_2{}^+ + \gamma_3{}^+ + \gamma_1{}^- + \gamma_2{}^- + \gamma_3{}^-$$
$st:$
$$3w_2{}^U + \frac{1}{2}w_3{}^U - 2w_1{}^L - \varepsilon_1{}^+ + \varepsilon_1{}^- = 0$$
$$\frac{1}{5}w_1{}^U + 2w_3{}^U - 2w_2{}^L - \varepsilon_2{}^+ + \varepsilon_2{}^- = 0$$
$$\frac{1}{4}w_1{}^U + \frac{1}{4}w_2{}^U - 2w_3{}^L - \varepsilon_3{}^+ + \varepsilon_3{}^- = 0$$
$$5w_2{}^L + 4w_3{}^L - 2w_1{}^U - \gamma_1{}^+ + \gamma_1{}^- = 0$$

$$\frac{1}{3}w_1{}^L + 4w_3{}^L - 2w_2{}^U - \gamma_2{}^+ + \gamma_2{}^- = 0$$
$$2w_1{}^L + \frac{1}{2}w_2{}^L - 2w_3{}^U - \gamma_3{}^+ + \gamma_3{}^- = 0$$
$$w_1{}^L + w_2{}^U + w_3{}^U \geq 1$$
$$w_2{}^L + w_1{}^U + w_3{}^U \geq 1$$
$$w_3{}^L + w_1{}^U + w_2{}^U \geq 1$$
$$w_1{}^U + w_2{}^L + w_3{}^L \leq 1$$
$$w_2{}^U + w_1{}^L + w_3{}^L \leq 1$$
$$w_3{}^U + w_1{}^L + w_2{}^L \leq 1$$
$$w_1{}^U - w_1{}^L \geq 0$$
$$w_2{}^U - w_2{}^L \geq 0$$
$$w_3{}^U - w_3{}^L \geq 0$$
$$W_U, W_L, E^+, E^-, \Gamma^+, \Gamma^- \geq 0$$

The solutions of the above model are the $\left[w_i{}^L, w_i{}^U\right]$ for $i = 1,...,n$. They are as follow:

$$\left[w_1{}^L, w_1{}^U\right] = [0.4208, 0.7035]$$
$$\left[w_2{}^L, w_2{}^U\right] = [0.2208, 0.2208]$$
$$\left[w_3{}^L, w_3{}^U\right] = [0.0757, 0.3583]$$

It is worth while pointing out here that for a crisp comparison matrix, Bryson [17] developed a different goal programming (GP) method for generating priority vectors.

Consider the following theorems on the above model.

Theorem1. Let W_L^* and W_U^* be the optimal solution of the GP model (27). If A is a crisp consistent comparison matrix, then we have $W_L^* = W_U^* = W^*$, where W^* is the principal right eigenvector of A.

Proof. If A is a crisp consistent comparison matrix, then there exists the eigenvalue equation: $AW^* = nW^*$, namely, $(A - nI)W^* = 0$. Let $W_L^* = W^*$ and $W_U^* = W^*$. It is easy to find that $W_L^* = W_U^* = W^*$ is a feasible solution of the GP model (27). Accordingly, we have $E = (A_L - I)W_U - (n-1)W_L = (A - nI)W^* = 0$ and $E = (A_U - I)W_L - (n-1)W_U = (A - nI)W^* = 0$, which leads to $J = \sum_{i=1}^{n}(|\varepsilon_i| + |\gamma_i|) = 0$. That is to say, $W_L^* = W_U^* = W^*$ is also the optimal solution to the GP model (21). So, $W_L^* = W_U^* = W^*$.

b. Model (2)

Sugihara et al. 0 developed the goal programming for deriving interval weight. They deal with interval judgments in two ways. One is called the lower approximation and the other is called the upper approximation. For the lower approximation, it is required that:

$$W_{ij^8} = \left[\frac{W_{i^*}^L}{W_{j^*}^U}, \frac{W_{i^*}^U}{W_{j^*}^L}\right] \subseteq a_{ij} = [l_{ij}, u_{ij}], \qquad \forall i, j \ (i \neq j), \quad (28)$$

This can be rewritten as

$$\frac{W_{i^*}^L}{W_{j^*}^U} \geq l_{ij} \quad \text{and} \quad \frac{W_{i^*}^U}{W_{j^*}^L} \leq u_{ij}, \qquad \forall i, j \ (i \neq j) \qquad (29)$$

Or

$$w_{i^*}^L - l_{ij}w_{j^*}^U \geq 0 \ \text{ and } \ w_{i^*}^U - u_{ij}w_{j^*}^L \geq 0, \qquad \forall i, j \ (i \neq j). \qquad (30)$$

For the upper approximation, it is required that

$$W_{ij}^* = \left[\frac{W_i^{L^*}}{W_j^{U^*}}, \frac{W_i^{U^*}}{W_j^{L^*}}\right] \supseteq a_{ij} = [l_{ij}, u_{ij}], \qquad \forall i, j \ (i \neq j), \quad (31)$$

This can be rewritten as

$$\frac{W_i^{L^*}}{W_j^{U^*}} \le l_{ij} \quad \text{and} \quad \frac{W_i^{U^*}}{W_j^{L^*}} \ge u_{ij}, \qquad \forall i, j \ (i \neq j), \qquad (32)$$

Or

$$w_i^{L^*} - l_{ij} w_j^{U^*} \le 0 \quad \text{and} \quad w_i^{U^*} - u_{ij} w_j^{L^*} \ge 0, \quad \forall i, j \ (i \neq j), \quad (33)$$

The lower and upper approximation models are respectively constructed as follows:

$$\text{Maximize} \quad J_* = \sum_{i=1}^{n} (w_{i^*}^U - w_{i^*}^L)$$

$$\text{s.t.} \quad \begin{cases} w_{i^*}^L - l_{ij} w_{j^*}^U \ge 0, & \forall i, j \ (i \neq j), \\ w_{i^*}^U - u_{ij} w_{j^*}^L \le 0, & \forall i, j \ (i \neq j), \\ w_{i^*}^L + \sum_{j=1, j\neq i}^{n} w_{j^*}^U \ge 1, & \forall i \\ w_{i^*}^U + \sum_{j=1, j\neq i}^{n} w_{j^*}^L \le 1, & \forall i \\ w_{i^*}^U - w_{i^*}^L \ge 0, & \forall i \\ w_{i^*}^U \ge \varepsilon, & \forall i, \end{cases} \qquad (34)$$

$$\text{Minimize} \quad J^* = \sum_{i=1}^{n} (w_i^{U^*} - w_i^{L^*})$$

$$\text{s.t.} \quad \begin{cases} w_i^{L^*} - l_{ij} w_j^{U^*} \le 0, & \forall i, j \ (i \neq j), \\ w_i^{U^*} - u_{ij} w_j^{L^*} \ge 0, & \forall i, j \ (i \neq j), \\ w_i^{L^*} + \sum_{j=1, j\neq i}^{n} w_j^{U^*} \ge 1, & \forall i, \\ w_i^{U^*} + \sum_{j=1, j\neq i}^{n} w_j^{L^*} \le 1, & \forall i, \\ w_i^{U^*} - w_i^{L^*} \ge 0, & \forall i, \\ W_i^{U^*} \ge \varepsilon, & \forall i, \end{cases} \qquad (35)$$

And ε is a small positive real number.

Compared with the above lower and upper approximation models, the GP model (27) differs from them in the following ways:

First of all, the GP model considers an interval comparison matrix as a whole and does not consider each judgment element individually, which makes the GP model to have less constraint, whereas the lower and upper approximation models deal with each judgment individually and therefore have more constraints than the GP model.

Second, the GP model (27) is applicable to any crisp and interval comparison matrices no matter whether they are consistent or not, while the lower approximation model is only applicable to consistent comparison matrices (crisp or interval) because there is no feasible solution that can be found for any inconsistent comparison matrix or inconsistent interval comparison matrix.

