Estimation of Parameters of the Power-Law-Non-Homogenous Poisson Process in the Case of Exact Failures Data

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KEYWORDS	ABSTRACT
Non-Homogenous, Poisson, Process, Exact Failure	*This expository article shows how the maximum likelihood estimation method and the Newton-Raphson algorithm can be used to estimate the parameters of the power-law Poisson process model used to analyze data from repairable systems.*

1. Introduction

A repairable system is often modeled as a counting failure process. Analysis of repairable system reliability must consider the effects of successive repair actions. When there is no trend in the system failure data, the failure process can often be modeled as a renewal process where successive repair actions render the system to be in "good as new" condition. The two principal classes of systems where this is not appropriate is (1) reliability improvement where design flaws are removed and the failure intensity is decreasing over time as the design evolves and improves, and (2) reliability deterioration when a system ages.

This article deals with the use maximum likelihood estimation method and Newton-Raphson algorithm to estimate the parameters of the power-law poisson process, the most commonly used model to analyze data from a repairable system. For systems undergoing reliability improvement testing, it is critically important to identify whether significant improvement is occurring. System reliability improvement can be detected by observing a significant trend of increasing successive time-between-failures, i.e., system failure inter-arrival times. For fielded systems, it is very important to detect when the system reliability is

deteriorating. Decisions for preventive maintenance and over-haul require this information. System reliability deterioration can be detected by observing a significant trend of decreasing successive time-between-failures. A non-homogeneous Poisson process (NHPP) is capable of modeling these situations. If the failure intensity function, m (t), is decreasing over time, the times between failures tend to be longer, and if it is increasing, the times between failures tend to be shorter.

If a system in service can be repaired to "good as new" condition following each failure, then the failure process is called a renewal process. For renewal processes, the times between failures are independent and identically distributed. A special case of this is the Homogeneous Poisson Process (HPP) which has independent and exponential times between failures.

2. HPP and NHPP Process

Notation:	
HPP	Homogeneous Poisson Process
NHPP	Non-homogeneous Poisson process
N(t)	number of observed failures in (0, t]
m(t)	The intensity function (sometimes called the instantaneous failure intensity)
M(t)	Expected (mean) number of failures by time t (sometimes called "the mean cumulative function MCF")
θ, β	model parameters (scale and shape parameter respectively) (θ >0, β > 0)
t	development test time or in-service time
AMSAA	Army Material Systems Analysis Activity

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2.1. Homogenous Poisson Process

A counting process, N (t), is a homogenous Poisson process with parameter $\lambda > 0$ if

• N (0)=0
• The process has independent increments
• The number of failures in any interval of length t is distributed as a Poisson distribution with parameter λt ($\lambda = 1/$)

There are several implications to this definition of Poisson process. First, the distribution of the number of events in (0, t] has the Poisson distribution with parameter λt. Second, the expected number of failures by time t, is $M(t) = E[N(t)] = \lambda t$, where λ is often called the failure intensity or rate of occurrence of failures (ROCOF). Therefore, the probability that N (t) is a given integer n is expressed by:

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n = 0,1,2,\dots \quad (1)$$

The intensity function is $m(t) = M'(t) = \lambda$. Therefore, if the inter-arrival times are independent and identically distributed exponential random variables, then N (t) corresponds to a homogenous Poisson process.

2.2. Non-Homogenous Poisson Process

A counting process, N (t), is a non-homogenous Poisson process if

• N (0) = 0
• The process has independent increments
• The number of failures in any interval of length t is distributed as a Poisson distribution with parameter M (t)
• Pr {N (t+h) - N(t) = 1} = m(t) + o(h)
• Pr {N (t+h) - N(t) ≥ 2} = o(h)

M (t) is the mean value function which describes the expected cumulative number of failures. m (t) is the intensity function. o (h) denotes a quantity which tends to zero for small h. Given m (t), the mean value function M (t) =E [N (t)] satisfies:

$$M(t) = \int_0^t m(s)\,ds \quad (2)$$

Inversely, knowing M (t), the instantaneous failure intensity at time t can be obtained as $m(t) = \frac{d}{dt}M(t)$.

