SEMI-RADICALS OF SUB MODULES IN MODULES

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Abstract: Let \( R \) be a commutative ring and \( M \) be a unitary \( R \)-module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula (s.r.s.r.f) and present the existence of such a module.

Keywords: Prime sub module, semiprime sub module, radical and semi-radical of a module, modules satisfying the semi-radical formula.

1. Introduction

In this paper all the rings are commutative with identity and all the modules are unitary. Let \( R \) be a ring and \( M \) be an \( R \)-module. If \( N \) is a submodule of \( M \) we use the notation \( N \subseteq M \). If the submodule \( N \) is generated by a subset \( S \) of \( M \), we write \( N = \langle S \rangle \). If \( N \) and \( K \) are sub modules of \( M \), then the set \( \{ r \in R | rK \subseteq N \} \) is denoted by \( (N : K) \) or simply by \( (N : M) \) which is clearly an ideal of \( R \). If \( I \) is an ideal of the ring \( R \), we write \( I \subseteq R \). In Section 2 we define prime and primary sub modules of an \( R \)-module \( M \). In Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if \( N \) is a semiprime submodule of an \( R \)-module \( M \), then \( (N : M) \) is a semiprime ideal of \( R \) but not conversely in general. In Lemma 2.8 we prove that the converse is also true if \( M \) is a multiplication module. In Section 3 we define radical of an \( R \)-module \( M \) and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule of a module and also modules satisfying the semi-radical formula which is abbreviated as (s.t.s.r.f) and in Theorem 3.9 we show that such a module does exist.

Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a \( P \)-semiprime submodule we consider some of its properties.

2. Some Elementary Results

We begin this section with the following definitions:

Definition 2.1. Let \( N \) be a proper submodule of an \( R \)-module \( M \).

(a) \( N \) is called a prime submodule of \( M \) if for any \( r \in R \) and \( m \in M \), \( rm \in N \) implies that \( m \in N \) or \( r \in (N : M) \).

(b) \( N \) is called a primary submodule of \( M \) if for any \( r \in R \) and \( m \in M \), \( rm \in N \) implies that \( m \in N \) or \( r^n \in (N : M) \) for some positive integer \( n \).

In (a) it can easily be shown that \( P = (N : M) \) is a prime ideal of \( R \) and we say that \( N \) is \( P \)-prime.

We recall that if \( I \) is an ideal of a ring \( R \), then the radical of \( I \), denoted by \( \sqrt{I} \), is defined as the intersection of all prime ideals containing \( I \).

Alternatively, we define the radical of \( I \) as:

\[
\sqrt{I} = \{ r \in R | r^n \in I \text{ for some positive integer } n \}.
\]

Also if \( N \) is a primary submodule of an \( R \)-module \( M \), then \( (N : M) \) is a primary ideal of \( R \) and \( P = (N : M) \) is a prime ideal. We describe this situation by saying that \( N \) is \( P \)-primary.

Lemma 2.2. Let \( N \) be a proper submodule of an \( R \)-module \( M \).

(i) \( N \) is a prime submodule of \( M \) if and only if \( ID \subseteq N \) (with \( I \) an ideal of \( R \) and \( D \) a submodule of \( M \)) implies that \( D \subseteq N \) or \( I \subseteq (N : M) \).

(ii) \( N \) is a primary submodule of \( M \) if and only if for any finitely generated ideal \( I \) of \( R \) and any submodule \( D \) of \( M \), \( ID \subseteq N \) implies that \( D \subseteq N \) or \( I^n \subseteq (N : M) \) for some positive integer \( n \).

(iii) Let \( P \) be a prime ideal of \( R \), then \( N \) is a \( P \)-primary submodule of \( M \) if and only if (a)
we must show that (a). Therefore 

\[ r \in (N : M) \]

which implies that \( r \in (N : M) \). Therefore \( N \) is a prime submodule of \( M \).

(ii) (\( \Rightarrow \)): Let \( D \subseteq M \) and \( I \) be a finitely generated ideal of \( R \) such that \( ID \subseteq N \).

Then by [5, Corollary 1, P.99], \( D \subseteq N \) or \( I \subseteq (N : M) \). Let \( D \subseteq N \), then \( I \subseteq (N : M) \) and by [5, Proposition 8. P.83], there exists a positive integer \( n \) such that \( I^n \subseteq (N : M) \).