Next, the upper approximation model aims at finding an interval weight vector

$$W = ([w_1^{L*}, w_1^{U*}], ..., [w_n^{L*}, w_n^{U*}])^T$$

$$a_{ij} = [l_{ij}, u_{ij}] \in W_{ij}^* = [\frac{w_i^{L*}}{w_j^{U*}}, \frac{w_i^{U*}}{w_j^{L*}}] \ \ for \ \forall i, j (i \neq j)$$

Due to the fact that the DM's judgments are subjective and cannot always be 100% precise, there is no guarantee that the DM's judgments will certainly fall within and will not exceed the real interval

$\overline{W}_{ij} = \left[\frac{\overline{w}_i^L}{\overline{w}_i^U}, \frac{\overline{w}_i^U}{\overline{w}_i^L}\right]$ Where $\overline{W} = ([\overline{w}_1^L, \overline{w}_1^U], \cdots, [\overline{w}_n^L, \overline{w}_n^U])^T$

is an unknown real weight.

On the other side, the GP model aims at finding an interval weight vector $W = ([w_1^L, w_1^U], ...[w_n^L, w_n^U])^T$ such that $W_{ij} = [\frac{w_i^L}{w_j^U}, \frac{w_i^U}{w_j^L}]$ are close to DM's judgments, but there is no requirement that W_{ij} must involve or be included within $a_{ij} = [l_{ij}, u_{ij}]$. So, the GP model sounds to be more logical and natural.

Finally, the lower and upper approximation models require extra constraints: $w_{i^*}^L \ge \varepsilon$ or $w_i^{L^*} \ge \varepsilon$ for $i = 1, ..., n$ to avoid the occurrence of zero weights, while the GP model has no such requirements.

c. Global Interval Weights

Suppose that $[w_j^L, w_j^U]$ is the normalized interval weight for criterion j $(j = 1, ..., m)$ and $[w_{ij}^L, w_{ij}^U]$ the normalized interval weight of alternative A_i with respect to the criterion $j (i = 1, ..., n; j = 1, ..., m)$ obtained using the GPM, as shown in Table 2. They satisfy the following normalization constraints:

$$\begin{aligned} & w_j^L + \sum_{k=1, k\neq j}^{m} w_k^U \ge 1, && j = 1, \cdots, m, \\ & w_j^U + \sum_{k=1, k\neq j}^{m} w_k^L \le 1, && j = 1, \cdots, m, \\ & w_{ij}^L + \sum_{k=1, k\neq j}^{m} w_{kj}^U \ge 1, && i = 1, \cdots, n; \ j = 1, \cdots, m, \\ & w_{ij}^U + \sum_{k=1, k\neq j}^{m} w_{kj}^L \le 1, && i = 1, \cdots, n; \ j = 1, \cdots, m. \end{aligned} \qquad (36)$$

Salo and Hamalainen 0 show by an example that interval arithmetic is unsuitable for the synthesis of interval weights. They propose a hierarchical decomposition method that decomposes a hierarchical composition problem into a series of linear programming problems over the feasible regions.

Tab. 2. Synthesis of interval weights

Composite weights	Criterion n	…	Criterion 2	Criterion 1	Alternatives
	$[w_m^L, w_m^U]$	…	$[w_2^L, w_2^U]$	$[w_1^L, w_1^U]$	
$[w_{A_1}^L, w_{A_1}^U]$	$[w_{1m}^L, w_{1m}^U]$	…	$[w_{12}^L, w_{12}^U]$	$[w_{11}^L, w_{11}^U]$	A_1
$[w_{A_2}^L, w_{A_2}^U]$	$[w_{2m}^L, w_{2m}^U]$	…	$[w_{22}^L, w_{22}^U]$	$[w_{21}^L, w_{21}^U]$	A_2
$\vdots$	$\vdots$	…	$\vdots$	$\vdots$	$\vdots$
$[w_A^L, w_{A2}^U]$	$[w_{nm}^L, w_{nm}^U]$	…	$[w_{n2}^L, w_{n2}^U]$	$[w_{n1}^L, w_{n1}^U]$	A_n

Bryson and Mobolurin 0 suggest a linear programming method, which seems simpler and is therefore adopted here. Their method treats the weights of criteria as decision variables and captures respectively the lower and upper bounds of the composite weight of each alternative $A_i\,(i=1,\ldots,n)$ by constructing the following pair of LP models:

$$\text{Minimize} \quad w_{A_i}^L = \sum_{j=1}^{m} w_{ij}^L w_j \tag{37a}$$

$$\text{s.t.} \quad W \in \Omega_W, \tag{37b}$$

$$\text{Maximize} \quad w_{A_i}^U = \sum_{j=1}^{m} w_{ij}^U w_j \tag{37c}$$

$$\text{s.t.} \quad W \in \Omega_W, \tag{37d}$$

Where w_j is the decision variable for the j th criterion weight $(j=1,\ldots,m)$ and

$$\Omega_W = \{W = (w_1,\ldots,w_m)^T \mid w_j^L \le w_j \le w_j^U, \sum_{j=1}^{m} w_j = 1, j = 1,\ldots,m\}$$

The above pair of LP models results in a global interval weight for each alternative A_i denoted by $[w_{A_i}^L, w_{A_i}^U](i=1,\ldots,n)$. The following theorem shows that the global interval weights are always normalized.

Theorem2. Let $[w_{A_i}^L, w_{A_i}^U](i=1,\ldots,n)$ be the global interval weights obtained by the LP models (34)–(37). Then there exist

$$\sum_{i=1}^{n} w_{A_i}^L \le 1 \quad \text{and} \quad \sum_{i=1}^{n} w_{A_i}^U \ge 1,$$

$$w_{A_i}^L + \sum_{j=1, j\ne i}^{n} w_{A_j}^U \ge 1, \quad i = 1,\cdots,n, \tag{38}$$

$$w_{A_i}^U + \sum_{j=1, j\ne i}^{n} w_{A_j}^L \le 1, \quad i = 1,\cdots,n.$$

Proof. Let $\tilde{W} = (\tilde{w}_1,\ldots,\tilde{w}_m)^T \in \Omega_W$ be an arbitrary feasible solution, which may be not optimal to any of w_{Ai}^L and w_{Ai}^U $(i=1,\ldots,n)$. Then, we have:

$$\begin{aligned}
&w_{A_i}^L = \underset{W\in\Omega_W}{\text{Min}} \sum_{j=1}^{m} w_{ij}^L w_j \le \sum_{j=1}^{m} w_{ij}^L \tilde{w}_j,\\
&w_{A_i}^U = \underset{W\in\Omega_W}{\text{Max}} \sum_{j=1}^{m} w_{ij}^U w_j \ge \sum_{j=1}^{m} w_{ij}^U \tilde{w}_j,\\
&\sum_{i=1}^{n} w_{A_i}^L \le \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^L \tilde{w}_j = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} w_{ij}^L\right)\tilde{w}_j \le \sum_{j=1}^{m} \tilde{w}_j = 1,\\
&\sum_{i=1}^{n} w_{A_i}^U \ge \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^U \tilde{w}_j = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} w_{ij}^U\right)\tilde{w}_j \ge \sum_{j=1}^{m} \tilde{w}_j = 1.
\end{aligned} \tag{39}$$

Denote by $X_i^* = (x_{i1}^*,\ldots,x_{im}^*)^T \in \Omega_W$ and $Y_i^* = (y_{i1}^*,\ldots,y_{im}^*)^T \in \Omega_W$ the optimal solutions of the LP model (37a), (37b) for w_{Ai}^L and the LP model (37c), (37d) respectively. Obviously, $X_i^* = (x_{i1}^*,\ldots,x_{im}^*)^T$ and $Y_i^* = (y_{i1}^*,\ldots,y_{im}^*)^T$ are not necessarily optimal w_{Ai}^L or w_{Ai}^U, $j=1,\ldots,n;\ j \ne i$.