As a general class of well-developed stochastic process models in reliability engineering, non-homogeneous Poisson process models have been successfully used in studying hardware and software reliability problems. NHPP models are especially useful to describe failure processes which possess trends such as reliability

improvement or deterioration. The cumulative number of failures to time t, N (t), follows a Poisson distribution with parameter M (t). The probability that N (t) is a given integer n is expressed by:

$$\Pr\{N(t) = n\} = \frac{[M(t)]^n}{n!} e^{-M(t)}, n = 0,1,2,\dots \quad (3)$$

The most commonly used and flexible model for the non-homogenous Poisson process is the power-law process and for which:

$$M(t) = E[N(t)] = \left(\frac{t}{\theta}\right)^\beta \quad (4)$$

$$m(t) = \frac{d}{dt}M(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \quad (5)$$

The intensity function represents the rate of failures or repairs. The value of the shape ( ) depends on whether the studied system is improving, deteriorating, or remaining stable.

If $0 < \;< 1$, the failure/repair rate is decreasing. Thus, the studied system is improving over time.

If $\;= 1$, the failure/repair rate is constant. Thus, the studied system is remaining stable over time (HPP process).

If $\;> 1$, the failure/repair rate is increasing. Thus, the studied system is deteriorating over time.

Note   With the (maximum likelihood) estimation method, the power-law process is commonly referred to as the AMSAA model. When only a single system is considered and the least squares estimation method is used, the power-law process is known as the Duane model.

3. Estimation Of Parameters For The Case Of Exact Failure Data

Let T_{ij} denote the time of occurrence of the jth failure for the ith system the pdf of T_{ij} at t_{ij} given the previous observation is:

$$f\left(t_{ij}|t_{i2},t_{i3},\dots,t_{ir} = y_i,\theta,\beta\right) = m(t_{ij})EXP\left\{-\left[M(t_{ij}) - M(t_{ij-1})\right]\right\} \quad (6)$$

Where y_i denotes the retirement time of the ith system. The joint density function or the likelihood function of $t_{i1},t_{i2},\dots t_{ir} = y_i$ is:

$$l_i(\theta,\beta) = EXP[-M(t_{ir} = y_i)]\prod_{j=1}^{r}[m(t_{ij})]^{N_{ij}} \quad (7)$$

Where N_{ij} denotes the frequency of failures at t_{ij}.
Note: it is not theoretically possible to have $N_{ij} > 1$.
    $N_{ri} = 0$ for all systems (retirement time).
The likelihood function for the N systems is:

$$l(\theta,\beta) = \prod_{i=1}^{N} \{ EXP[-M(y_i)] \prod_{j=1}^{r_i} [m(t_{ij})]^{N_{ij}} \} \qquad (8)$$

$$l(\theta,\beta) = \prod_{i=1}^{N} \{ EXP\left[-\left(\frac{y_i}{\theta}\right)^{\beta}\right] \prod_{j=1}^{r_i} \left[\frac{\beta}{\theta^{\beta}} t_{ij}^{\beta-1}\right]^{N_{ij}} \} \qquad (9)$$

The log likelihood function is:

$$L(\theta,\beta) = -\frac{1}{\theta^{\beta}} \sum_{i=1}^{N} y_i^{\beta} + (\ln\beta) \sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} - (\beta\ln\theta) \sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} + (\beta - 1) \sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \ln t_{ij} \qquad (10)$$

Taking the first partial derivatives of L(,) according to , yields:

$$\frac{\partial L(\theta,\beta)}{\partial \theta} = \frac{\beta}{\theta^{\beta-1}} \sum_{i=1}^{N} y_i^{\beta} - \frac{\beta}{\theta} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) \qquad (11)$$

$$\frac{\partial L(\theta,\beta)}{\partial \beta} = -\frac{1}{\theta^{\beta}} \sum_{i=1}^{N} y_i^{\beta} \ln y_i + \frac{\ln\theta}{\theta^{\beta}} \sum_{i=1}^{N} y_i^{\beta} + \frac{1}{\beta} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) - \ln\theta \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) + \sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \ln t_{ij} \qquad (12)$$

