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Heuristic Methods Based on MINLP Formulation for Reliable Capacitated Facility Location Problems

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ABSTRACT

This paper addresses a reliable facility location problem with considering facility capacity constraints. In reliable facility location problem some facilities may become unavailable from time to time. If a facility fails, its clients should refer to other facilities by paying the cost of retransfer to these facilities. Hence, the fail of facilities leads to disruptions in facility location decisions and this problem is an attempt to reducing the impact of these disruptions. In order to formulate the problem, a new mixed-integer nonlinear programming (MINLP) model with the objective of minimizing total investment and operational costs is presented. Due to complexity of MINLP model, two different heuristic procedures based on mathematical model are developed. Finally, the performance of the proposed heuristic methods is evaluated through executive numerical example. The numerical results show that the proposed heuristic methods are efficient and provide suitable solutions.

1. Introduction

The facility location problem is one of the important combinatorial optimization problems that try to determine location of facilities among a set of candidate places. These models have various applications in regional and urban planning as well as in transportation, distribution, and energy management.[1] In addition, it determines how to assign "clients" to the facilities, so that the short-term or long-term interests of system are achieved, for example the total cost of system is minimized or the total profit is maximized. Facility location problems have been studied in literature extensively and different types of facilities, such as factories, warehouses, stores, airports, hospitals, emergency departments and so on were examined. Hakimi [2] was first one who formulated the problem as a theoretical approach. He considered the emergency facility location on a network to minimize its maximum distance to the demands points. Facility location problems and its derivatives such as capacitated or uncapacitated state are NP-Hard [3]. Hence, numerous approaches for solving this problem, including exact methods, approximate and

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heuristic methods, meta-heuristics and etc. were offered by many researchers.

Most of the researches done in the field of facility location problems have been formed on a primary assumption. This assumption is that the facilities are always available and they are completely reliable (they would never fail). In this paper we have ignored of this default assumption and try to design a reliable network where the facilities are unreliable. This problem called the "*reliable facility location*" problem. The reliable facility location problems are located in a class of location problems with uncertainty. The uncertainties in facility location problems can be generally classified into three categories: provider-side uncertainty, receiver-side uncertainty, and in-between uncertainty. The provider-side uncertainty may capture the randomness in facility capacity and the reliability of facilities, etc.; the receiver-side uncertainty can be the randomness in demands; and the in-between uncertainty may be represented by the random travel time, transportation cost, etc [3].

The reliability facility location model is one of the provider-side uncertainty problems that was first studied by Snyder and Daskin [4]. They explained that in reality, facilities may fail from time to time due to poor weather, labor actions, changes of ownership, or other factors. Such failures may lead to excessive transportation costs as customers must be served from facilities much farther than their regularly assigned facilities [4]. They have raised the reliability of location network by allocation of backup facility to each client. For this purpose, they have presented a linear model for choosing facility locations to minimize cost, while also taking into account the expected transportation cost after failures of facilities. The objective function of their model was a two-part objective function including the cost of pre-fail and post-fail and it was minimized by Lagrangian relaxation solution method. In this paper, if a facility fails, the failure costs include the clients transfer cost to other active facilities or penalty cost resulting from non-fulfillment of clients demand. In other words, at the time of the failure the decision maker chooses one of these cost (the transfer cost or penalty cost), by creating a balance between them.

After Snyder and Daskin, other researchers began to investigate the reliable facility location problems. Number of related researches is very limited since the history of issue is less than a decade. Thus, it looks that further investigations in this area is essential. Berman et al. [5] developed a *p*-median problem with unreliable facilities which is focused on finding *m* facility locations to minimize the expected sum of the weighted travel distances from demands originating at customer locations to the closest operating facility. They also assumed that the probabilities of facilities failure are the same similar to Snyder and Daskin [4]. They have developed several exact and heuristic solution approaches to solve model. Cui et al. [6] proposed a compact mixed integer programming (MIP) formulation and a continuum approximation (CA) model to study the reliable uncapacitated fixed charge location problem which seeks to minimize initial setup costs and expected transportation costs in normal and failure scenarios. The CA model predicts the total system cost without details about facility locations and customer assignments, and it provides a fast heuristic to find near-optimum solutions. They solved the MIP model by a custom-designed Lagrangian Relaxation (LR) algorithm. Their computational results showed that the LR algorithm is efficient for mid-sized reliable uncapacitated location problems. They also illustrated that the CA is a suitable alternative to the LR algorithm for large-scale problems. Li and Ouyang [7] studied the reliable uncapacitated fixed charge location problem where facilities are subject to spatially correlated disruptions that occur with location-dependent probabilities. They also developed a CA model to minimize the sum of initial facility construction costs and expected customer transportation costs under normal and failure scenarios. Shen et al. [3] formulated this problem as a two-stage stochastic model and then as a nonlinear integer programming model. They provided a 4-approximation algorithm while assumed the probability of a facility failure is constant and independent of the facility. An et al. [8] proposed a set of two-stage robust optimization models to design reliable *p*-median facility location networks subject to disruptions. They analyzed structural properties of the problem and applied the column-and-constraint generation method with customized enhancement strategies.