So, we have

$$w_{A_i}^L = \sum_{j=1}^{m} w_{ij}^L x_{ij}^*$$

$$w_{A_i}^U = \sum_{j=1}^{m} w_{ij}^U y_{ij}^*, \; w_{A_k}^L = \underset{W\in\Omega_W}{\text{Min}} \sum_{j=1}^{m} w_{kj}^L w_j \le \sum_{j=1}^{m} w_{kj}^L y_{ij}^* \; \text{and} \; w_{A_k}^U = \underset{W\in\Omega_W}{\text{Max}} \sum_{j=1}^{m} w_{kj}^U w_j \ge \sum_{j=1}^{m} w_{kj}^U y_{ij}^*$$

Furthermore, we have

$$\sum_{k=1,k\neq i}^{n} w_{A_k}^{U} \geq \sum_{k=1,k\neq i}^{n}\left(\sum_{j=1}^{m} w_{kj}^{U} x_{ij}^{*}\right) = \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{U}\right)x_{ij}^{*},$$
$$\sum_{k=1,k\neq i}^{n} w_{A_k}^{L} \leq \sum_{k=1,k\neq i}^{n}\left(\sum_{j=1}^{m} w_{kj}^{L} y_{ij}^{*}\right) = \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{L}\right)y_{ij}^{*}. \quad (40)$$

By (36c) and (36d), we get

$$w_{A_i}^{L} + \sum_{k=1,k\neq i}^{n} w_{A_k}^{U} \geq \sum_{j=1}^{m} w_{ij}^{L} x_{ij}^{*} + \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{U}\right)x_{ij}^{*} \geq \sum_{j=1}^{m} w_{ij}^{L} x_{ij}^{*} + \sum_{j=1}^{m}(1-w_{ij}^{L})x_{ij}^{*} = \sum_{j=1}^{m} x_{ij}^{*} = 1,$$
$$w_{A_i}^{U} + \sum_{k=1,k\neq i}^{n} w_{A_k}^{L} \leq \sum_{j=1}^{m} w_{ij}^{U} y_{ij}^{*} + \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{L}\right)y_{ij}^{*} \leq \sum_{j=1}^{m} w_{ij}^{U} y_{ij}^{*} + \sum_{j=1}^{m}(1-w_{ij}^{U})y_{ij}^{*} = \sum_{j=1}^{m} y_{ij}^{*} = 1. \quad (41)$$

In the same way, we can derive similar inequalities for all other intervals $[w_{Aj}^{L}, w_{Aj}^{U}]$ $(j=1,\dots,n;\ j\neq i)$. So, inequalities (38b) and (38c) hold for all $i=1,\dots,n$.

3.1.2. Lexicographic Goal Programming Method

According to Arbel 0, Arbel and Vargas 0 interpretation, interval judgments may be considered as constraints on weights. Accordingly, (1) may be expressed as

$$l_{ij} \leq w_i / w_j \leq u_{ij},\quad i,j=1,\dots,n \quad (42)$$

$$l_{ij} w_j \leq w_i \leq u_{ij} w_j,\quad i,j=1,\dots,n \quad (43)$$

Inequality (42) or (43) holds only for consistent judgments, but they does not hold for conflicting (inconsistent) judgments. In the presence of conflicting judgments, deviation variables p_{ij} and q_{ij} could be introduced into (43), which lead to:

$$l_{ij} w_j - p_{ij} \leq w_i \leq u_{ij} w_j + q_{ij},\quad i,j=1,\dots,n \quad (44)$$

Where p_{ij} and q_{ij} are both nonnegative real numbers, but can't be positive at the same time, i.e. $p_{ij}.q_{ij}=0$. It is desirable that the deviation variables p_{ij} and q_{ij} are kept to be as small as possible, which leads to the following lexicographic goal programming (LGP) model

$$\begin{aligned}
\text{Min}\quad & J=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(p_{ij}+q_{ij})\\
\text{s.t}\quad & w_i - l_{ij}w_j + p_{ij} \geq 0,\quad i=1,\dots,n-1;\ j=i+1,\dots,n,\\
& w_i - u_{ij}w_j - q_{ij} \geq 0,\quad i=1,\dots,n-1;\ j=i+1,\dots,n,\\
& \sum_{i=1}^{n} w_i = 1,\\
& w_i, p_{ij}, q_{ij} \geq 0,\quad \text{for all } i \text{ and } j.
\end{aligned} \quad (45)$$

The LGP model considers only the upper triangular judgments of interval comparison matrices when generating weights because no new information is embodied in the lower triangular judgments. It will be proved in the next section that LGP models are in general not equivalent when the upper or lower triangular judgments are used.

3-1-3. Two-Stage Logarithmic Goal Programming Method

Ying-Ming Wang et al 0 ,applied multiplicative constraint to interval comparison matrix in AHP and developed the following goal programming models to estimate weights. Multiplicative constraint $\prod_{i=1}^{n}\ln w_i = 1$ is equivalent to $\sum_{i=1}^{n}\ln w_i = 0$. Since interval judgments may be interpreted as constraints on weights, accordingly, (1) may be expressed as

$$l_{ij} \leq w_i / w_j \leq u_{ij},\quad i,j=1,\dots,n \quad (46)$$

That can be expressed as the following:

$$\ln l_{ij} \leq \ln w_i - \ln w_j \leq \ln u_{ij},\quad i,j=1,\dots,n \quad (47)$$

Inequality (47) holds only for consistent judgments. To generate a set of unified inequality constraints holding for both consistent and inconsistent judgments, deviation variables p_{ij} and q_{ij} are introduced into the following relation:

$$\ln l_{ij} - p_{ij} \leq \ln w_i - \ln w_j \leq \ln u_{ij} + q_{ij},\quad i,j=1,\dots,n \quad (48)$$

Where p_{ij} and q_{ij} are both nonnegative real numbers, but only one of them can be positive, i.e. $p_{ij}.q_{ij}=0$. For consistent judgments, both p_{ij} and q_{ij} are set to be zero. In the presence of inconsistent judgments, only one of p_{ij} or q_{ij} may be unequal to zero. So, (48) holds for both consistent and inconsistent judgments. It is desirable that the deviation variables p_{ij} and q_{ij} are kept to be as small as possible, which means to minimize the inconsistency of interval comparison matrices, thus leading to the following objective function and goal programming (GP) model:

$$\begin{aligned} &\text{Min } J=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(p_{ij}+q_{ij}) \\ &\text{s.t. } \ln w_i-\ln w_j+p_{ij}\ge \ln l_{ij}, \quad i,j=1,\cdots,n, \\ &\qquad \ln w_i-\ln w_j-q_{ij}\le \ln u_{ij}, \quad i,j=1,\cdots,n, \\ &\qquad \sum_{i=1}^{n}\ln w_i \\ &\qquad p_{ij},q_{ij}\ge 0 \quad \text{and} \quad p_{ij}q_{ij}=0, \quad i,j=1,\cdots,n. \end{aligned} \tag{49}$$

Since the value of $\ln w_i$ is nonnegative when $w_i \ge 1$ and negative when $w_i<1$, the following nonnegative variables are introduced:

$$\begin{aligned} x_i&=\frac{\ln w_i+|\ln w_i|}{2}, \quad i=1,\cdots,n, \\ y_i&=\frac{-\ln w_i+|\ln w_i|}{2}, \quad i=1,\cdots,n. \end{aligned} \tag{50}$$

Based on x_i and y_i, $\ln w_i$ can be expressed as

$\ln w_i = x_i - y_i,\ i=1,...,n$

Where $x_i . y_i = 0$. Thus, the above GP model (49) can be expressed and simplified as

$$\begin{aligned} &\text{Min } J=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(p_{ij}+q_{ij}) \\ &\text{s.t. } x_i-y_i-x_j+y_j+p_{ij}\ge \ln l_{ij}, \quad i=1,\cdots,n-1, \quad j=i+1,\cdots,n, \\ &\qquad x_i-y_i-x_j+y_j-q_{ij}\le \ln u_{ij}, \quad i=1,\cdots,n-1, \quad j=i+1,\cdots,n, \\ &\qquad \sum_{i=1}^{n}(x_i-y_i)=0, \\ &\qquad x_i,y_i\ge 0, \quad x_iy_i=0, \quad i=1,\cdots,n, \\ &\qquad p_{ij},q_{ij}\ge 0, \quad p_{ij}q_{ij}=0, \quad i=1,\cdots,n-1, \quad j=i+1,\cdots,n \end{aligned} \tag{51}$$

Or

$$\begin{aligned} &\text{Min } J=\sum_{i=2}^{n}\sum_{j=1}^{i-1}(p_{ij}+q_{ij}) \\ &\text{s.t. } x_i-y_i-x_j+y_j+p_{ij}\ge \ln l_{ij}, \quad i=2,\cdots,n, \quad j=1,\cdots,i-1, \\ &\qquad x_i-y_i-x_j+y_j-q_{ij}\le \ln u_{ij}, \quad i=2,\cdots,n, \quad j=1,\cdots,i-1, \\ &\qquad \sum_{i=1}^{n}(x_i-y_i)=0, \\ &\qquad x_i,y_i\ge 0, \quad x_iy_i=0, \quad i=1,\cdots,n, \\ &\qquad p_{ij},q_{ij}\ge 0, \quad p_{ij}q_{ij}=0, \quad i=2,\cdots,n, \quad j=1,\cdots,i-1. \end{aligned} \tag{52}$$

About the above GP models, there exist the following theorems.