\((\Leftarrow)\): Let \( r \in R \), \( a \in M \) be such that \( ra \in N \) and let \( a \notin N \). By taking \( I = (r) \) and \( D = Ra \) we see that \( ID \subseteq N \). But \( D \subseteq N \) and hence \( I \subseteq (N : M) \).

Now let \( r \in (N : M) \), that is \( c \in (N : M) \) be a positive integer such that \( c^a \in (N : M) \), that is \( c \in (N : M) \) for some positive integer \( n \). Hence \( cm \notin N \).

\( (\Leftarrow)\): Assume that (a), (b) hold. Let \( r \in R \) and \( m \in M \), \( rm \in N \). Assume further that \( m \notin N \), then by (b), \( r \) must belong to \( P \) and so \( r \in (N : M) \) by (a). Therefore \( N \) is a primary submodule of \( M \). Next we must show that \( P = (N : M) \).

Let \( r \in (N : M) \), then \( r^a \in (N : M) \) for some positive integer \( n \), and so \( r^a M \subseteq N \). Since \( N \) is proper, there exist \( x \in M \setminus N \). Now \( r^a x \in N \) and \( x \notin N \) so by (b) we conclude that \( r^a \in P \) and, as \( P \) is prime, \( r \in P \). We find that \( (N : M) = P \) and therefore \( N \) is \( P \)-primary.

The following definition is essential in the rest of the paper.

**Definition 2.3.** A proper submodule \( N \) of an \( R \)–module \( M \) is said to be semiprime in \( M \), if for every ideal \( \mathcal{I} \) of \( R \) and every submodule \( K \) of \( M \), \( \mathcal{I}^2 K \subseteq N \) implies that \( IK \subseteq N \). Note that since the ring \( R \) is an \( R \)–module by itself, a proper ideal \( \mathcal{I} \) of \( R \) is semiprime if for every ideals \( J \) and \( K \) of \( R \), \( J^2 K \subseteq I \) implies that \( JK \subseteq I \).

**Proposition 2.4.** Let \( M \) be an \( R \)–module.

(i) If \( N \) is a prime submodule of \( M \), then \( N \) is semiprime.

(ii) If \( N \) is a semiprime submodule of \( M \), then \( (N : M) \) is semiprime ideal of \( R \).

**Proof.** (i) Let \( I \subseteq R \) and \( D \subseteq M \) be such that \( ID \subseteq N \). Then by [5, Corollary 1, P.99], \( I \subseteq (N : M) \).

(ii) (\( \Rightarrow \)): Let \( I \subseteq (N : M) \). Then \( I (IK) \subseteq N \) and since \( N \) is prime, \( I \subseteq (N : M) \) or \( IK \subseteq N \). But \( (N : M) \subseteq (N : K) \) and hence \( I \subseteq (N : K) \), and so \( IK \subseteq N \). In any case we see that \( IK \subseteq N \), and therefore \( N \) is semiprime.

**Example 2.5.** Let \( R = Z \), \( M = Z \oplus Z \) and \( B = \langle 9, 0 \rangle \).

Then it is clear that \( (B : M) = (0) \). Since \( Z \) is an integral domain, \( (B : M) = (0) \) is a prime ideal and hence a semiprime ideal of \( Z \). But \( B \) is not a semiprime submodule of \( M \); because if we take \( I = (3) \) and \( K = \langle (2, 0) \rangle \). Then:

\[ I^2 K = \left\{ \langle 18q, 0 \rangle \mid q \in Z \right\} \]  

But:

\[ IK = \left\{ \langle 6q, 0 \rangle \mid q \in Z \right\} \]  

is not a subset of \( B \).

It is clear that if \( N \) is a semiprime submodule of an \( R \)–module \( M \) and \( I \subseteq R \), \( K \subseteq M \) be such that \( I^n K \subseteq N \) for some positive integer \( n \), then \( IK \subseteq N \).

**Theorem 2.6.** Let \( N \) be a proper submodule of an \( R \)–module \( M \). Then the following statements are equivalent:

(i) \( N \) is semiprime.