Wide researches have been done in facility location problem with capacity constraint (see for example [9] and [10]). But common point of all researches done in the field of reliable facility location problems is that all of them investigated the problems with uncapacitated facilities pattern. Although the assumption itself is very common in the facility location models, it may be unrealistic in practice. There is no facility with infinite capacity in fact. The uncapacitated facilities pattern is true when that the amount of demands is very small respect to the quantity supplied. This condition is also economically not accepted. Few researches have been done in the capacitated reliable facility location problem. Aydin and Murat [11] developed a scenario based model and presented a novel hybrid method, swarm intelligence based sample average approximation (SIBSAA), to solve the capacitated reliable facility location problem. In the reliable capacitated facility location problem (RCFLP), if a facility fails, its next-level backup facility can accept the clients of failed facility only if it has sufficient capacity to satisfy the additional demand. The capacitated model is much more complex than uncapacitated model. In this paper, we added the facilities capacity constraints in the model. In the other words, the investigated problem is a reliable capacitated fixed charge location problem (RCFL). On the complexity of the reliable facility location problems, Li et al. [12] proved that a reliable uncapacitated fixed-charge location problem (RUFL) is NP-Hard. Since the problem studied in this paper has more complexity than the RUFL problem due to adding capacity constraints, it can be concluded that the RCFL problem is NP-hard as well.

To solve the model we have presented two powerful heuristic methods based on MINLP formulation that are called "*relax and fix heuristic*" and "*relax and round heuristic*". To the best of our knowledge, the heuristic algorithm for reliable facility location problem was only used by Shen et al. [3]. The literature on heuristic algorithms for facility location problems is extensive. However, these heuristic methods mainly dealt with deterministic problems. Shmoys et al. [13], Korupolu et al. [14], Mahdian et al. [15], Resende and Werneck [16], Du et al. [17] and Sinha [18] are some researchers that used the heuristic or approximation approaches for facility location problems in deterministic pattern.

The use of heuristic or approximation methods to solve facility location problems with uncertainty is increasing recently. In receiver-side uncertainty class, Gabor and Van Ommeren [19] proposed a 2-approximation algorithm for a facility location problem with stochastic demands and inventories. Murali et al. [20] proposed locate-allocate heuristic to solve capacitated facility location problem in order to maximize coverage, taking into account a distance-dependent coverage function and demand uncertainty. For in-between uncertainty class, Eiselt et al. [21] proposed a heuristic algorithm to optimally locate *p* facilities on a network where one link can fail. Their presented algorithm can solve optimality the problem in polynomial time.

ReVelle et al. [22] presented a heuristic approach for discrete facility location problems where reliability and uncertainty are addressed by chance constrained covering and maximal expected covering models.

The rest of this paper is organized as follows: In section 2, we formulate a reliability model based on MINLP mathematics model. We solve this model using two heuristic approaches in Section 3. Section 4 provides the computation results, and finally Section 5 outlines the conclusion and some suggestions for future studies.

2. The Problem Formulation

So far, two common forms of modeling that presented for reliable facility location problem are the Scenario-based model and the failure probability-based model. In Scenario-Based model there is a finite set of scenarios, where each scenario specifies the set of operational facilities and happens with a specific probability [3].

One of the major drawbacks of the Scenario-based model is that with the increasing number of possible scenarios the model becomes too complex to be solved. This situation is exacerbated when the failure probabilities are independent. So, the failure probability-based model can be a suitable alternative for Scenario-based model.

This form of modeling first presented by Snyder and Daskin [4] is based on independent probabilities of facilities failure (q_j) whereas $(0 \leq q_j \leq 1)$. Since then, the other researchers such as Shen et al. [3] offered different types of probability-based modeling.

In this section, a new nonlinear integer programming formulation model based on failure probability pattern is presented. This model is an extension of Snyder and Daskin [4] model in which the capacity constraint is considered. Moreover, it is assumed that failure probabilities of the facilities are different and failure to meet the clients demand has been allowed by paying penalty costs.

This model has two sub-problems including locating sub-problem and assigning sub-problem while each sub-problem is introduced by individual decision variables. Before presenting the MINLP model, we introduce the notations of indices, parameters and variables that will be used throughout the model.

The indices of the model are:

- i Index of clients
- j Index of potential facility locations
- r Index of multiple levels for allocation of facilities to clients
- NF The set of candidate facilities that may not fail ("nonfailable" facilities)

The parameters of the model are:

- h_j The facility cost to open facility *j*
- d_i The demand of client *i*
- $c_{i,j}$ The service cost if client *i* is serviced by facility *j*
- q_j The probability that facility *j* fails (q_j is equal to zero for $j \in NF$)
- B_j The capacity of facility *j*
- g_i The penalty for client *i* if its demands are not met
- R The number of facilities that need to be opened

The variables of the model are:

- y_j The binary variable that is equal to 1 if facility *j* is opened; otherwise 0.
- $x_{i,j,r}$ The binary variable that is equal to 1 if demand client *i* is assigned to facility *j* in the r^{th} level assignment; otherwise 0.
- $z_{i,r}$ The binary variable that is equal to 1 if client *i* has (*r-1*)th backup facility but has no *r*th backup facility; otherwise 0.

The MINLP model:

$$min \sum_{j=1}^{J} h_j y_j + \sum_{i=1}^{I}\sum_{r=1}^{R}\sum_{j=1}^{J} d_i c_{i,j} x_{i,j,r}(1-q_j)\prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}} + \sum_{j=1}^{J}\sum_{r=1}^{R+1}\prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}} d_i g_i z_{i,r} \tag{1}$$

s.t.