Theorem1. $A=(a_{ij})_{n\times n}$ is a consistent interval comparison matrix if and only if $J^*=0$, where J^* is the optimal value of objective function (51a) or (52a).

Proof. If A is a consistent interval comparison matrix, then the convex feasible region S_w is nonempty, which means that $l_{ij}\le w_i/w_j\le u_{ij}$ holds for all the judgments, equivalently, $\ln l_{ij}\le \ln w_i-\ln w_j\le \ln u_{ij},\ i,j=1,...,n$. So $p_{ij}=0$, $q_{ij}=0$ for all the judgments, which is equivalent to $J^*=0$. If $J^*=0$, then $p_{ij}=0$ and $q_{ij}=0$ hold for all $i,j=1,...,n$. Accordingly, inequality (47) holds for all the judgments. This means that (46) holds for all $i,j=1,...,n$. In other words, there is no contradiction among all the judgments. So, the convex feasible region S_w cannot be empty when $J^*=0$. By Definition 6, A is a consistent interval comparison matrix .

Theorem2. GP models (51) and (52) are equivalent.

Proof. Consider a reciprocal pair of interval judgments, say, $l_{ij}\le a_{ij}\le u_{ij}$ and $1/u_{ij}\le a_{ij}\le 1/l_{ij}$. By introducing the deviation variables, the above reciprocal pair of interval judgments can be transformed to the following pair of inequality constraints:

$$\ln l_{ij}-p_{ij}\le \ln w_i-\ln w_j\le \ln u_{ij}+q_{ij} \tag{53}$$

$$-\ln u_{ij} - p_{ji} \le \ln w_i - \ln w_j \le -\ln l_{ij} + q_{ji}$$

Inequality (53a) may be further written as

$$\ln u_{ij} - q_{ij} \le \ln w_i - \ln w_j \le \ln l_{ij} + p_{ji} \qquad (54)$$

Let $p_{ij} = q_{ji}$ and $q_{ij} = p_{ji}$. Then, inequality constraints (53a) and (40) are indeed equivalent. Besides, since $p_{ij} + q_{ji} = p_{ji} + q_{ji}$, the contributions of deviation variables $p_{ij}, q_{ji\,ji}$ and $p_{ji} + q_{ji}$ to their respective objective functions are also equivalent. Since the above discussion can be applied to all the reciprocal pairs of interval judgments, it can be concluded that models (51) and (52) are in fact equivalent.

Note that Theorem 1 shows how to check if an interval comparison matrix is consistent or not. Theorem 2 ensures that using the upper or lower triangular judgments of an interval comparison matrix will always lead to the same results, Since Models (51) and (52) are equivalent in nature, we will consider only GP model (51) in the rest. Generally speaking, there may be multiple solutions to the GP model, which leads to intervals of weights. In order to find a feasible interval for each weight $w_i = (i = 1, \cdots, n)$, we keep the optimal objective function value unchanged and use it as a constraint to construct the following pairs of GP models:

$$\begin{aligned}
&\text{Min/Max} \quad \ln w_i = x_i - y_i \\
&\text{s.t.} \quad x_i - y_i - x_j + y_j + p_{ij} \ge \ln l_{ij}, \quad i = 1, \cdots, n-1, \quad j = i+1, \cdots, n, \\
&\qquad x_i - y_i - x_j + y_j - q_{ij} \le \ln u_{ij}, \quad i = 1, \cdots, n-1, \quad j = i+1, \cdots, n, \\
&\qquad \sum_{i=1}^{n} (x_i - y_i) = 0, \\
&\qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_{ij} + q_{ij}) = j^*, \\
&\qquad x_i, y_i \ge 0, \quad x_i y_i = 0, \quad i = 1, \cdots, n, \\
&\qquad p_{ij}, q_{ij} \ge 0, \quad p_{ij} q_{ij} = 0, \quad i = 1, \cdots, n-1, \quad j = i+1, \cdots, n.
\end{aligned} \qquad (55)$$

Where J^* is the optimal value of the objective function of the GP model (51). Note that the complementary constraints $x_i y_i = 0$ ($p_{ij} q_{ij} = 0$) can always be satisfied without x_i and y_i (p_{ij} and q_{ij}) being simultaneously selected as basic variables in a simplex method. The optimal objective values of the above pairs of GP models (55) consist of the possible intervals of the logarithmic weights $\ln w_i$ ($i = 1,...,n$), which are denoted by the logarithmic weight intervals $[\ln w_i^L, \ln w_i^U]$ ($i = 1,...,n$) accordingly, the weight intervals $[w_i^L, w_i^U]$ can be obtained from logarithmic weight intervals, where $w_i^L = \exp(\ln w_i^L)$ and $w_i^U = \exp(\ln w_i^U)$. Since the whole solution process for generating weights includes two stages, the method is thus referred to as the two-stage logarithmic goal programming (TLGP) Method.

Theorem3. If $J^* = 0$, then TLGP degenerates to solving the following pairs of GP models:

$$\begin{aligned}
&\text{Min/Max} \ln w_i = x_i - y_i \\
&\text{s.t.} \quad x_i - y_i - x_j + y_j \ge \ln l_{ij}, \quad i = 1, \cdots, n-1, \quad j = i+1, \cdots, n, \\
&\qquad x_i - y_i - x_j + y_j \le \ln u_{ij}, \quad i = 1, \cdots, n-1, \quad j = i+1, \cdots, n, \\
&\qquad \sum_{i=1}^{n} (x_i - y_i) = 0, \\
&\qquad x_i, y_i \ge 0, \quad x_i y_i = 0, \quad i = 1, \cdots, n.
\end{aligned} \qquad (56)$$

The proof of Theorem 3 is straightforward. This theorem shows that if an interval comparison matrix A already been known to be consistent, then only the GP model of the second stage will need to be solved.

3.2. Linear Programming Method

There are several methods that can be used to derive priorities from interval comparison matrices. Arber's preference programming method is the simplest yet most effective way to derive priorities from consistent interval comparison matrices. The method can generate consistent interval weights that can satisfy all judgments in a consistent interval comparison matrix.

So, it is recommended to use the theorem 1 to judge if an interval comparison matrix is consistent or not.

The method was originally developed to find the vertices of the convex feasible region,

$$S_w = \{w = (w_1, \ldots, w_n)\} \mid l_{ij} \le w_i / w_j \le u_{ij}, \sum_{i=1}^{n} w_i = 1, w_i > 0, j = 1, \ldots n\}$$

If all the vertices prefer w_l to $w_k (l, k = 1, \cdots, n)$, then any convex linear combination of all the vertices would prefer w_l to w_k. Usually, the priority vector was generated as a convex linear combination of all the vertices. Since the convex combination produces only a point estimation of priorities, we propose to generate interval weights as the final priorities, which can be obtained by solving the following pairs of linear programming (LP) models:

$$\begin{aligned}&\text{Min/Max} \quad w_i\\&\text{s.t.} \quad W \in S_W,\end{aligned} \tag{57}$$

Where $W = (w_1, w_2, ..., w_n)^T$. The solutions to the above pairs of LP models form the weight intervals denoted by $[w_i^L, w_i^U](i = 1, \cdots, n)$.

3.2.1. Numerical Example

Consider the following comparison matrix:

$$\begin{pmatrix} 1 & \left[\frac{1}{2}, 2\right] & \left[\frac{1}{6}, 6\right] \\ \left[\frac{1}{2}, 2\right] & 1 & \left[\frac{1}{3}, 3\right] \\ \left[\frac{1}{6}, 6\right] & \left[\frac{1}{3}, 3\right] & 1 \end{pmatrix}$$

As can be seen, the above matrix is consistent.