(ii) Whenever \( r/m \in N \) for some \( r \in R \), \( m \in M \) and \( t \in Z^+ \), then \( rm \in N \).
Proof. (i) \((\Rightarrow)\) (ii). Let \( r^m \in N \) where \( r \in R \), \( m \in M \) and \( t \in Z^+ \). Taking \( I = (r) \) and \( K = (m) \) we have \( I^2K \subseteq N \) and so \( IK \subseteq N \) which implies that \( rm \in N \).

(ii) \((\Rightarrow)\). Let \( I \subseteq R \) and \( K \leq M \) be such that \( I^2K \subseteq N \). Consider the set:

\[
S = \{ ra | r \in I, a \in K \}
\]

(3)

Then for every \( r \in I, a \in K \) we have \( r^2a \in I^2K \subseteq N \) and hence \( ra \in N \). This implies that \( S \subseteq N \) and since \( IK \) is the submodule of \( M \) generated by \( S \), we must have \( IK \subseteq N \). Therefore \( N \) is semiprime.

Definition 2.7. An \( R \)-module \( M \) is said to be a multiplication module if for each submodule \( N \) of \( M \), \( N = IM \) for some ideal \( I \) of \( R \).

It can be easily shown that, an \( R \)-module \( M \) is a multiplication module if and only if \( N = (N : M)M \) for every submodule \( N \) of \( M \).

Lemma 2.8. Let \( M \) be a multiplication \( R \)-module. Then a submodule \( N \) of \( M \) is semiprime if and only if \( (N : M) \) is a semiprime ideal of \( R \).

Proof. (\(\Rightarrow\)): This is clear from Proposition 2.4 (ii).

(\(\Leftarrow\)): Let \( I \subseteq R \), \( K \leq M \), be such that \( I^2K \subseteq N \). Hence:

\[
(I^2K : M) \subseteq (N : M).
\]

(4)

It can be shown that:

\[
I^2(K : M) \subseteq (I^2K : M)
\]

and so we obtain:

\[
I^2(K : M)M \subseteq (N : M).
\]

(5)

But \( (N : M) \) is a semiprime ideal of \( R \) and hence \( I(K : M) \subseteq (N : M) \). Thus we conclude that:

\[
I(K : M)M \subseteq (N : M)M,
\]

(6)

and using the fact that \( M \) is a multiplication \( R \)-module we have \( IM \subseteq N \). Therefore \( N \) is a semiprime submodule of \( M \).

The following lemma shows that the same situation, as above, holds for prime and primary sub modules.

Lemma 2.9. Let \( M \) be a multiplication \( R \)-module. Then:

(a) A submodule \( N \) of \( M \) is prime if and only if \( (N : M) \) is a prime ideal of \( R \).

(b) A submodule \( N \) of \( M \) is primary if and only if \( (N : M) \) is a primary ideal of \( R \).

Proof. (a) \((\Rightarrow)\): Clear.

(\(\Leftarrow\)): Let \( I \subseteq R \), \( D \leq M \) be such that \( ID \subseteq N \), then \( (ID : M) \subseteq (N : M) \). But \( (D : M) \subseteq (ID : M) \) and so \( (D : M) \subseteq (N : M) \). Since \( (N : M) \) is a prime ideal of \( R \) we have \( I \subseteq (N : M) \) or \( D : M \subseteq (N : M) \). Suppose that \( I \subseteq (N : M) \). Then \( (D : M) \subseteq (N : M) \) and from the above we have \( (D : M)M \subseteq (N : M)M \), that is, \( D \subseteq N \). Hence \( N \) is a prime submodule of \( M \) by Lemma 2.2 (i).

(b) \((\Rightarrow)\): Clear.