$$\sum_{j=1}^{J} x_{i,j,r} + \sum_{j=1\ \&\ j\in NF}^{J}\sum_{s=1}^{r-1} x_{i,j,s} + \sum_{s=1}^{r} z_{i,s} = 1 \qquad i = 1,\dots,I;\ \ r = 1,\dots.,R+1, \tag{2}$$

$$\sum_{i=1}^{I}\sum_{r=1}^{R} d_i x_{i,j,r}(1-q_j)\prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}} \le B_j \qquad j = 1,\dots,J, \tag{3}$$

$$x_{i,j,r} \le y_j \qquad j = 1,\dots,J;\ \ i = 1,\dots,I;\ \ r = 1,\dots.,R+1, \tag{4}$$

$$\sum_{r=1}^{R} x_{i,j,r} \le 1 \qquad j = 1,\dots,J;\ \ i = 1,\dots,I, \tag{5}$$

$$\sum_{j=1}^{J} y_j = R \quad (6)$$

$$x_{i,j,R+1} = 0 \qquad j = 1, \dots, J;\ i = 1, \dots, I, \quad (7)$$

$$x_{i,j,r}, y_j, z_{i,r} \in \{0,1\} \qquad j = 1, \dots, J;\ i = 1, \dots, I;\ r = 1, \dots, R+1. \quad (8)$$

The relation (1) indicates the problem's objective function, which comprises the sum of facility setup cost, expected service cost, and the expected penalty cost. Constraints (2) ensure that client i is assigned to a facility in level r or facility assignment is stopped in previous levels (levels 1 to r-1) by assignment of a nonfailable facility or by paying penalty cost. Constraints (3) ensure that the expected demand that met by facility j is not larger than its capacity. Constraints (4) prevent an assignment to a facility that has not been opened. Constraints (5) prohibit the assignment of a facility to a client more than one time. Constraints (6) ensure that R facilities to be opened. Constraints (7) prevent the assignment of facilities to clients in level (R+1). Constraints (8) define the model variables.

In the above formulation, solution space of the model can be linear by changing in constraints (3). With this change, we can greatly reduce the complexity of the model. For this purpose, it is necessary to replace the constraints (3) with the constraints (9) or (10).

$$\sum_{i=1}^{I}\sum_{r=1}^{R} d_i x_{i,j,r}\left(1-q_j\right)\left(Avg_{j'=1}^{J} q_{j'}\right)^{r-1} \le B_j - \varepsilon \qquad j = 1, \dots, J, \quad (9)$$

$$\sum_{i=1}^{I}\sum_{r=1}^{R} d_i x_{i,j,r}\left(1-q_j\right)\left(Max_{j'=1}^{J} q_{j'}\right)^{r-1} \le B_j \qquad j = 1, \dots, J. \quad (10)$$

In constraints (9), the average of q_j $(j = 1, \dots, J)$ were replaced as a suitable approximation to the failure probability of individual facilities. The ε value was added to the right of this relation to prevent from violation of the capacity levels. This value can be determined experimentally. This margin of safety was created in constraints (10) using maximum of q_{j} $(j = 1, \dots, J)$. Note that to determine the average or maximum values in constraints (9) and (10), the out-of-range probability values are ignored.

3. The MINLP-Based Heuristic Methods

Real-world MILP problems are computationally very complex; hence employing heuristic approaches is prevalent to tackle them. Heuristic methods are divided into two general categories:

(1) “Improvement Heuristics” that in which an initial solution is improved (2) “Constructive Heuristics” that build a solution, step by step, according to a set of rules defined before-hand.

In this paper, two distinct Constructive Heuristics methods that called “*relax and fix*” and “*relax and round*” heuristics are developed. These methods are effective approaches to find feasible solutions for MILP or MINLP formulations. These methods have been described as follows:

3.1- The “Relax and Fix” Heuristic

The “relax and fix” heuristic is based on decomposing the original problem into subsets that can be solved more easily by an iterative pattern. In this paper, the original problem is divided into series of sub-problems consisting of earlier and later of backup levels for use of “relax and fix” heuristic. Next, this approach is implemented based on a detailed planning for Initial levels and macro planning for later levels. The quality of final solution and complexity of problem are influenced by the size and number of the sub-problems.

In the proposed heuristics the main model is divided to R sub problems. Each sub problem is equivalent to an individual backup level in original model. Figure 1 shows how to implement the proposed heuristics in an iterative pattern. This approach starts from backup level 1 and continues until the last backup level (level R). This is due to the important and influence of decisions in initial levels rather than the higher levels. In other words, the cost of services and penalty in higher levels is very low due to sequential multiplying of facilities failure probabilities at each other.

Three strategies include “freezing strategy”, “solving whole model” and “relaxation strategy” can be executed at per iteration. In the first iteration, the model related to the sub problem1 (backup level 1) is considered as a whole and in the other sub problems (levels 2 to R), the relaxation strategy simplifies the model by relaxing binary variables including $x_{i,j,r}$ and $z_{i,r}$. Similarly, in k^{th} iteration, the freezing strategy is applied for sub problem $k-1$. In freezing strategy, the value of binary variables in level $k-1$ ($x_{i,j,k-1}$ and $z_{i,k-1}$) are fixed based on the result of solving the whole model in $(k-1)^{th}$ iteration. Also, the model related to the time interval k is considered as whole model and the models related to sub problems $k+1$ to R are simplified according to the relaxation strategy. Note that the variable y_j in all iterations are exempted from relaxation or freezing strategies.