$\max_{k=1,2,3}(l_{ik} l_{kj}) \le \min_{k=1,2,3}(u_{ik} u_{kj})$

The linear programming models for w_1, for example, is as follows:

$$\begin{aligned}&\min/\max w_1\\&st:\\&\frac{1}{2} \le \frac{w_1}{w_2} \le 2\\&\frac{1}{6} \le \frac{w_1}{w_3} \le 6\\&\frac{1}{2} \le \frac{w_2}{w_1} \le 2\\&\frac{1}{3} \le \frac{w_2}{w_3} \le 3\\&\frac{1}{3} \le \frac{w_3}{w_2} \le 3\\&w_1 + w_2 + w_3 = 1\\&w_1, w_2, w_3 > 0\end{aligned}$$

The objective function $\min w_1$, leads to w_1^L and the objective function $\max w_1$, leads to w_1^U.

The interval for w_1 is as follows: $\left[w_1^L, w_1^U\right] = [0.11, 0.6]$.

3.3. Nonlinear Programming Method

If an interval comparison matrix is judged to be inconsistent using Theorem 1, it appears particularly important to derive the priorities with pre-determined consistency requirements being met. The literature review shows that this problem has not been addressed. This subsection is devoted to investigating this problem and proposing an Eigenvector Method (EM)-based nonlinear programming method, which can be used to generate satisfactory interval weights from inconsistent interval comparison matrices.

From the principal right eigenvector method, it is known that

$$\hat{A}W = \lambda_{\max} W \tag{58}$$

Where $\hat{A}$ is a crisp comparison matrix; $\lambda_{\max}$ is the maximum eigenvalue of the comparison matrix $\hat{A}$; W is the principal right eigenvector corresponding to $\lambda_{\max}$. The relationship between $\lambda_{\max}$ and the consistency ratio (CR) can be described by:

$$\text{CR} = \frac{\text{CI}}{\text{RI}} = \frac{\lambda_{\max} - n}{n-1} \Big/ \text{RI} = \frac{\lambda_{\max} - n}{(n-1)\text{RI}}, \tag{59}$$

Where RI is an average random consistency index [15] that depends on the particular AHP scale used. It is suggested that if CR ≤ 0.1 the comparison matrix is believed to have satisfactory consistency and to be acceptable and that if CR > 0.1 it has poor consistency and needs to be revised. Formula (59) may be further written as:

$$\lambda_{\max} = n + (n-1)\text{RI.CR}. \tag{60}$$

Substituting (10) into (8) produces

$$\hat{A}W = [n + (n-1)\text{RI.CR}]W. \tag{61}$$

Relation (61) is derived from crisp comparison matrices and can be extended to the interval comparison matrices. Suppose $\hat{A} = (\hat{a}_{ij})_{n\times n}$ is a crisp comparison matrix, which is randomly generated from the interval comparison matrix A with $l_{ij} \le \hat{a}_{ij} \le u_{ij}$ and $\hat{a}_{ij} = 1/\hat{a}_{ij}$. Then relation (61) holds for $\hat{A}$. Thus, the following pairs of nonlinear programming models, which are based on Saaty's principal right eigenvector method, can be developed to generate the weight intervals with satisfactory consistency:

$$\begin{aligned}&\text{Min/Max} \, w_i\\&\text{s.t.} \quad \sum_{j=1}^{i-1} \frac{w_j}{\hat{a}_{ij}} - (n-1)(1+\text{RI.CR})w_i + \sum_{j=i+1}^{n} \hat{a}_{ij} w_j = 0, \quad i = 1, \cdots, n\\&\qquad \sum_{i=1}^{n} w_i = 1,\\&\qquad l_{ij} \le \hat{a}_{ij} \le u_{ij}, \quad i = 1, \cdots, n-1; \; j = i+1, \cdots, n,\\&\qquad \text{CR} \le \delta,\end{aligned} \tag{62}$$

Where (61b) is the expansion of (61) and the (62c) is interval constraints, d is the level of satisfactory consistency (e.g. $\delta \le 0.1$), CR, $w_i\,(i=1,...,n)$ and $\hat{a}_{ij}\,(i=1,...,n-1;\ j=i+1,...,n)$ are all decision variables. The purpose of imposing constraint condition (62d) on the above NLP models is to derive the weights satisfying the level of satisfactory consistency. The optimal objective values of the above pair of NLP models consist of the possible interval of w_i, that are denoted by $[w_i^L, w_i^U]$. Repeating the above solution process for each weight $w_i\,(i=1,...,n)$, all the priority intervals that meet the requirement of the satisfactory consistency can be obtained.

The numerical value of δ in (62d) may be determined or adjusted according to the actual requirements of decision analysis. Moreover, the following NLP model, which minimizes the inconsistency of an interval comparison matrix, may also be utilized to derive the weights from an inconsistent interval matrix:

$$\begin{aligned}
&\text{Min CR}\\
&\text{s.t.}\ \sum_{j=1}^{i-1}\frac{w_j}{\hat{a}_{ij}} - (n-1)(1+\text{RI.CR})w_i + \sum_{j=i+1}^{n}\hat{a}_{ij}w_j = 0,\quad i=1,\cdots,n,\\
&\qquad \sum_{i=1}^{n} w_i = 1,\\
&\qquad l_{ij} \le \hat{a}_{ij} \le u_{ij},\quad i=1,\cdots,n-1;\ j=i+1,\cdots,n.
\end{aligned} \tag{63}$$

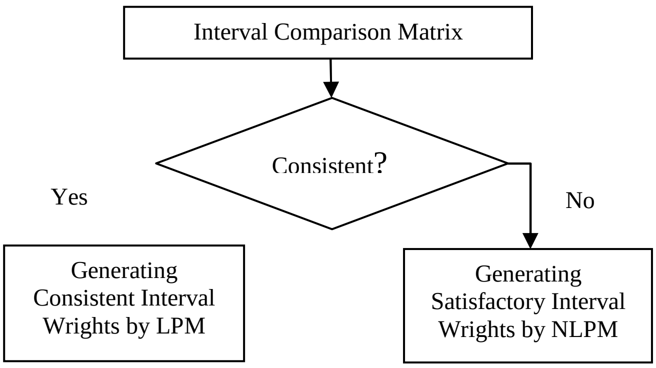

Fig. 2. Process for generating priority from interval comparison matrices

Such a NLP model usually leads to one crisp set of weights, which has the minimum inconsistency. The whole process introduced above for generating interval weights from interval comparison matrices is summarized in Fig.2.

3.4. Stochastic Approach

Stan Lipovetsky and Asher Tishler, 1999 treat the elements of the pairwise comparison matrices as realizations of random variables and discusse about the strongest features of the AHP that generates numerical priorities from the subjective knowledge expressed in the estimates of pairwise comparison matrices.

For this purpose, they propose that the values given by the judge for the ratios of priorities are considered as the random variables. Clearly, if a_{ji} in relation (2) is a random variable, so $a_{ji} \equiv \frac{1}{a_{ij}}$ is also a random variable. Lipovetsky and Tishler 0, analyze the AHP for five types of random variables (distributions): triangle, beta, normal, Laplace, and Cauchy. They show that the probability density function (p.d.f.) of $\frac{1}{a_{ij}}$ is not the same as the p.d.f. of a_{ij} when the distribution of a_{ji} is normal, triangle, Laplace, or beta. If, however, the p.d.f. of a_{ji} is Cauchy, then the p.d.f. of the random variable $a_{ji} \equiv \frac{1}{a_{ij}}$ is also Cauchy [8].

This property of the Cauchy p.d.f. is crucial to the analysis of a Saaty matrix, because the ratios of the priorities defined in relation (2) and evaluated by a judge depend on the order in which the pairs of objects were compared, which is arbitrary. That is, the judge could either be asked to assess the value of $\frac{\alpha_j}{\alpha_i}$, or the value of $\frac{\alpha_i}{\alpha_j}$. But, it is reasonable to suppose that the distribution function of the priority ratio should not depend on the order in which the objects are presented for consideration. That is to say, if the Cauchy p.d.f. accurately describes the process by which the judge estimates the value a_{12} (the ratio of the priorities of the first and second objects), then the p.d.f. of the random variable, a_{21}, in the lower triangle of the Saaty matrix will also be Cauchy.