Taking the second partial derivatives of L(,) yields:

$$\frac{\partial^2 L(\theta,\beta)}{\partial \theta^2} = -\frac{\beta(\beta+1)}{\theta^{\beta+2}} \sum_{i=1}^{N} y_i^{\beta} + \frac{\beta}{\theta^2} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) \qquad (13)$$

$$\frac{\partial^2 L(\theta,\beta)}{\partial \beta^2} = -\sum_{i=1}^{N} \left(\frac{y_i}{\theta}\right)^{\beta} \ln^2 \left(\frac{y_i}{\theta}\right) - \frac{1}{\beta^2} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) \qquad (14)$$

$$\frac{\partial^2 L(\theta,\beta)}{\partial \theta \partial \beta} = \frac{\beta}{\theta^{\beta-1}} \sum_{i=1}^{N} y_i^{\beta} \ln y_i - \frac{\beta\ln\theta}{\theta^{\beta-1}} \sum_{i=1}^{N} y_i^{\beta} + \frac{1}{\theta^{\beta-1}} \sum_{i=1}^{N} y_i^{\beta} - \frac{1}{\theta} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) \qquad (15)$$

The estimation of the parameters , can be done by tow methods:

The first method is by equating (11) & (12) to zero and this yields:

$$\theta = \left[\frac{\sum_{i=1}^{N} y_i^{\beta}}{\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij}} \right]^{1/\beta} \qquad (16)$$

$$\left[\frac{1}{\theta^{\beta}} \sum_{i=1}^{N} y_i^{\beta} \ln y_i - \sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \ln t_{ij} \right]^{-1} \left(\sum_{i=1}^{N} \sum_{j=1}^{r_i} N_{ij} \right) = \beta \qquad (17)$$

Every solution $\hat\theta, \hat\beta$ of (16) & (17) are estimates of ,

The second method is by using the Newton-Raphson algorithm:

1- Evaluate the first partial derivatives L/ and L/ at $=\hat\theta i$ and $=\hat\beta i$.

2- Evaluate the second partial derivatives $^2L/$ ^{2}, $^2L/$ ^{2} and $^2L/$ at $=\hat\theta i$ and $=\hat\beta i$.

3- Solve the linear equations for the adjustments a_i and b_i :

$$\frac{\partial L(\theta,\beta)}{\partial \theta} = \left(-\frac{\partial^2 L(\theta,\beta)}{\partial \theta^2} \right) a_i + \left(-\frac{\partial^2 L(\theta,\beta)}{\partial \theta \partial \beta} \right) b_i \qquad (18)$$

$$\frac{\partial L(\theta,\beta)}{\partial \beta} = \left(-\frac{\partial^2 L(\theta,\beta)}{\partial \theta \partial \beta} \right) a_i + \left(-\frac{\partial^2 L(\theta,\beta)}{\partial \beta^2} \right) b_i \qquad (19)$$

4- Calculate the new estimates $\hat\theta_{i+1} = \hat\theta_i + a_i$ and $\hat\beta_{i+1} = \hat\beta_i + b_i$

5-Continue steps (1) through (4) until the estimates meet a convergence criterion. For example, stop when a_{i+1} and b_{i+1} are small, say, each a small fraction of the standard errors of $\hat\theta_{i+1}$ and $\hat\beta_{i+1}$. Alternatively stop when $L(\hat\theta_{i+1}, \hat\beta_{i+1}) - L(\hat\theta_i, \hat\beta_i)$ is statistically small, say, less than 0.01.

4. Estimation of the Standard Errors of Parameters and Their Confidence Limits.

The estimation of the standard errors of $\hat\theta, \hat\beta$ are as follows:

First we calculate the estimate of the Fisher matrix

$$\hat{I} = \begin{pmatrix} -\frac{\partial^2 L(\theta,\beta)}{\partial \theta^2} \big|_{\hat\theta,\hat\beta} & -\frac{\partial^2 L(\theta,\beta)}{\partial \theta \partial \beta} \big|_{\hat\theta,\hat\beta} \\ -\frac{\partial^2 L(\theta,\beta)}{\partial \theta \partial \beta} \big|_{\hat\theta,\hat\beta} & -\frac{\partial^2 L(\theta,\beta)}{\partial \beta^2} \big|_{\hat\theta,\hat\beta} \end{pmatrix} \qquad (20)$$

Second using the asymptotic variance-covariance matrix:

$$\hat{\Sigma} = \begin{pmatrix} \hat{Var}(\theta) & \hat{Cov}(\theta,\beta) \\ \hat{Cov}(\theta,\beta) & \hat{Var}(\beta) \end{pmatrix} \quad (21)$$