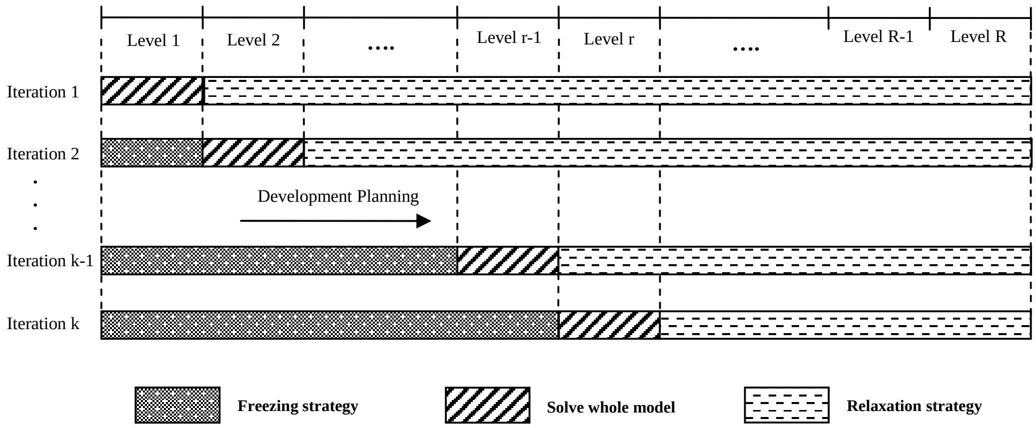

Fig. 1. Illustration of the proposed *Relax and Fix* heuristic in an iterative pattern

3.1.1- The Constraints of New Model

To implement the "*relax and fix*" heuristic, the constraints in the original model needs to be changed. The constraints of new model in the t^{th} iteration of heuristic are as follows:

$$\sum_{j=1}^{J} x_{i,j,r} + \sum_{j=1\ \&\ j\in NF}^{J} \sum_{s=1}^{r-1} x_{i,j,s} + \sum_{s=1}^{r} z_{i,s} = 1 \qquad i=1,\dots,I;\ r=1,\dots.,t, \tag{11}$$

$$\sum_{j=1}^{J} xx_{i,j,r} + \sum_{j=1\ \&\ j\in NF}^{J} \left(\sum_{s=1}^{t-1} x_{i,j,s} + \sum_{s=t}^{r-1} xx_{i,j,s} \right) + \sum_{s=1}^{t} z_{i,s} + \sum_{s=t+1}^{r} zz_{i,s} = 1 \qquad i=1,\dots,I;\ r=t+1,\dots.,R+1, \tag{12}$$

$$\sum_{i=1}^{I} \left(\sum_{r=1}^{t} d_i x_{i,j,r} \left(Avg_{j'=1}^{J} q_{j'} \right)^{r-1} + \sum_{r=t+1}^{R} d_i xx_{i,j,r} \left(Avg_{j'=1}^{J} q_{j'} \right)^{r-1} \right) (1-q_j) \le B_j - \varepsilon \qquad j=1,\dots,J, \tag{13}$$

$$x_{i,j,r} \le y_j \qquad j=1,\dots,J;\ i=1,\dots,I;\ r=1,\dots.,t, \tag{14}$$

$$xx_{i,j,r} \le y_j \qquad j=1,\dots,J;\ i=1,\dots,I;\ r=t+1,\dots.,R+1, \tag{15}$$

$$\sum_{r=1}^{t} x_{i,j,r} + \sum_{r=t+1}^{R} xx_{i,j,r} \le 1 \qquad j=1,\dots,J;\ i=1,\dots,I, \tag{16}$$

$$xx_{i,j,R+1} = 0 \qquad j = 1, \dots, J;\; i = 1, \dots, I\,, \tag{17}$$

$$0 \le xx_{i,j,r} \le 1 \text{ and } 0 \le zz_{i,r} \le 1 \qquad j = 1, \dots, J;\; i = 1, \dots, I;\; r = 1, \dots, R+1\,. \tag{18}$$

The constraints (2) of the original model are replaced by constraints (11) and (12) in the new model. Also, the constraints (3) are replaced by constraints (13); constraints (4) are replaced by constraints (14) and (15); constraints (5) are replaced by constraints (16) and constraints (17) and (18) are added to the new model.

3.1.2- The New Objective Function

The new formulation of objective function in t^{th} iteration of heuristic is equal to:

$$\begin{aligned} min \sum_{j\in J} h_j y_j &+ \sum_{i=1}^{I}\sum_{r=1}^{t}\sum_{j=1}^{J} d_i c_{ij} x_{i,j,r}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}} \\ &+ \sum_{i=1}^{I}\sum_{r=t+1}^{R}\sum_{j=1}^{J} d_i c_{ij} xx_{i,j,r}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{s=1}^{t} x_{i,k,s}+\sum_{s=t+1}^{r-1} xx_{i,k,s}} \\ &+ \sum_{j=1}^{J}\sum_{r=1}^{t}\prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}}\, d_i g_i z_{i,r} + \sum_{j=1}^{J}\sum_{r=1}^{t}\prod_{k=1}^{J} q_k^{\sum_{s=1}^{t} x_{i,k,s}+\sum_{s=t+1}^{r-1} xx_{i,k,s}}\, d_i g_i zz_{i,r}\,. \end{aligned} \tag{19}$$

The remarkable point is that changing the objective function of the original model leads to increasing model complexity unlike the changing in the original model constraints. This is opposite of what we expect. Therefore, it is necessary to use a simpler form of the objective function. This issue is provided by offering a lower bound.