We believe that the use of other distribution functions such as the normal, Laplace, triangle or beta distribution to generate the priority ratios in the lower triangle of the Saaty matrices is less desirable; since the p.d.f. of the priority ratios derived using these distributions depends on the order in which the objects were presented to the judge. Note also that if the priorities α_1 and α_2 are drawn from a normally distributed population, then the marginal p.d.f. of the ratio $\frac{\alpha_1}{\alpha_2}$ is Cauchy [8]. The p.d.f. of the inverse, $\frac{\alpha_2}{\alpha_1}$, is also Cauchy. Thus, if the priorities are normally distributed, their ratios - the values a_{ij} and a_{ji} in relations (1)-(3) are described by a Cauchy distribution. Specifically, each element of the upper triangle of matrix (1) is assumed to be a random variable with the following p.d.f.:

$$f(x) = \frac{1}{\pi b}\frac{1}{1+\left[\frac{x-a}{b}\right]^2} \tag{64}$$

The mode (maximal probability) of the p.d.f. (64) is at the central point $x = a$, and the parameter b serves as a mean deviation from the central value.
We shall consider only positive values of x in order to obtain meaningful (positive) elements in relation (1).
Since each element in the upper triangle (1) is a random variable distributed according to relation (64), then each element in the lower triangle of relation (1) (obtained by relation (3)), is also a random variable.
The p.d.f. of each element of the lower triangle of relation (1) is derived as follows. Let *x* be a random variable. Then its inverse is given by

$$y = u(x) \equiv \frac{1}{x} \tag{65}$$

The p.d.f. of the random variable y in relation (65) is given by [8].

$$g(y) = f\left[w(y)\right]\left|w'(y)\right| \tag{66}$$

The inverse function of *y* is denoted by $x = w(y)$,and its derivative is $\frac{dx}{dy} = w'(y)$. For the function $y = \frac{1}{x}$, the inverse and the derivative are $x = \frac{1}{y}$, and $\frac{dx}{dy} = -\frac{1}{y^2}$ respectively. Thus, the p.d.f. (66) can be written as following:

$$g(y) = \frac{1}{y^2} f[\frac{1}{y}] \tag{67}$$

The p.d.f. (67) of the inverse of the variable, whose probability distribution is given by relation (64), is as follows. First, to simplify the analysis and ensure that the a_{ij} s in relation (1) are positive, we introduce the following notation:

$$z \equiv ay \tag{68}$$

$$\xi \equiv \frac{b}{a},\ 0 \le \xi \le 1 \tag{69}$$

Assume that the range of *n* in relation (69) is $[0,1]$ that implies that *a*, the center of the random variable representing the evaluations of the priorities, is in the range $[0;2a]$, and $b \le a$. This assumption ensures that the evaluations in relation (1) are non-negative, and that the deviations are symmetrically distributed around the center of a. transforming the Cauchy p.d.f. (64), using formula (67), yields the following p.d.f. for the inverse random variable (65).

$$g(y) = a\frac{\xi(1+\xi^2)}{\pi}\frac{1}{\xi^2 + [z(1+\xi^2)-1]^2},\ z > 0. \tag{70}$$

Comparison of relation (64) with relation (70) shows that x and $\frac{1}{x}$ possess a similar form.

In practice, the elements of the lower triangle in relation (1) are not estimated directly by a judge. Rather, they are assumed to be the reciprocals of the values in the upper triangle of (1). Thus, if the values of the elements of the upper triangle of the Saaty matrix are functions of random variables which are described by the distribution (64), then their reciprocals, the elements of the lower triangle of the Saaty matrix, must be estimated using the corresponding p.d.f. in relation (70).
Thus, we suggest the following procedure to estimate the a_{ij} s in (1). It is usually assumed that the parameter of the center of the p.d.f., that coincides with the point of the maximum probability of the p.d.f., is equal to the value of the random variable $x = a = a_{ij}$. That is, it coincides with the value given by the judge to the a_{ij} in the upper triangle of the matrix (1). Thus, we propose to estimate the center of the inverse variable, $y = \frac{1}{x}$, by the mode, or the maximal probability, of *y*. The point of maximal probability of the p.d.f. (70) is obtained by solving $\frac{\partial g(y)}{\partial y} = 0$. For the transformed Cauchy p.d.f. we have:

$$Z^* = \frac{1}{1+\xi^2},\quad 0 \le \xi \le 1. \tag{71}$$

Clearly, the maximal probability of $\frac{1}{x^*} = y^* = \frac{z^*}{a}$ is less than $\frac{1}{a}$ for all $\xi > 0$. A similar result is obtained for the beta, normal, triangle and Laplace distributions 0. This result implies that the centers of the random variables in the lower triangle of (1) are generally not reciprocally symmetric with the centers of the original random variables in the upper triangle of the pairwise comparison matrix (1).
In fact, the influence of the elements of the lower triangle of the Saaty matrix on the estimated priority vector will be smaller when we compute these elements using the mode of y; that is,

$$a_{ij} = \frac{1}{a_{ij}} Z^* \tag{72}$$

Instead of the reciprocal values (3) . it can be proved that Cauchy distribution is very useful for randomizing data 0. The shape of the Cauchy p.d.f. resembles to a normal p.d.f., but it diminishes less steeply (exhibits "fat tails"). For small deviations, $|x-a|<b$, expansion of the p.d.f. by a Taylor series (as a first approximation to a Cauchy p.d.f., or to a normal p.d.f.) yields a parabolic dependence similar to the structure of a beta distribution. The parameter representing the center of the Cauchy p.d.f. coincides with the mean value of the Cauchy p.d.f.. At the modal point, $x=a$, the value $f(a)$ is equal to $\frac{1}{(\pi b)}$. At the points $x=a\pm b$, the p.d.f. equals half the height of the peak, $f(a\pm b)=\frac{1}{(2\pi b)}=f(a)/2$. Also, $f(a\pm 2b)=f(a)/5$, and $f(a\pm 3b)=f(a)/10$. Thus, the parameter b can be interpreted as a measure of the mean deviation from the center of the p.d.f.. Using the relation (67) to transform a random variable x with a Cauchy p.d.f. (See relation (64)), we obtain after simple transformations, the following p.d.f. for $y\equiv\frac{1}{x}$,

$$g(y)=\frac{1}{\pi b}\frac{1}{1+[\frac{y^{-1}-1}{b}]^2}\frac{1}{y^2}=\frac{1}{\pi\tilde{b}}\frac{1}{1+[\frac{y-\tilde{a}}{\tilde{b}}]^2}, \qquad (73)$$

With the following new values for the parameters:

$$\tilde{a}\equiv\frac{a}{a^2+b^2}=\frac{1}{a}\frac{1}{1+\xi^2}, \qquad (74\text{ a})$$

$$\tilde{b}\equiv\frac{b}{a^2+b^2}=\frac{1}{a}\frac{\xi}{1+\xi^2}, \qquad (74\text{ b})$$

Clearly, formula (73) is another representation of the Cauchy p.d.f. (70) where the parameter ξ is defined in formula (69). Relations (73) and (74 a) reveals that $\tilde{a}$, the value of the center of y, is smaller than $\frac{1}{a}$ by the term z^* defined in the relation (71). The p.d.f. of the inverse variable with the distribution (14) reproduces the initial Cauchy p.d.f. (64). This can be seen by applying the transformation (15a), (15b) to the parameters in relations (74 a) and (74 b):

$$\tilde{\tilde{a}}=\frac{\tilde{a}}{\tilde{a}^2+\tilde{b}^2}=\frac{a}{a^2+b^2}/\frac{a^2+b^2}{(a^2+b^2)^2}=a. \qquad (75)$$

Analogously, $\tilde{\tilde{b}}=b$. Thus, relations (5) and (14) are p.d.f.s of mutually reciprocal random variables.