Remark. Some authors define a semiprime submodule as an intersection of prime sub modules. But by our
3. Radicals and Semi-Radicals

Let $M$ be an $R$-module and $N$ a submodule of $M$. If there exists a prime submodule of $M$ which contain $N$, then the intersection of all prime sub modules containing $N$, is called the $M$-radical of $M$ and is denoted by $\text{rad}_M N$, or simply by $\text{rad} N$. If there is no prime submodule containing $N$, then we define $\text{rad} M = M$; in particular $\text{rad}_M M = M$. An ideal $I$ of a ring $R$ is called a radical ideal if $\sqrt{I} = I$. Similarly, we say that a submodule $B$ of an $R$-module $M$ is a radical submodule if $\text{rad} B = B$. It is easy to see that an ideal $I$ of a ring $R$ is semiprime if and only if it is radical. Because, let $I$ be semiprime, and let $x \in \sqrt{I}$. Then $x^k \in I$ for some positive integer $k$. So $x^k 1 \in I$, and since $I$ is semiprime we have $x.1 = x \in I$. Therefore $I = \sqrt{I}$.

On the other hand, if $I = \sqrt{I}$ then by definition of $\sqrt{I}$ and Propositions 2.4 (i) and 2.10, $I$ is semiprime. Finally by Propositions 2.4 (i) and 2.10 we see that for any submodule $B$ of an $R$-module $M$, $\text{rad} B$ is a semiprime submodule whenever $\text{rad} B \neq M$.

Theorem 3.1. Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if it is radical.

Proof. Since $\text{ann}_R(M) \subseteq (N : M)$, by [2, Theorem 3, P.216],

$$\sqrt{(N : M) M} = \text{rad} (N : M) M .$$

As $M$ is a multiplication module we have $(N : M) M = M$, and if $N$ is semiprime, $(N : M)$ is a radical ideal. Therefore

$$\sqrt{(N : M) M} = \text{rad} (N : M) M$$

iff

$$(N : M) M = \text{rad} (N : M) M .$$

If $N = \text{rad} M$ this implies that $N$ is a radical submodule of $M$, that is, $N = \text{rad} N = \cap P (P$ is a prime submodule of $M$ containing $N$). Hence by Propositions 2.4 (1) and 2.10 $N$ is a semiprime submodule of $M$. The proof is now complete.

After Remark 2.11 we may ask under what condition a semiprime submodule is the intersection of prime submodules containing it. The following corollary can be considered as an answer.

Corollary 3.2. Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if $N = \cap P (P$ is a prime submodule of $M$ containing $N$).

Proof. $(\Rightarrow)$: If $N$ is semiprime then by Theorem 3.1, it is radical, that is, $N = \cap P$ (a prime submodule of $M$ containing $N$).

$(\Leftarrow)$: By Propositions 2.4 (i) and 2.10, $N$ is semiprime.

Proposition 3.3. If $M$ is a finitely generated $R$-module, then every proper submodule of $M$ is contained in a semiprime sub module.

Proof. By Corollary of [3, Proposition 4, P.63], every proper submodule of $M$ is contained in a prime submodule. So by Proposition 2.4 (i), we have the result.

Definition 3.4. (1) A semiprime submodule $P$ of an $R$-module $M$ is called a minimal semiprime of a proper submodule $N$ if $N \subseteq P$ and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of $0 = <0_M>$ is called a minimal semiprime submodule of $M$.

Theorem 3.5. Let $M$ be an $R$-module. If a submodule $N$ of $M$ is contained in a semiprime submodule $P$, then $P$ contains a minimal semiprime submodule of $N$.

Proof. It is similar to the proof of [5, Theorem 4, P.84].

Proposition 3.6. Every proper submodule of a finitely generated $R$-module $M$ possesses at least one minimal semiprime submodule of $M$.

Proof. Let $N$ be a proper submodule of $M$, then by Proposition 3.3, $N$ is contained in a semiprime submodule of $M$.

Corollary 3.7. Every semiprime submodule of an $R$-module $M$ contains at least one minimal semiprime submodule of $M$.

Proof. Let $P$ be a semiprime submodule of $M$ and take $N = <0_M>$ in the Theorem 3.5. Then $P$ contains a minimal semiprime submodule of $<0_M>$, and so a minimal semiprime submodule of $M$.

Definition 3.8. Let $M$ be an $R$-module and $N \leq M$. If there exists a semiprime submodule of $M$ which contains $N$, then the intersection of all semiprime sub modules containing $N$ is called the semi-radical of $N$ and is denoted by $S = \text{rad} M N$, or simply by $S = \text{rad} N$. If there is no semiprime submodule containing $N$, then we define...
$S - radN = M$, in particular $S - radM = M$. We call $S - rad\{0\}$ the semiprime radical of $M$.