Theorem: The optimal value of objective function (19) is bounded from below by relation (20).

$$\begin{aligned} \min \sum_{j\in J} h_j y_j &+ \sum_{i=1}^{I}\sum_{r=1}^{t}\sum_{j=1}^{J} d_i c_{ij} x_{i,j,r}\left(1-q_j\right) q_{min}^{r-1} + \sum_{i=1}^{I}\sum_{r=t+1}^{R}\sum_{j=1}^{J} d_i c_{ij} xx_{i,j,r}\left(1-q_j\right) q_{min}^{r-1} \\ &+ \sum_{j=1}^{J}\sum_{r=1}^{t} q_{min}^{r-1}\, d_i g_i z_{i,r} + \sum_{j=1}^{J}\sum_{r=1}^{t} q_{min}^{r-1}\, d_i g_i zz_{i,r}\,, \end{aligned} \tag{20}$$

where:

$$q_{min}^{r-1} = Min_{j=1}^{J} q_j\,. \tag{21}$$

Proof: To prove the above theorem, we must show the following statements are always true.

$$q_{min}^{r-1} \le \prod_{k=1}^{J} q_k^{\sum_{s=1}^{r-1} x_{i,k,s}} \qquad i = 1, \dots, I;\; r = 1, \dots, t\,, \tag{22}$$

$$q_{min}^{r-1} \leq \prod_{k=1}^{J} q_k^{\sum_{s=1}^{t} x_{i,k,s}+\sum_{s=t+1}^{r-1} xx_{i,k,s}} \qquad j = 1, \dots, J;\ \ i = 1, \dots, I; \quad r = t+1, \dots, R+1 . \tag{23}$$

To prove relations (23) and (24) it is assumed that there are n backup facilities for r^{th} level of client i. According to constraints (11) and (12), the maximum value of n is equal to $r-1$ (So, we have $n \leq r-1$). We show these backup facilities by $k_1, k_2, \dots, k_n$ indices. It is clear that $q_{min} \leq q_{k_l}$ $(l = 1, \dots, n)$.Therefore, since $0 \leq q_j \leq 1$ $(j = 1, \dots, J)$ and $n \leq r-1$ can be found that: $\prod_{j=1}^{J} q_j^{\sum_{s=1}^{r-1} x_{i,j,s}} = q_{k_1} q_{k_2} \dots q_{k_n} \geq q_{min}^{r-1}$, and $\prod_{k=1}^{J} q_k^{\sum_{s=1}^{t} x_{i,k,s}+\sum_{s=t+1}^{r-1} xx_{i,k,s}} \geq q_{k_1} q_{k_2} \dots q_{k_n} \geq q_{min}^{r-1}$.

The advantage of this lower bound is changing the MINLP model to the MILP model.

3.2. The "Relax and Round" Heuristic

The goal of rounding heuristics is to convert a fractional solution $\bar{x}$ of the system $Ax \leq b$; $l \leq x \leq u$ into an integral solution. The *simple rounding* heuristic is one of the conventional rounding heuristic. This heuristic uses the notion of *up* and *down locks* to determine if a rounding is possible. In *simple rounding* heuristic, rounding up a fractional solution z_j might violate the respective row $a_i^T x \leq b$ if $a_{i,j} > 0$ and rounding down of this fractional solution might violate the respective row $a_i^T x \leq b$ if $a_{i,j} < 0$ [23]. In this section we present the special rounding schema which guarantee to keep all linear constraints satisfied. So, first we need to relax the original model and solve it. The relaxation of problems is usually used to obtain lower bounds from complex problems, and also upper bounds by modifying the solution of relaxed problem. To implement the rounding heuristics for RCFLP, the $x_{i,j,r}$ and $z_{i,r}$ variables are relaxed in first step and then rounded by an efficient heuristic. Before presenting the rounding scheme, we show that the following statement is true.

Theorem: In optimal solution of RCLP for a client l if there are two facilities j and m that $x_{l,j,r} = x_{l,m,r+1} = 1$ then $c_{lj} < c_{lm}$.

Proof: We use the contradiction technique to prove the proposition. Suppose that in optimal solution $x_{l,j,r} = x_{l,m,r+1} = 1$ and $c_{lj} > c_{lm}$. Then, the new value of objective function is considered by "swapping" the assignment of facilities j and m; namely assignment of facility j to level $r+1$ and assignment of facility m to level r. We show the objective function value before and after the swapping by f_1 and f_2, respectively. The value of f_1 and f_2 is equal to:

$$f_1 = \sum_{j=1}^{J} h_j y_j + \sum_{i=1}^{I} \sum_{s=1}^{r-1} \sum_{j=1}^{J} d_i c_{i,j} x_{i,j,s} (1-q_j) \prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}$$
$$+ \sum_{i=1\ \&\ i \neq l}^{I} \sum_{s=r}^{r+1} \sum_{j=1}^{J} d_i c_{i,j} x_{i,j,s} (1-q_j) \prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}$$

$$+d_l c_{l,j} x_{l,j,r}(1-q_l)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}} + d_l c_{l,m} x_{l,m,r+1}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r} x_{l,k,h}}$$

$$+\sum_{i=1}^{I}\sum_{s=r+2}^{R}\sum_{j=1}^{J} d_i c_{i,j} x_{i,j,s}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}} + \sum_{j=1}^{J}\sum_{s=1}^{R+1}\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}\, d_i g_i z_{i,s}\ .$$

$$f_2 = \sum_{j=1}^{J} h_j y_j + \sum_{i=1}^{I}\sum_{s=1}^{r-1}\sum_{j=1}^{J} d_i c_{i,j} x_{i,j,s}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}$$

$$+\sum_{i=1\ \&\ i\neq l}^{I}\sum_{s=r}^{r+1}\sum_{j=1}^{J} d_i c_{ij} x_{i,j,s}(1-q_j)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}$$

$$+d_l c_{l,m} x_{l,m,r}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}} + d_l c_{l,j} x_{l,j,r+1}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r} x_{l,k,h}}$$

$$+\sum_{i=1}^{I}\sum_{s=r+2}^{R}\sum_{j=1}^{J} d_i c_{i,j} x_{i,j,s}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}} + \sum_{j=1}^{J}\sum_{s=1}^{R+1}\prod_{k=1}^{J} q_k^{\sum_{h=1}^{s-1} x_{i,k,h}}\, d_i g_i z_{i,s}\ .$$