3-4-1. The Construction of Stochastic Salty Matrices

We are now ready to use the Cauchy p.d.f. to generate the lower triangle elements of a Saaty matrix. Accounting the stochastic nature of the process , a value is assigned to each element a_{ij}, and we evaluate the Saaty matrix (1) for three values: the actual a_{ij}, and $a_{ij}\pm b$ (b represents the mean deviation from the center a). First, select a value for ξ. For example, $b=a/2$, $\xi=1/2$ or (see relation (4)). Then, change each element of the upper triangle of the Saaty matrix (1) to a number in the range $[a-b;a+b]$. For example,

$$a_{ij}\rightarrow\{(1-\xi)a_{ij};a_{ij};(1+\xi)a_{ij}\}=\{0.5a_{ij},a_{ij},1.5a_{ij}\}. \quad (76)$$

Using relations (73) and (74) we obtain the corresponding values for the lower triangle of the Saaty matrix. These values are within the range $\left[\tilde{a}+\tilde{b};\tilde{a}-\tilde{b}\right]$, and the maximal (minimal) value of a_{ji} corresponds to the minimal (maximal) value of a_{ij} in relation (76):

$$a_{ij}\Rightarrow\{\frac{1+\xi}{1+\xi^2}\frac{1}{a_{ij}};\frac{1}{1+\xi^2}\frac{1}{a_{ij}};\frac{1-\xi}{1+\xi^2}\frac{1}{a_{ij}}\}=\{1.2\frac{1}{a_{ij}};0.8\frac{1}{a_{ij}};0.4\frac{1}{a_{ij}}\}. \qquad (77)$$

If all elements are chosen using the same value for n, then the resulting Saaty matrix obtained by relations (76) and (77) can be represented as a weighted sum of the diagonal, the upper triangle and the lower triangle portions of a Saaty matrix (1):

$$\hat{A}=I_n+k\begin{bmatrix}0 & a_{12} & \dots & a_{1n}\\ & 0 & .. & a_{2n}\\ & & \dots & \vdots\\ & & & 0\end{bmatrix}+qz^*\begin{bmatrix}0 & & & \\ a_{21} & 0 & & \\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & \dots & 0\end{bmatrix} \qquad (78)$$

Where I_n is an identity diagonal matrix of order n, the term z^* is as defined in relation (71), and $a_{ji}=\frac{1}{a_{ij}}$ (see relation (3)). The terms k and q define the location of the random values, i.e., the left edge, the center, or the right edge of the range (76) (the right edge, the center, or the left edge of the range (77)). Specifically:

$$\begin{cases}\text{left edge}\\ k=1-\xi\\ q=1+\xi\end{cases}; \quad \begin{cases}\text{center}\\ k=1\\ q=1\end{cases}; \quad \begin{cases}\text{right edge}\\ k=1+\xi\\ q=1-\xi\end{cases} \qquad (79)$$

The random values in the lower triangle Saaty matrix are smaller than the inverse of the corresponding values in the upper triangle.

That is, the priorities must be estimated using a reciprocally nonsymmetrical matrix. Thus, the "observed" values- i.e., those given explicitly by the judge, and arranged in the upper triangle of the pair wise comparison matrix - seem to play a more significant role in determining the priority vector than do the ``unobserved" elements in the lower triangle.

The conventional AHP method of priority evaluation (4) was constructed specifically for reciprocally symmetric pair wise comparison matrices (Saaty, 1980). We have just shown that the AHP has to be modified to account for nonsymmetrical matrices whose structure is given by relation (78).

3.4.2. The AHP with Reciprocally Nonsymmetrical Matrices

This section discuss about reciprocally nonsymmetrical Saaty matrices in four variants of the AHP.

3.4.2.1. The Conventional AHP

When all the a_{ij} s are defined as ratios of priorities (see relation (2)), multiplication of the random matrix (78) by the vector of priorities yields:

$$\hat{A}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}=\begin{bmatrix}[1+k(n-1)]\alpha_1\\ [1+k(n-2)+qz^*]\alpha_2\\ \vdots\\ [1+(n-1)qz^*]\alpha_n\end{bmatrix}\equiv D\alpha \tag{80}$$

Where D is the following diagonal matrix:

$$D\equiv \operatorname{diag}\{[1+k(n-1)],[1+k(n-2)+qz^*], \ldots,[1+(n-1)qz^*]\}. \tag{81}$$

Since the elements in the matrix (78) are real evaluations and not the theoretically exact ratios of priorities, relation (80) can be replaced by the generalized eigen problem

$$\hat{A}\alpha=\lambda D\alpha, \tag{82}$$

For which the maximal eigenvalue λ can differ from 1 (due to inconsistencies in the judge's estimates of the initial priority ratios in relation (1)). Elements of the diagonal matrix D can be written in the general form:

$$d_{ii}=1+qz^*(i-1)+k(n-i), \qquad i=1,\ldots,n. \tag{83}$$

If the matrix $\hat{A}$ is not random (and hence, it is reciprocally symmetric), then $\xi=0$ (see relation (69)), $Z^*=1$ (see relation (71)) and $k=q=1$ (see relation (79)). In this case, the matrix D is a scalar matrix with all the elements given by $d_{ij}=n$, and the generalized eigen problem (82) reduces to the conventional AHP eigen problem (4).

3-4-2-2. The Logarithmic Method

The so-called logarithmic method (LN), or multiplicative mode, of estimating priorities in the AHP is described in [7, 16, and 21].

The LN method yields a solution in which the priorities are proportional to the geometric mean values of the elements in the rows of the Saaty matrix (1). That is,

$$\alpha_i=\left(\prod_{j=1}^{n}a_{ij}\right)^{1/n}, \qquad i=1,\ldots,n. \tag{84}$$

The reciprocal symmetry of the elements of a conventional Saaty matrix allows priorities to be estimated by the **LN** method using the elements in the columns of matrix (1). The vector β is the reciprocal of the vector α in relation (84), and is given by (up to a multiplicative normalization):

$$\alpha_i=\beta_i^{-1}, \qquad i=1,\ldots,n. \tag{85}$$

Using a random matrix (78) instead of relation (1), i.e., using a reciprocally nonsymmetrical matrix, results in the model,

$$\hat{a}_{ij}=\frac{\tilde{\alpha}_i}{\tilde{\beta}_i^{-1}}(1+\varepsilon_{ij}), \tag{86}$$

The elements $\hat{a}_{ij}$ (the random pair wise comparisons) are represented as ratios of the estimates of the $\tilde{\alpha}_i$ s and the dual estimates, the $\tilde{\beta}_j^{-1}$ s, of the priorities in the random data evaluation.

The ε_{ij} s are random relative deviations. Note that the dual vector $\tilde{\beta}$ is analogous to the left eigenvector (the eigenvector of a transposed Saaty matrix), which is also used in the conventional AHP (Saaty, 1980). Minimizing the objective function:

$$\mathrm{LN}=\sum_{ij}(\ln(1+\varepsilon_{ij}))^2\to\min \tag{87}$$

Yields priority estimates that are the geometric means of the elements in the rows and columns of the random matrix (78):

$$\tilde{\alpha}_i=\left(\prod_{j=1}^{n}\hat{a}_{ij}\right)^{1/n}, \qquad \tilde{\beta}_i=\left(\prod_{i=1}^{n}\hat{a}_{ij}\right)^{1/n} \tag{88}$$

Clearly, $\tilde{\alpha}_i\neq\tilde{\beta}_i^{-1}$ since relation (3) does not hold.

When only one priority vector is used in relation (86) we have:

$$\tilde{\alpha}_{ij}=\frac{\tilde{\alpha}_i}{\tilde{\alpha}_j}(1+\varepsilon_{ij}) \tag{89}$$

And the solution of the objective function (87) for the model (89) yields priority estimates of the form

$$\tilde{\alpha}_k = \left(\prod_{j=1}^{n} \hat{a}_{kj} \Big/ \prod_{i=1}^{n} \hat{a}_{ik} \right)^{1/2n} = (\tilde{\alpha}_k \tilde{\beta}_k^{-1})^{1/2} \tag{90}$$

Thus, if one uses priority ratios as in relation (78), the **LN** solution can be written as a geometric mean of the $\tilde{\alpha}_i$ s and the corresponding reciprocal estimators $\tilde{\beta}_i^{-1}$ s, where $\tilde{\alpha}$ and $\tilde{\beta}$ are defined in relation (88).

