A Numerical Method for Backward Inverse Heat Conduction Problem With two Unknown Functions

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Abstract: This paper considers a linear one dimensional inverse heat conduction problem with non constant thermal diffusivity and two unknown terms in a heated bar with unit length. By using the WKB method, the heat flux at the end of boundary and initial temperature will be approximated, numerically. By choosing a suitable parameter in WKB method the ill-posedness of solution will be improved. Finally, a numerical example will be presented.

Keywords: Inverse heat conduction, Ill-posed problem, Finite difference method

1. Statement of the problem

This section deals with a linear heat equation

\[
\frac{\partial u(x,t)}{\partial t} = a(t) \frac{\partial^2 u(x,t)}{\partial x^2} - q(t) u(x,t),
\]  

\[
D = \{(x,t) | 0 < x < 1, 0 < t < T\},
\]

with boundary conditions

\[ u(0,t) = g(t), \quad 0 \leq t \leq T, \]

\[ u(1,t) = h(t), \quad 0 \leq t \leq T, \]

and the initial condition

\[ u(x,0) = u_o(x), \quad 0 \leq x \leq 1, \]

where \( T \) is a given positive constant number, \( q(t) \), \( a(t) \) and \( g(t) \) are known functions on \([0,T]\), and \( h(t) \), \( u_o(x) \) and \( u(x,t) \) are unknown functions. To solve the above problem, we use the following extra conditions

\[ \frac{\partial u(0,t)}{\partial t} = 0, \quad 0 \leq t \leq T, \]

\[ u(x,T) = f(x), \quad 0 \leq x \leq 1, \]

where \( f(x) \) is an known function.

Clearly, the problem (1)-(6) is inverse heat conduction problem with two unknown terms.

In fact, these problems are of two types: backward inverse heat conduction problems (BIHCP) and determining unknown temperature histories and heat flux on a part of boundary, from known values in the body is the idea of many mathematicians. Physically, let us consider a heated bar with unit length in one dimensional space. A direct measurement of the heat flux or temperature in a boundary and initial time of hot body is almost impossible. Consequently, this problem is an inverse heat conduction problem (IHCP). Some numerical and theoretical approaches for solving IHCPs when the histories of temperature at the initial time or in a boundary, not both, are unknown have been summarized in [1, 2, 3, 4, 5, 6, 7]. Beck in [1,2] has shown that, if an error is made in a known boundary condition, then there will be errors in the unknown heat flux on the other boundary. In [4], an estimated of the solution an BIHCP, by using the regularization method, is derived. These results are consistent with earlier observations that small values of time can produce large errors in surface flux.

In this article, we discrete the variable \( t \) and reduce the problem (1)-(6) to a system of linear, inhomogeneous secondary order differential equations. Then, we express a relation of a parameter and increment time. By using this parameter we modify the instability of solution.

2. Time Variable Discretization

In this section, by given the following theorem we shall prove the existency and unicity for the solution of problem (1)-(6).

Theorem 1 For any final time \( T \), let \( g(t) \in H^1([0,T]), a(t) \in L^\infty([0,T]), \) such that, \( a(t) > k > 0 \) where \( k \) is a constant number, \( q(t) \) be an
integrable positive function in \([0, \, T]\), and \(f(x)\) be an analytical function for any \(0<x<1\), then there exist a unique weak solution \(u \in L^2(0,\, T; \mathcal{H}^1((0,1)))\) and Hölder continuous function \(h(t)\), for the problem (1)-(6).

**Proof** In order to prove this theorem, let us consider the transformation

\[
v(x,t) = u(x,t) \exp \left\{ \int_0^t q(\tau) \, d\tau \right\}.
\]

By using this transformation, the problem (1)-(2) and (5)-(6) becomes

\[
\frac{\partial v(x,t)}{\partial t} = a(t) \frac{\partial^2 v(x,t)}{\partial x^2} - q(t) v(x,t),
\]

\[D = \{ (x,t) \mid 0 < x < 1, 0 < t < T \},
\]

\[v(x,T) = u_M(x) \exp \left\{ -\int_0^T q(\tau) \, d\tau \right\} = f_i(x), \quad 0 < x < 1,
\]

\[v(0,t) = g(t) \exp \left\{ -\int_0^t g(\tau) \, d\tau \right\} = g_i(t), \quad 0 < t < T,
\]

\[\frac{\partial v(0,t)}{\partial x} = 0, \quad 0 \leq t \leq T.
\]

Because, \(q(t)\) is a positive function and integrable in its domain, if \(g\) and \(f(x)\) may be satisfied in the assumptions of theorem 1, then \(g_1\) and \(f_1(x)\) satisfying in these assumptions, too. Consequently by using [1, 4, 6, 8, 9] the proof of this statement will be completed. In continuation, assume that \(M \in \mathbb{R}^+\), \(\Delta t_M = T/M\), and \(t_i = i \Delta t_M\).