The difference between the values of f_2 and f_1 is equal to:

$$f_2 - f_1 = d_l c_{l,m}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}} + d_l c_{l,j}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r} x_{l,k,h}} - d_l c_{l,j}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}$$

$$-d_l c_{lm}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r} x_{l,k,h}} = d_l c_{lm}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}} + d_l c_{lj}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}.q_m$$

$$-d_l c_{lj}\left(1-q_j\right)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}} - d_l c_{lm}(1-q_m)\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}.q_j$$

$$=\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}\, d_l\left(c_{lm}(1-q_m) + c_{lj}\left(1-q_j\right)q_m - c_{lj}\left(1-q_j\right) - c_{lm}(1-q_m)q_j\right)$$

$$=\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}\, d_l\left(c_{lm} - c_{lm}q_m + c_{lj}q_m - c_{lj}q_jq_m - c_{lj} + c_{lj}q_j - c_{lm}q_j + c_{lm}q_mq_j\right)$$

$$=\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}\, d_l\left(\left(c_{lm} - c_{lj}\right).(1 + q_mq_j - q_j - q_m)\right)\ .$$

Since $min\left(q_mq_j - q_j - q_m | 0 \le q_m\ \&\ q_j \le 1\right) = -1$, the value of $(1 + q_mq_j - q_j - q_m)$ is positive. Moreover, $\prod_{k=1}^{J} q_k^{\sum_{h=1}^{r-1} x_{l,k,h}}\, d_l > 0$ and $\left(c_{lm} - c_{lj}\right) < 0$. So, it can be concluded $f_2 < f_1$.

According to the above theorem, the pseudo-code of the proposed rounding heuristic in figure 2:

```
For  i=1 to I
  For  j=1 to J
    n=0;
    While ∑_{r∈R} x_{i,j,r} > 0  Then
      n=n+1;
      R_j(n) = {s | x_{i,j,s} = max_{r=1}^{R} x_{i,j,r}}  and
      x_{i,j,s} = 0;
    End while
  End for
  For  j=1 to J
    For m=1 to J
      IF R_j(1) = R_m(1)  Then
          IF c_{ij} < c_{im}  Then
              R_m(1) = [ ];   Else   R_j(1) = [ ];

          End if
      End if
    End for
    x_{i,j,R_j(1)} = 1;
  End for
End for
```

Fig. 2. The pseudo-code of the proposed rounding heuristic

4. Computational Results

The performance of the proposed heuristic methods and the credibility and performance of the proposed mathematical model are evaluated and compared in this section. The proposed nonlinear integer programming models and two heuristic methods are coded in the GAMS 24.1.2 and Matlab 7.10 software and solved on a PC with 2.66 GHz processor and 4 G of RAM. The "*Relax and Fix*" and "*Relax and Round*" heuristics are noted by *H1* and *H2*. Two *Relative Percentage Deviation* criteria that called RPD1 and RPD2 are used as performance measures which are calculated based on the deviation of solutions to the best solutions and average of solutions that achieved by mathematical model, *H1* and *H2*. (Note that the index A denotes a solution method.)

$$RPD1_A = \frac{(objectve\ function)_A - Min(objectve\ function)}{Min(objectve\ function)} \tag{25}$$

$$RPD2_A = \frac{(objectve\ function)_A - Avg(objectve\ function)}{Avg(objectve\ function)} \tag{26}$$

4.1. Generating Random Instances

Since there is no benchmark for reliable facility location problems by capacity constraint and different probabilities of facilities failure, it is necessary to generate random instances. Therefore, several random instances are generated according to what is outlined in Table 1. The Snyder and Daskin [3] methods were used to generate the value of demands and fixed cost.

Tab.1. Simulation Parameters of Test Cases

Parameter	notation	produced by
The facility cost to open facility *j*	h_j	$U[500,1500]$ and rounded to the nearest integer
The demand of client *i*	d_i	$U[0,1000]$ and rounded to the nearest integer
The service cost if client *i* is serviced by facility *j*	$c_{i,j}$	$U[10,500]$ and rounded to the nearest integer
The probability that facility *j* fails	q_j	$U[0.05,0.45]$ with two decimal places
The capacity of facility *j*	B_j	normal distributions with $\mu = 5000$ and $\delta = 1000$
The penalty for client *i* if its demands are not met	g_i	normal distributions with $\mu = 1000$ and $\delta = 150$

Besides, the value of *R* parameter was determined according to the instances size. Ultimately, 12 random instances with the above conditions is generated and each instance is labeled with (α, β), which respectively indicate the number of candidate sites and number of clients. Both the solution quality and the efficiency of the proposed procedures depend on the size of these parameters. In order to determine the best trade-off between algorithms' speed and solution quality the runtime limit of 1800 seconds is imposed on the *H1*, *H2* and MINLP models. This limit is increased to 3600 seconds for larger instances.

4.2- Comparison and Evaluation of the Proposed Solution Methods

In order to assess and contrast the performance of the developed heuristics and verification of mathematical model, 12 test problems were solved by them. The results were juxtaposed by results of MINLP model in Table 2. According to the RPD1 and RPD2 values, the performance of the proposed heuristics is clearly convincing and our computational experiments show that *H1* and *H2* heuristics are very effective. This assumption is valid in both quality of results (according to RPD1 and RPD2 criteria) and CPU time.