If $N \leq M$, then the envelope of $N$, denoted by $E(N)$, is defined as:

$$E(N) = \left\{ x \in M \mid x = ra \text{ for some } r \in R, a \in M \text{ and } r^n a \in N \text{ for some } n \in \mathbb{Z} \right\}$$  \hspace{1cm} (10)

We say that $M$ satisfies the semi-radical formula, $M$ (s.t.s.r.f) if for any $N \leq M$, the semi-radical of $N$ is equal to the submodule generated by its envelope, that is, $S - radN = \{E(N)\}$. We already know that $\{E(N)\} \subseteq radN$, by [4, P.1815]. Now let $x \in E(N)$ and $P$ be a semiprime submodule of $M$ containing $N$. Then $x = ra$ for some $r \in R, a \in M$ and for positive integer $n, r^n a \in N$. But $r^n a \in P$ and since $P$ is semiprime we have $ra \in P$. Hence $E(N) \subseteq P$. We conclude that $E(N) \subseteq (P)$ (P is a semiprime submodule containing $N$). So $E(N) \subseteq S - radN$. On the other hand, since every prime submodule of $M$ is clearly semiprime, we have $S - radN \subseteq radN$. We see that:

$$\{E(N)\} \subseteq S - radN \subseteq radN$$  \hspace{1cm} (11)

Now we present an $R$-module which satisfies the semi-radical formula.

**Theorem 3.9.** Let $M$ be a finitely generated multiplication $R$-module. Then $M$ satisfied the semi-radical formula.

**Proof.** Let $N \leq M$, then by [4, Theorem 4.4], we have $\langle \{E(N)\} : M \rangle = \langle radN : M \rangle$.

Hence $\langle \{E(N)\} : M \rangle M = \langle radN : M \rangle M$ and since $M$ is a multiplication $R$-module, $\{E(N)\} = radN$. Next from (*) we have:

$$\langle \{E(N)\} : M \rangle M \subseteq \langle S - radN : M \rangle M \subseteq \langle radN : M \rangle M$$  \hspace{1cm} (12)

that is,

$$\langle \{E(N)\} \rangle \subseteq S - radN \subseteq radN.$$  \hspace{1cm} (13)

Thus we find that $S - radN = \{E(N)\}$.

**Remark.** Under the conditions of Theorem 3.9, we see that for any submodule $N \neq M$ of $M$, we always have $RadN = S - RadN$.

**Proposition 3.10.** Let $M$ be a finitely generated $R$-module. Then the semi-radical of a proper submodule $N$ of $M$ is the intersection of its minimal semiprime submodules.

**Proof.** This is clear by using Theorem 3.5 and Proposition 3.6.

For the rest of this section we state and prove some properties of semi-radical of sub modules.

**Theorem 3.11.** Let $B$ and $C$ be sub modules of an $R$-module $M$. Then,

1. $B \subseteq S - radB$.
2. $S - rad(S - radB) = S - radB$.
3. $S - rad(B \cap C) \subseteq S - radB \cap S - radC$, and we have the equality when for every semiprime submodule $P$, $B \cap C \subseteq P$ implies that $B \subseteq Por \subseteq P$.
4. $S - rad(B + C) = S - rad(S - radB + S - radC)$.
5. $\sqrt{S - radB} : M)$.
6. If $M$ is finitely generated, then $S - radB = M$ if and only if $B = M$.
7. If $M$ is finitely generated, then $B + C = M$ if and only if $S - RadB + S - RadC = M$.
8. $S - radM = S - rad\sqrt{I}$ for every ideal $I$ of $R$.

**Proof.** (1) clear.

(2) Since $S - RadB$ is semiprime by Proposition 2.10, we have:

$$S - Rad(S - RadB) = S - RadB.$$  \hspace{1cm} (14)

(3) Let $P$ be a semiprime submodule of $M$ such that $B \subseteq P$, so $B \cap C \subseteq P$ and hence $S - rad(B \cap C) \subseteq P$. But $P$ is arbitrary, therefore $S - rad(B \cap C) \subseteq S - radB$. By a similar argument we have $S - rad(B \cap C) \subseteq S - radC$. Now let $P$ be a semiprime submodule of $M$ such that $B \cap C \subseteq P$ and assume that $B \subseteq P$. Then $S - radB \subseteq P$ and so $S - radB \cap S - radC \subseteq P$. Since $P$ is arbitrary this implies that $S - radB \cap S - radC \subseteq S - rad(B \cap C)$ and hence we have the equality.