We can solve the equation $\hat{\Sigma} = \hat{I}^{-1}$ and find the estimation of the standard errors of $\hat{\theta}, \hat{\beta}$:

$$SE(\hat{\theta}) = \sqrt{\hat{Var}(\hat{\theta})}$$

$$SE(\hat{\beta}) = \sqrt{\hat{Var}(\hat{\beta})}$$

The tow sided 100 % confidence limits of θ, β are:

$$\theta_L = \hat{\theta}EXP\left[-\frac{z_{1-\alpha/2}SE(\theta)}{\theta}\right] \quad (22)$$

$$\theta_U = \hat{\theta}EXP\left[+\frac{z_{1-\alpha/2}SE(\theta)}{\theta}\right] \quad (23)$$

$$\beta_L = \hat{\beta}EXP\left[-\frac{z_{1-\alpha/2}SE(\beta)}{\beta}\right] \quad (24)$$

$$\beta_U = \hat{\beta}EXP\left[+\frac{z_{1-\alpha/2}SE(\beta)}{\beta}\right] \quad (25)$$

5. Numerical Example.

In this section we will show through an example how the statistical methods and numerical calculations illustrated in the previous sections can be used to analyze exact failure data from a repairable system with the power-law process as a chosen model.

5.1. Failure/Rtirement Data

Informative data *Table I.3* displays the needed informative data collected on three systems that will be used and analysed to estimate the parameters of the chosen model (the power-law process model)**.** This data are a sequence of failure times on three systems in addition to the retirement time of each system.

Tab. 1. Informative data

Failure/ retirement time	Retirement	frequency	System ID
1	1	2	1
5	1	1	1
9	0	1	1
4	1	2	2
7	1	1	2
8	0	1	2
3	1	2	3
6	1	1	3
10	0	1	3

Time the failure or retirement time of each sample.

System ID identifies each system within a sample. we do not need a system column when we have only one system.

Retirement indicates whether the data in each corresponding row is a failure time or a retirement time. Typically, the column will contain two distinct values; one representing failure times, one representing retirement times. The lower value indicates the retirement time for a system.
Frequency the total frequency of failures at a particular time. it is not theoretically possible to have multiple failures at any one instant for a given system.

5.2. Estimating of the Shape and Scale and Their Standard Errors Confidence Limits

Using equations (11) through (15) for the first and second partial derivatives of the log likelihood function L(,) according to , ; and the Newton-Raphson method by the linear equations (18), (19) for the adjustments a_i and b_i with = 0.5, =1 as a start values we will reach a good estimate of   and through six iteratios:

$$\hat{\beta} = 0.948227904$$
$$\hat{\theta} = 2.824738618$$

The following table shows the evolution of estimates through these six iterations:

Tab. 2. Sequential iterations

Iteration i	$\hat{\beta}_i$	$\hat{\theta}_i$	$\hat{\beta}_{i+1}$	$\hat{\theta}_{i+1}$
1	0.5	1	0.736837559	2.039875655
2	0.736837559	2.039875655	0.900782511	2.700238496
3	0.900782511	2.700238496	0.945284096	2.817096422
4	0.945284096	2.817096422	0.94821643	2.824709958
5	0.94821643	2.824709958	0.948227903	2.824738618
6	0.948227903	2.824738618	**0.948227904**	**2.824738618**

As we mentioned   the Newton-Raphson method requires the evaluation of L/ , L/ , ^{2} L/ ^{2}, ^{2} L/ ^{2} and ^{2} L/   at = and = and repeat that for each iteration (six iteration in our example); but in spite of the complexity of these calculations, they can easily be done using EXCEL. The following tables display these calculations for each iteration as has been done by EXCEL:

Tab.3. Calculations for 1st and 2nd iteration

$\theta_{input}=1, \beta_{input}=0.5$		$\theta_{input}=2.039875655, \beta_{input}=$	
L/	-0.004647608	L/	-0.018473295
^{2}L/^{2}	-2.243028589	^{2}L/^{2}	-1.15856988
L/	8.562284837	L/	2.814636774
^{2}L/^{2}	-79.47979851	^{2}L/^{2}	-36.41917458
^{2}L/	9.868022781	^{2}L/	4.779346856
a_i	1.039875655	a_i	0.660362841
b_i	0.236837559	b_i	0.163944952
θ_{output}	2.039875655	θ_{output}	2.700238496
β_{output}	0.736837559	β_{output}	0.900782511