The performance of H2 heuristic surpasses the other methods and the objective values of the heuristic H2 is less than those obtained by *H1* and MINLP. By increasing the size of test problems the quality of H2 heuristic is increased; so that the average of RPD2 values for the first six test problems is equal to (-2.4%) and this value is equal to (-9.5%) for the second six test problems. Also, the *H1* heuristic algorithm shows similar results; such that the average of RPD2 value for the first six test problems is equal to (1.1%) and this value is equal to (-4.0%) for the second six test problems. Furthermore, the performance of *H1* in terms of CPU time shows that *H1* is the superior method. The average of CPU time for *H1* heuristic is greater than 1605 while this amount is greater than 2480 and 2645 for *H2* and MINLP model, respectively.

Tab. 2. Comparing the performance of the developed heuristics versus MINLP Model

TP(α,β)	MINLP Model				H1				H2			
	Time (s)	Obj	RPD1 (%)	RPD2 (%)	Time (s)	Obj	RPD1 (%)	RPD2 (%)	Time (s)	Obj	RPD1 (%)	RPD2 (%)
TP(5,10)	43	79462	0.0%	-12.3%	2	97412	22.6%	7.5%	2	95021	19.6%	4.8%
TP(10,10)	1103	92766	0.0%	-7.7%	24	106018	14.3%	5.5%	753	102705	10.7%	2.2%
TP(10,15)	>1800	114169	6.7%	1.2%	29	117235	9.6%	3.9%	1674	106997	0.0%	-5.1%
TP(10,20)	>180	186920	8.4%	4.9%	302	175441	1.8%	-1.6%	>1800	172359	0.0%	-3.3%
TP(15,15)	>1800	144079	12.3%	7.2%	362	130659	1.8%	-2.7%	>1800	128315	0.0%	-4.5%
TP(20,20)	>3600	229631	25.5%	14.7%	824	187804	2.7%	-6.2%	2141	182911	0.0%	-8.6%
TP(20,30)	>3600	278606	27.6%	12.5%	1141	245912	12.6%	-0.7%	>3600	218381	0.0%	11.8%
TP(25,30)	>3600	334527	36.5%	20.1%	2187	256293	4.6%	-8.0%	>3600	245065	0.0%	12.0%
TP(25,40)	>3600	389726	29.3%	15.0%	>3600	325069	7.8%	-4.0%	>3600	301522	0.0%	11.0%
TP(30,40)	>3600	458633	25.9%	14.2%	>3600	381554	4.7%	-5.0%	>3600	364267	0.0%	-9.3%
TP(25,50)	>360	571543	15.4%	8.4%	>360	514920	4.0%	-2.3%	>3600	495301	0.0%	-6.1%
TP(30,50)	>3600	602747	18.9%	10.8%	>3600	522781	3.1%	-3.9%	>3600	506987	0.0%	-6.8%
Average	>264	-	17.2%	7.6%	>160	-	7.5%	-1.5%	>2480	-	2.5%	-6.0%

Figure 3 shows a comparison between *H1*, *H2* and MINLP model in terms of RPD2 criterion. In this figure is clearly evident that the objective values of the proposed heuristics are below the average rate in most cases. This issue proves that there is a significant difference between the results of the proposed heuristics and MINLP model. Due to complexity of the model and its non-linear property, the performance of the proposed model deteriorates for problem scales larger than TP(30,50). This issue is confirmed by the convergence in RPD2 criterion for problems with larger scale (form TP(24,40) to TP(30,50)).

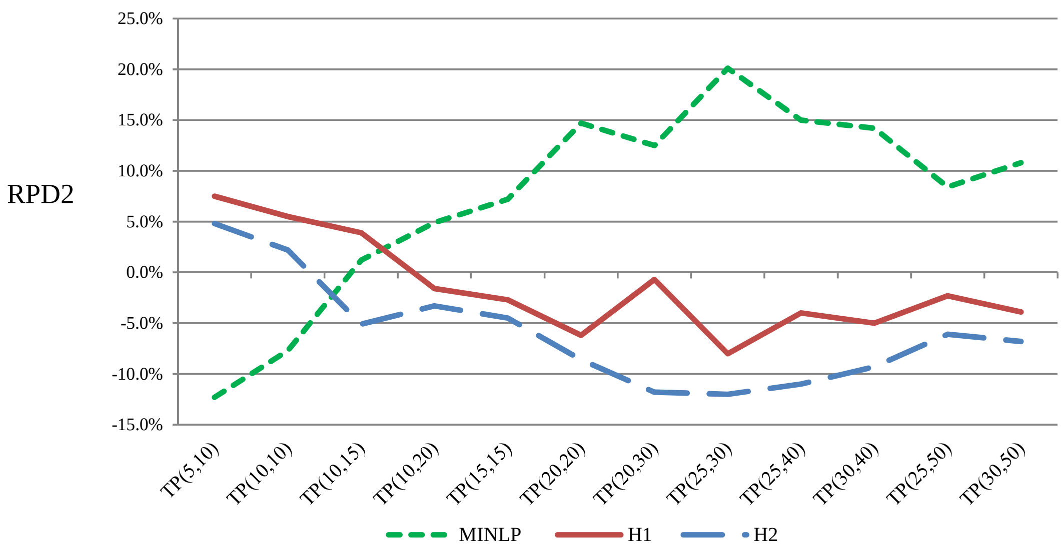

Fig. 3. Comparing the RPD2 criterion of *H1*, *H2* and MINLP model

5. Conclusion and Future Researches

In this paper, we studied a combination of reliability concept into classical facility location problems. The main point that distinguishes this paper than the other related papers are (i) considering capacity constraint and (ii) the assumption that failable facilities all have the different probabilities. In order to formulate the problem, a new mixed-integer nonlinear programming (MINLP) model was proposed based on concept of model presented by Snyder and Daskin [3]. Key to this type of formulation is the concept of "backup" assignments, which represents the facilities to which customers are assigned when closer facilities have failed. The cost of facilities opening (fixed cost) and the expected transportation cost, taking into account the costs that resulted from facility failures (operational cost), were included in the objective function. Due to complexity of MINLP model, two different heuristic procedures ("*Relax and Fix*" heuristic and "*Relax and Round*" heuristic) based on mathematical model were developed. In general, as the computational results showed, the proposed heuristic methods were able to find efficient solutions. Since the lower bounds that provide the values much closer to the objective function of the original model can create the better results, one suggestion for the future research can be the extension of the "*Relax and Fix*" heuristic by providing more powerful lower bounds.