(4) Let $P$ be a semiprime submodule of $M$ such that $(S - radB + S - radC) \subseteq P$. So $S - radB \subseteq P$ and $S - radC \subseteq P$. Hence $B \subseteq C$ and $C \subseteq P$ which implies $B + C \subseteq P$. Therefore $S - rad(B + C) \subseteq P$.

But $P$ is chosen arbitrary, so:

$$S - rad(B + C) \subseteq S - rad(S - radB + S - radC).$$  \hspace{1cm} (15)

Now suppose that $P$ be a semiprime submodule such that $B + C \subseteq P$. So $B \subseteq P$, and $C \subseteq P$. Hence $S - radB \subseteq P$ and $S - radC \subseteq P$ and therefore $S - radB + S - radC \subseteq P$.

But $S - rad(S - radB + S - radC) \subseteq P$ and we conclude that:

$$S - rad(S - radB + S - radC) \subseteq S - rad(B + C).$$  \hspace{1cm} (16)
(5) If \( S - \text{rad}B = M \), then we have the result. So let \( P \) be a semiprime submodule of \( M \) such that \( B \subseteq P \). So \( (B:M) \subseteq (P:M) \). We know that \((P:M)\) is a semiprime ideal of \( R \) and we have shown that \( \sqrt{(P:M)} = (P:M) \). Hence \( \sqrt{(B:M)} \subseteq \sqrt{(P:M)} = (P:M) \) implies that: \( \sqrt{(B:M)} M \subseteq (P:M) P \subseteq P \), and since \( P \) can be any semiprime submodule of \( M \) containing \( B \), we have \( \sqrt{(B:M)} M \subseteq S - \text{rad}B \), that is, \( \sqrt{(B:M)} M \subseteq (S - \text{rad}M : M) \).

(6) If \( B = M \), then \( S - \text{rad}B = S - \text{rad}M = M \). Conversely, let \( S - \text{rad}B = M \), but \( B \neq M \). Since \( M \) is finitely generated, it contains a prime so a semiprime submodule \( P \) containing \( B \), by Corollary after Proposition 4 of [3]. Hence \( S - \text{rad}B \neq M \), a contradiction.

(7) Using parts (4) and (6) we have:
\[
B + C = M \iff S - \text{rad} (B + C) = M
\]
\[
\text{iff } S - \text{rad} (S - \text{rad}B + S - \text{rad}C) = M
\]
\[
\text{iff } S - \text{rad}B + S - \text{rad}C = M.
\]

(8) If \( M \) has no semiprime submodule containing \( IM \), then \( S - \text{rad}IM = M \) and we have:
\[
I \subseteq \sqrt{I} \implies IM \subseteq \sqrt{IM} \implies S - \text{rad}IM \subseteq S - \text{rad}\sqrt{IM} \implies M \subseteq S - \text{rad}\sqrt{IM} \implies M = S - \text{rad}\sqrt{IM} \tag{17}
\]
\[
= S - \text{rad}IM.
\]

Now let \( P \) be a semiprime submodule of \( M \) such that \( IM \subseteq P \), so \( I \subseteq (IM : M) \subseteq (P : M) \) and since \((P : M)\) is semiprime \( \sqrt{P} \subseteq \sqrt{(P : M)} = (P : M) \).

So \( \sqrt{IM} \subseteq P \) and hence \( S - \text{rad} \sqrt{IM} \subseteq P \). Since \( P \) is arbitrary we have:
\[
S - \text{rad} \sqrt{IM} \subseteq S - \text{rad} M.
\]

Therefore \( S - \text{rad}IM = S - \text{rad} \sqrt{IM} \). The proof is now complete.

**Corollary 3.12.** Let \( M \) be an \( R \)-module and \( I \) an ideal of \( R \). Then \( S - \text{rad}I^n M = S - \text{rad}M \) for every positive integer \( n \).