Also, we use \(\hat{u}_i(x)\) instead of the approximate \(u(x, i \Delta t_M)\), and \(a_i = a(t_i)\) for any \(0 \leq i \leq M\).

Obviously, we have \(u_M(x) = f(x)\). Now, apply the semi-implicit finite difference method in the form

\[\hat{u}_{i+1}(x) = \hat{u}_i(x) + \theta \frac{\partial \hat{u}_i(x,t_i)}{\partial t} + \theta \frac{\partial \hat{u}_i(x,t_i)}{\partial t} \Delta t_M,
\]

where \(\theta > 0\) and \(\theta' = 1 - \theta\). Then, by substituting (1)-(6) into (7) we drive the following ordinary differential equations system

\[
\frac{d^2 \hat{u}(x)}{dx^2} = -\lambda^2 A \hat{u}(x) + \lambda^2 f(x),
\]

where \(\lambda = (\Delta t_M)^{-2}\) and \(A = B^T C\), such that

\[
\hat{u}(x) = [\hat{u}_0(x), K' \hat{u}_{M-1}(x)]^T_{x=M},
\]

\[
f(x) = \left[ \begin{array}{c} 0 \ \mathcal{K} \ 0 \ f_M \\ f_M \end{array} \right]_{x=M},
\]

\[
f_M = (1 + \theta q \Delta t_M)u_M(x) - \Delta t_M \theta a_M \hat{u}^\prime_M(x),
\]

\[
\mathcal{C} = [C_{ij}]_{M \times M},
\]

where

\[
[C_{ij}]_{M \times M} = \begin{cases} 1 - \theta q_{i-1} \Delta t_M & j = i \\
(1 - \theta q_i \Delta t_M) & j = i + 1 \\
0 & \text{else where}
\end{cases}
\]

and

\[
\mathcal{B} = \begin{pmatrix}
\theta a_0 & \theta a_1 & 0 & 0 & L & 0 \\
0 & \theta a_1 & \theta a_2 & 0 & L & 0 \\
0 & 0 & \theta a_1 & \theta a_2 & L & 0 \\
M & M & M & O & O & M \\
0 & 0 & L & L & \theta a_{M-2} & \theta a_{M-1} \\
0 & 0 & 0 & L & 0 & \theta a_{M-1}
\end{pmatrix}
\]

Consequently, we have

\[
\hat{u}(0) = [g_0, K, g_{M-1}]^T
\]

and

\[
\hat{u}'(0) = [0, L, 0]^T.
\]

Now, let us \(f(x) = 0\), then, for the solution of the equations system (8) may be in the form

\[
\hat{u}(x) = \cos(\lambda S(x))\left[ f_0(x) + \lambda^{-1} f_1(x) + L \right],
\]

or

\[
\hat{u}(x) = \sin(\lambda S(x))\left[ f_0(x) + \lambda^{-1} f_1(x) + L \right].
\]

Where \(S(x)\) is an unknown function and \(f_0(x), f_1(x), L\), are unknown vector-functions. By substituting (9) and (10) into the ordinary differential equations system (8), cancel the cosine or sine term and simplifying the produced results, then we obtain a recurrent system of equations

\[
(A - S^{-2}(x)I)f_0(x) = 0,
\]

\[
(A - S^{-2}(x)I)f_1(x) = f_0(x)S'(x) + 2f'_0(x)S(x),
\]

\[
(A - S^{-2}(x)I)f'_0(x) = f_{i-2}(x)S'(x) + 2f'_{i-2}(x)S(x) + f''_{i-2}(x), \quad k \geq 2.
\]

If \(a(t)\) is a monotone function, then the characteristic equation (8), has not turning points for any \(x \in [0,1]\) ([5]).
Then $A$ has $M$ unequal eigenvalues and $M$ linear independent eigenvectors corresponding to eigenvalues of matrix $A$.