If all the elements of a Saaty matrix are generated with the same value of ξ (see relation (69)) (as is done in relations(78)-(82)), then expressions (88) and (90) can be represented by a simple form.

Using the elements of the matrix (78) in relation (88) yields the following evaluations (up to normalization):

$$\tilde{\alpha}_i = \alpha_i (\frac{qz^*}{k})^{1/n}, \qquad \tilde{\beta}_i = \beta_i (\frac{k}{qz^*})^{1/n}, \qquad i = 1,...,n \tag{91}$$

Te α_i s and β_i s are the **LN** solutions (84) and (85) for the nonrandom data, z^* is as defined in Eq. (71), and the parameters k and q are as in relation (79). Clearly, the estimates (91) satisfy relation (87), as do the nonrandom priorities, i.e., $\tilde{\alpha}_i = \tilde{\beta}_i^{-1}$ (up to a normalization). This means that $\tilde{\alpha}_i$ in relation (90) coincides with $\tilde{\alpha}_i$ in relation (91). An interesting property of the **LN** method with random data is obtained when the $\tilde{\alpha}_i$ s in relation (91)

An interesting property of the **LN** method with random data is obtained when the $\tilde{\alpha}_i$ s in relation (91) are evaluated for the edges and the center of the relevant random values. Using definitions (79) for the location parameters, yields:

$$(\tilde{\alpha}_i)_{\text{left}} = \alpha_i (\frac{1+\xi}{1-\xi} z^*)^{i/n}; \qquad (\tilde{\alpha}_i)_{\text{center}} = \alpha_i (z^*)^{i/n};$$
$$(\tilde{\alpha}_i)_{\text{right}} = \alpha_i (\frac{1-\xi}{1+\xi} z^*)^{i/n}. \tag{92}$$

Thus, the priority weights at the center and at the edges of the random data in the generalized **LN** method are connected by the following geometric mean relation:

$$(\tilde{\alpha}_i)_{\text{center}} = \left[(\tilde{\alpha}_i)_{\text{left}}, (\tilde{\alpha}_i)_{\text{right}}\right]^{1/2}, \qquad i = 1,...,n. \tag{93}$$

Clearly, as is usually done in AHP solutions, all the **LN** priority vectors need to be normalized (the sum of their elements is set equal to 1).

3.4.2.3. The Synthetic Hierarchy Method

In this section we will show how to modify the Synthetic Hierarchy Method (SHM) 0 to accommodate random data. The SHM, in linearized form, may be represented as follows:

$$a_{ik}\gamma_k = \gamma_i + \varepsilon_{ik}, \qquad i,k = 1,...,n. \tag{94}$$

Least-squares (LS) estimation of the parameters in relation (94) is defined by

$$\text{LS} = \|\varepsilon\|^2 = \sum_{i,k=1}^{n} (a_{ik}\gamma_k - \gamma_i)^2 \to \min \tag{95}$$

Subject to the normalizing condition

$$\sum_{i=1}^{n} \gamma_i^2 = 1. \tag{96}$$

Optimization problems (95) and (96) imply

$$F \equiv \text{LS} - \lambda \left[\sum_{i=1}^{n} \gamma_i^2 - 1 \right] \to \min. \tag{97}$$

The first order conditions of relation (97) yield the eigen problem:

$$S\gamma = \lambda\gamma \tag{98}$$

Where

$$S \equiv \text{diag}(\overline{A}A) - \overline{A}, \tag{99}$$

A is defined by relation (1), and $\overline{A} \equiv A + A'$. The SHM can easily be modified to account for random data by replacing the Saaty matrix A in relation (1) by $\hat{A}$ in relation (78). The first order conditions of relation (97) then yield the eigen problem:

$$\tilde{S}\gamma = \lambda\gamma \tag{100}$$

Where

$$\tilde{S} \equiv nI_n + \text{diag}(\hat{A}'\hat{A}) - (\hat{A}' + \hat{A}) \tag{101}$$

And I_n is an identity matrix of order *n*.

3.4.2.4. Nonlinear Approximation of Priority Ratios

To assess the results of priority evaluation with random data we use a nonlinear approximation of the priority ratios in the Saaty matrix. This nonlinear approximation can be viewed as a regression on dummy variables. It produces, together with the priorities, the relevant standard deviations and t-statistics. Thus, it allows the construction of confidence intervals for the priorities.

Specifically, consider the following nonlinear model [14].

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \varepsilon_{ij}, \tag{102}$$

That corresponds to the definition of a pair wise comparison of the elements a_{ij} as ratios of the priorities of the objects under consideration. Minimization of the sum of deviations yields the nonlinear LS problem:

$$S = \sum_{i,j=1}^{n} \varepsilon_{ij}^2 = \sum_{i,j=1}^{n} (a_{ij} - \frac{\alpha_i}{\alpha_j})^2 \to \min \qquad (103)$$

The first order conditions of relation (103) can be expressed as a system of nonlinear equations.

Clearly, the solution of relation (103) requires some normalization, due to the homogeneity of degree 0 of the parameters in relation (103). One can use, for example,

$$\alpha_n = 1, \text{ or } \sum_i \alpha_i = 1, \text{ or } \sum_i \alpha_i^2 = 1 \qquad (104)$$

Minimization of relation (103) can be described as a regression on a set of dummy variables. Indeed, all the elements of matrix (1) can be considered as observed values of a function *y*, which is approximated by a theoretical model of the form

$$\hat{y}_{ij} = \sum_{k,l=1}^{n} \frac{\alpha_k}{\alpha_l} \delta_{ki} \delta_{lj}, \qquad (105)$$

Where δ_{ki} and δ_{lj} are Kronecker delta functions.

The function (104) transforms the *ij* th element in relation (103) into the ratio of different pairs of the unknown coefficients $\frac{\alpha_i}{\alpha_j}$, which corresponds to the multiplication of all the ratios in relation (104) by the values of dummy variables. To clarify, consider an example with a third-order Saaty matrix (1), with six values of the function *y*, and six dummy arguments $x_1, x_2, ..., x_6$. Using these dummy variables we construct the following regression model

$$\hat{y} = \frac{\alpha_1}{\alpha_2} x_1 + \frac{\alpha_1}{\alpha_3} x_2 + \frac{\alpha_2}{\alpha_3} x_3 + \frac{\alpha_2}{\alpha_1} x_4 + \frac{\alpha_3}{\alpha_1} x_5 + \frac{\alpha_3}{\alpha_2} x_6 \qquad (106)$$

There is no need to use the diagonal elements of the Saaty matrix in relation (105) since all these elements are equal to 1.

4. Conclusion

The most of decision making problems must be applied to uncertain real word conditions. Classical solving methods are not usually proper to these types of problems. In these paper new methods for interval weighted comparison matrices are discussed and their advantages and disadvantages are shown. Some criteria must be considered in the selection process of methods. That's Calculation complexity, CPU time, Precision of solution and interaction with decision maker.

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Appendix 1.

Tab. 1. Models with different input and output

Output (Weight)			Input (Comparison Matrix)			Year	Method
fuzzy	**Interval**	**Precise**	**fuzzy**	**Interval**	**Precise**		
		✓	✓			1983	Van Laarhoven & Pedryce
		✓	✓			1985	Buckley
		✓	✓			1996	Xu and Zhai
✓			✓			2000	Leung and Cao
✓			✓			2001	Buckley et al.
✓			✓			2001	Csutora and Buckley
	✓			✓		1987	Saaty and Vargas
		✓		✓		1991	Arbel
	✓			✓		1995	Salo et al.
	✓			✓		1993	Arbel and Vargas
		✓		✓		1993	Moreno et al.
	✓			✓		1997	Islame et al.
		✓		✓		1998	Haines
✓				✓		2004	Mikhailov
	✓			✓		2007	Ying-Ming Wang et al.
	✓			✓		2004	Sugihara et al.
	✓			✓		1997	Bryson and Mobolurin
	✓			✓		2005	Ying-Ming Wang et al
		✓		✓		1997	Lipovetsky and Tishler