References

[1] Nookabadi, A. S., Yadollahpour, M. R., & Kavosh, S. Optimal Capacities in Discrete Facility Location Design Problem. *International Journal of Industrial Engineering & Production Research*, *24*(1), (2013). pp. 81-89.

[2] Hakimi, S. L. Optimum locations of switching centers and the absolute centers and medians of a graph. *Operations Research*, *12*(3), (1964), pp. 450-459.

[3] Shen, Z. J. M., Zhan, R. L., & Zhang, J. The reliable facility location problem: Formulations, heuristics, and approximation algorithms. *INFORMS Journal on Computing*, *23*(3), (2011). pp. 470-482.

[4] Snyder, L. V., & Daskin, M. S. Reliability models for facility location: the expected failure cost case. *Transportation Science*, *39*(3), (2005). pp.400-416.

[5] Berman, O., Krass, D., & Menezes, M. B. (2007). Facility reliability issues in network p-median problems: strategic centralization and co-location effects. *Operations Research*, *55*(2), (2007). pp. 332-350.

[6] Cui, T., Ouyang, Y., & Shen, Z. J. M. (2010). Reliable facility location design under the risk of disruptions. *Operations Research*, *58*(4-part-1), (2010). pp. 998-1011.

[7] Li, X., & Ouyang, Y. A continuum approximation approach to reliable facility location design under correlated probabilistic disruptions. *Transportation Research part B: methodological*, *44*(4), (2010). pp. 535-548.

[8] An, Y., Zeng, B., Zhang, Y., & Zhao, L. Reliable p-median facility location problem: two-stage robust models and algorithms. *Transportation Research Part B: Methodological*, *64(c)*, (2014). pp.54-72.

[9] Jabalameli, M. S., Tabrizi, B. B., & Javadi, M. M. Capacitated facility location problem with variable coverage radius in distribution system. *International Journal of Industrial Engineering & Production Research*, *21*(4), (2010). pp. 231-237.

[10] Tavakkoli-Moghaddam, R., Baboli, A., & Vahdani, B. A Robust Optimization Approach for a p-Hub Covering Problem with Production Facilities, Time Horizons and Transporter. *International Journal of Industrial Engineering & Production Research*, *25*(4), (2014). pp. 317-331.

[11] Aydin, N., & Murat, A. A swarm intelligence based sample average approximation algorithm for the capacitated reliable facility location problem. *International Journal of Production Economics*, *145*(1), (2013). pp. 173-183.

[12] Li, Q., Zeng, B., & Savachkin, A. Reliable facility location design under disruptions. *Computers & Operations Research*, *40*(4), (2013). pp. 901-909.

[13] Shmoys, D. B., É. Tardos, K. Aardal. Approximation algorithms for facility location problems. *Proc. 29th ACM STOC, El Paso, TX, ACM, New York*, (1997). pp. 265–274.

[14] Korupolu, M. R., Plaxton, C. G., & Rajaraman, R. Analysis of a local search heuristic for facility location problems. *Journal of Algorithms*, *37*(1), (2000). pp. 146-188.

[15] Mahdian, M., Ye, Y., & Zhang, J. Approximation algorithms for metric facility location problems. *SIAM Journal on Computing*, *36*(2), (2006). pp. 411-432.

[16] Resende, M. G., & Werneck, R. F. A hybrid multi start heuristic for the uncapacitated facility location problem. *European Journal of Operational Research*, *174*(1), (2006). pp. 54-68.

[17] Du, D., Lu, R., & Xu, D. A primal-dual approximation algorithm for the facility location problem with submodular penalties. *Algorithmica*, *63*(1-2), (2012). pp. 191-200.

[18] Sinha, P. Observations on some heuristic methods for the capacitated facility location problem. *OPSEARCH*, *49*(1), (2012). pp. 86-93.

[19] Gabor, A. F., & Van Ommeren, J. C. W. An approximation algorithm for a facility location problem with stochastic demands and inventories. *Operations Research Letters*, *34*(3), (2006). pp. 257-263.

[20] Murali, P., Ordóñez, F., & Dessouky, M. M. Facility location under demand uncertainty: Response to a large-scale bio-terror attack. *Socio-Economic Planning Sciences*, *46*(1), (2012). pp. 78-87.

[21] Eiselt, H. A., Gendreau, M., & Laporte, G. Optimal location of facilities on a network with an unreliable node or link. *Information Processing Letters*, *58*(2), (1996). pp. 71-74.

[22] ReVelle, C. S., Williams, J. C., & Boland, J. J. Counterpart models in facility location science and reserve selection science. *Environmental Modeling & Assessment*, *7*(2), (2002). pp. 71-80.

[23] Hendel, G. New rounding and propagation heuristics for mixed integer programming Doctoral dissertation, Bachelor's thesis, *Technische Universität* Berlin (2011).