By using (11)-(13) we derive $2M$ independent solutions for (8). It follows from (11) that $S'^2(x)$ is an eigenvalue, and $f'_i(x)$ is an eigenvector of $A$. Let $\{e_0(x), K, e_{M-1}(x)\}$ be a base of eigenvectors. Then, we derive

$$S_j(x) = x\sqrt{\frac{1 - q_j \Delta t_M}{\theta a_j}}, \quad 0 \leq j \leq M - 1,$$

and

$$f'_i(x) = \alpha_{i,j}(x) e_j(x), \quad i \geq 0, 0 \leq j \leq M - 1,$$

where

$$\alpha_{i,j}(x) = \frac{\alpha_{i,j}(x)}{2} \int_0^x \alpha_{i,j}(s) ds = 0,$$

$$0 \leq j \leq M - 1, \quad i > 1.$$

Then, for finding $\hat{u}_i(x)$ for any $i = 0, 1, K, M - 1$, setting

$$\hat{u}(x) = \sum_{i=0}^{M-1} C^{(1)}_i \hat{u}^{(1)}(x) + \sum_{j=0}^{M-1} C^{(2)}_j \hat{u}^{(2)}(x),$$

where

$$\hat{u}^{(1)}(x) = \sin \left( x\sqrt{\frac{1 - q_j \Delta t_M}{\theta a_j}} \right) f_0^{(1)}(x),$$

$$\hat{u}^{(2)}(x) = \cos \left( x\sqrt{\frac{1 - q_j \Delta t_M}{\theta a_j}} \right) f_0^{(1)}(x),$$

such that, $C^{(1)}_i$ for any $i = 0, K, M - 1$, and $j = 1, 2$ are unknown constants and will be found from initial conditions

$$\hat{u}(0) = \left[ g_0, K, g_{M-1} \right]^T$$

and

$$\hat{u}'(0) = \left[ 0, L, 0 \right]^T.$$
Table. 1. Exact and Estimate of the Temperature in $x=1$ with $\Delta t_M = 0.1, \theta = 10$.

<table>
<thead>
<tr>
<th>t</th>
<th>$u(1,t)$</th>
<th>$\hat{u}(1,t)$</th>
<th>relative error</th>
<th>$|u(x,t_i)|_{L^2[0,1]}$</th>
<th>$|u(x,t_i) - \hat{u}(x,t_i)|_{L^2[0,1]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.540302</td>
<td>0.438671</td>
<td>18.8%</td>
<td>0.806089</td>
<td>0.035381</td>
</tr>
<tr>
<td>0.1</td>
<td>0.563181</td>
<td>0.428510</td>
<td>23.9%</td>
<td>0.830280</td>
<td>0.04682</td>
</tr>
<tr>
<td>0.2</td>
<td>0.627258</td>
<td>0.482832</td>
<td>23.0%</td>
<td>0.926704</td>
<td>0.05019</td>
</tr>
<tr>
<td>0.3</td>
<td>0.727453</td>
<td>0.591342</td>
<td>18.7%</td>
<td>1.085646</td>
<td>0.04729</td>
</tr>
<tr>
<td>0.4</td>
<td>0.859802</td>
<td>0.745534</td>
<td>13.2%</td>
<td>1.299360</td>
<td>0.03970</td>
</tr>
<tr>
<td>0.5</td>
<td>1.020319</td>
<td>0.936711</td>
<td>8.1%</td>
<td>1.560007</td>
<td>0.02904</td>
</tr>
<tr>
<td>0.6</td>
<td>1.204232</td>
<td>1.154731</td>
<td>4.1%</td>
<td>1.858290</td>
<td>0.01718</td>
</tr>
<tr>
<td>0.7</td>
<td>1.405515</td>
<td>1.387395</td>
<td>1.2%</td>
<td>2.182683</td>
<td>0.00627</td>
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<tr>
<td>0.8</td>
<td>1.616714</td>
<td>1.620597</td>
<td>0.24%</td>
<td>2.519261</td>
<td>0.00137</td>
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<tr>
<td>0.9</td>
<td>1.829042</td>
<td>1.839611</td>
<td>0.57%</td>
<td>2.852268</td>
<td>0.00370</td>
</tr>
</tbody>
</table>

4. Conclusion
In this paper we shown that, if we choose the appropriate of parameter $\theta$ such that, the estimated solution of this problem well-posed, then we can tend $\Delta t_M$ to zero and we derive the convergency and stability of this problem.
In order to, reduce of effect measurements error in the final time and boundary, we use the source term $q(t) u(x,t)$ in the problem (1)-(6).

References