**Proof.** We know that \( \sqrt{I^n} = \sqrt{I} \), so by part (8) of Theorem 3.11:
\[
S - \text{rad} I^n M = S - \text{rad} \sqrt{I^n} M = S - \text{rad} \sqrt{I} M = S - \text{rad} M.
\]

**Proposition 3.13.** Let \( Q \) be a \( P \)-primary submodule of an \( R \)-module \( A \). Then \( S - \text{rad}Q = S - \text{rad}(Q + PA) \).

**Proof.** We have \( Q \subseteq Q + PA \), so \( S - \text{rad}Q \subseteq S - \text{rad}(Q + PA) \). Let \( S - \text{rad}Q = \bigcap_{i \in I} P_i \), where any \( P_i \) is a semiprime submodule of \( A \) containing \( Q \). We see that
\[
P = \sqrt{(Q : A)} \subseteq \sqrt{(P_i : A)} = (P_i : A) \tag{19}
\]
implies \( PA \subseteq P_i \). So \( (Q + PA) \subseteq P_i \), for every \( i \in I \) and hence \( S - \text{rad}(Q + PA) \subseteq P_i \). Therefore \( S - \text{rad}(Q + PA) \subseteq \bigcap_{i \in I} P_i \) and so \( S - \text{rad}Q = S - \text{rad}(Q + PA) \).

**Definition 3.14.** Let \( N \) be a semiprime submodule of an \( R \)-module \( M \), and let \( P = \sqrt{(N : M)} = (N : M) \).

We call \( N \) a \( P \)-semiprime submodule of \( M \), if \( P \) is prime ideal of \( R \).

**Lemma 3.15.** Let \( M \) be a finitely generated \( R \)-module and let \( K \) be a maximal ideal of \( R \). If \( Q \) is a \( K \)-primary submodule of \( M \), then \( S - \text{rad}Q \) is a \( K \)-semiprime sub module.

**Proof.** By Theorem 3.11, part (5), we have \( K = \sqrt{(Q : M)} \subseteq (S - \text{rad}Q : M) \). But \( K \) is a maximal ideal of \( R \), so \((S - \text{rad}Q : M) = R \) or \((S - \text{rad}Q : M) = K \). If \((S - \text{rad}Q : M) = R \) then \( S - \text{rad}Q = M \) and by Theorem 3.11, part (6) we have \( Q = M \) which is primary. Hence \((S - \text{rad}Q : M) = K \) and since \( S - \text{rad}Q \) is an intersection of semiprime sub modules containing \( Q \) it is semiprime and in fact \( K \)-semiprime.

**Proposition 3.16.** Let \( N_1, N_2, \ldots, N_t \), be \( P \)-semiprime sub modules of an \( R \)-module \( M \). Then \( N = N_1 \cap N_2 \cap \cdots \cap N_t \) is also \( P \)-semiprime.

**Proof.** By Proposition 2.10, \( N \) is semiprime and we have:
\[
(N : M) = (N_1 : M) \cap \cdots \cap (N_t : M) \tag{20}
\]
\[
= P \cap \cdots \cap P = P. \text{Therefore } N \text{ is } P \text{-semiprime.}
\]

**Lemma 3.17.** Let \( M \) be a multiplication \( R \)-module and \( L, N \) be sub modules of \( M \). Also let \( K \) be a prime ideal of \( R \) and \( P \) be a \( K \)-semiprime submodule of \( M \) such that \( N \cap L \subseteq P \). If \( (N : M) \not\subseteq K \) then \( L \not\subseteq P \).

**Proof.** We have \( N \cup L \subseteq P \Rightarrow \bigcap_{L} (N \cup L : M) \subseteq (P : M) = K \Rightarrow (N : M) \cap (L : M) \subseteq K \).

and since \( K \) is a prime ideal of \( R \), \( (N : M) \subseteq K \) or \((L : M) \subseteq K \). Since \((N : M) \not\subseteq K \), we find that \((L : M) \subseteq K \). From this we conclude that \((L : M) M \subseteq KM \), that is, \( L \subseteq KM \). But \((P : M) = K \) implies that \( KM \subseteq P \). Therefore \( L \subseteq KM \subseteq P \).
4. Conclusion

In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

References