Tab. 4. Calculations for 3d and 4d iteration

$\theta_{input}=2.700238496, \beta_{input}=$		$\theta_{input}=2.817096422, \beta_{input}=$	
L/	-0.04334695	L/	-0.00259991
^{2}L/^{2}	-0.971049445	^{2}L/^{2}	-1.011564759
L/	0.660294165	L/	0.038803797
^{2}L/^{2}	-24.09120349	^{2}L/^{2}	-22.35444383
^{2}L/	3.523959293	^{2}L/	3.513070606
a_i	0.116857926	a_i	0.007613536
b_i	0.044501585	b_i	0.002932334
θ_{output}	2.817096422	θ_{output}	2.824709958
β_{output}	0.945284096	β_{output}	0.94821643

Tab. 5. Calculations for 5th and 6th iteration

$\theta_{input}=2.824709958, \beta_{input}=$		$\theta_{input}=2.824738618, \beta_{input}=$	
L/	-1.12307E-5	L/	-1.80417E-09
^{2}L/^{2}	-1.014159873	^{2}L/^{2}	-1.01417158
L/	0.000154589	L/	1.28802E-08
^{2}L/^{2}	-22.24674147	^{2}L/^{2}	-22.24633438
^{2}L/	3.512145998	^{2}L/	3.512147386
a_i	2.866E-5	a_i	4.98796E-10
b_i	1.14735E-5	b_i	6.57728E-10
θ_{output}	2.824738618	θ_{output}	2.824738618
β_{output}	0.948227903	β_{output}	0.948227904

From the calculation results in *table V* and equation (20) the estimate of the Fisher matrix at $\hat{\theta}=2.824738618, \hat{\beta}=0.948227904$ is:

$$\hat{I}=\begin{pmatrix} 1.01417158 & -3.512147386 \\ -3.512147386 & 22.24633438 \end{pmatrix}$$

As the inverse of a matrix of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ so according to that, the inverse of the estimate of the Fisher matrix mentioned above is:

$$\hat{I}^{-1}=\frac{1}{(1.01417158*22.24633438)-(3.512147386^2)}*\begin{pmatrix} 22.24633438 & 3.512147386 \\ 3.512147386 & 1.01417158 \end{pmatrix}$$

$$=\frac{1}{10.22642083}\begin{pmatrix} 22.24633438 & 3.512147386 \\ 3.512147386 & 1.01417158 \end{pmatrix}$$

$$=\begin{pmatrix} 2.17538 & 0.34344 \\ 0.34344 & 0.0991717 \end{pmatrix}$$

Now using the asymptotic variance-covariance matrix in equation (21) and letting $\hat{\Sigma}=\hat{I}^{-1}$ we find:

$$\widehat{Var}(\hat{\theta})=2.17538 \to SE(\hat{\theta})=\sqrt{2.17538}=1.475$$

$$\widehat{Var}(\hat{\beta})=0.0991717 \to SE(\hat{\beta})=\sqrt{0.0991717}=0.315$$

$$\widehat{Cov}(\hat{\theta},\hat{\beta})=0.34344$$

Using equations (22) through (25) for the 95% confidence limits of the shape and scale yields:

$$\theta_L=2.824738618 EXP\left[-\frac{1.96*1.475}{2.824738618}\right]=1.01207$$

$$\theta_U=2.824738618 EXP\left[+\frac{1.96*1.475}{2.824738618}\right]=7.86068$$

$$\beta_L=0.948227904 EXP\left[-\frac{1.96*0.315}{0.948227904}\right]=0.49443$$

$$\beta_U=0.948227904 EXP\left[+\frac{1.96*0.315}{0.948227904}\right]=1.81838$$

6. Concluding Remarks

There are many software packages like Minitab that can do this work. But we wanted to gain more understanding of the methods and numerical calculations so we illustrated a numerical example for the case of exact failure data; and step by step we did this work using EXCEL for the complicated calculations; and also we solved the same example using Minitab and compare the results of the tow calculations.

Parameter Estimates

Parameter	Estimate	Standard Error	95% Normal CI Lower	Upper
Shape	0.948228	0.315	0.494562	1.81805
Scale	2.82474	1.475	1.01515	7.86008

We can see how these results are very close to the results of our calculations.

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