SEMI-RADICALS OF SUB MODULES IN MODULES

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Abstract: Let \( R \) be a commutative ring and \( M \) be a unitary \( R \)-module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula (s.t.s.r.f) and present the existence of such a module.

Keywords: Prime sub module, semiprime sub module, radical and semi-radical of a module, modules satisfying the semi-radical formula.

1. Introduction

In this paper all the rings are commutative with identity and all the modules are unitary. Let \( R \) be a ring and \( M \) be an \( R \)-module. If \( N \) is a submodule of \( M \) we use the notation \( N \leq M \). If the submodule \( N \) is generated by a subset \( S \) of \( M \), we write \( N = \langle S \rangle \).

If \( N \) and \( K \) are submodules of \( M \), then the set \( \{ r \in R \mid rK \subseteq N \} \) is denoted by \( (N : K) \) or simply by \( (N : K) \) which is clearly an ideal of \( R \). If \( I \) is an ideal of the ring \( R \), we write \( I \leq R \). In Section 2 we define prime and primary sub modules of an \( R \)-module \( M \) and in Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if \( N \) is a semiprime submodule of an \( R \)-module \( M \), then \( (N : M) \) is a semiprime ideal of \( R \) but not conversely in general. In Lemma 2.8 we prove that the converse is also true if \( M \) is a multiplication module. In Section 3 we define radical of an \( R \)-module \( M \) and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule of a module and also modules satisfying the semi-radical formula which is abbreviated as (s.t.s.r.f) and in Theorem 3.9 we show that such a module does exist.

Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a \( P \)-semiprime submodule we consider some of its properties.

2. Some Elementary Results

We begin this section with the following definitions:

Definition 2.1. Let \( N \) be a proper submodule of an \( R \)-module \( M \).

(a) \( N \) is called a prime submodule of \( M \) if for any \( r \in R \) and \( m \in M \), \( rm \in N \) implies that \( m \in N \) or \( r \in (N : M) \).

(b) \( N \) is called a primary submodule of \( M \) if for any \( r \in R \) and \( m \in M \), \( rm \in N \) implies that \( m \in N \) or \( r^n \in (N : M) \) for some positive integer \( n \).

In (a) it can easily be shown that \( P = (N : M) \) is a prime ideal of \( R \) and we say that \( N \) is \( P \)-prime.

We recall that if \( I \) is an ideal of a ring \( R \), then the radical of \( I \), denoted by \( \sqrt{I} \), is defined as the intersection of all prime ideals containing \( I \). Alternatively, we define the radical of \( I \) as:

\[ \sqrt{I} = \{ r \in R \mid r^n \in I \text{ for some positive integer } n \} \]

Also if \( N \) is a primary submodule of an \( R \)-module \( M \), then \( (N : M) \) is a primary ideal of \( R \) and \( P = \sqrt{(N : M)} \) is a prime ideal. We describe this situation by saying that \( N \) is \( P \)-primary.

Lemma 2.2. Let \( N \) be a proper submodule of an \( R \)-module \( M \).

(i) \( N \) is a prime submodule of \( M \) if and only if \( ID \subseteq N \) (with \( I \) an ideal of \( R \) and \( D \) a submodule of \( M \)) implies that \( D \subseteq N \) or \( I \subseteq (N : M) \).

(ii) \( N \) is a primary submodule of \( M \) if and only if for every finitely generated ideal \( I \) of \( R \) and any submodule \( D \) of \( M \), \( ID \subseteq N \) implies that \( D \subseteq N \) or \( I^n \subseteq (N : M) \) for some positive integer \( n \).

(iii) Let \( P \) be a prime ideal of \( R \), than \( N \) is a \( P \)-primary submodule of \( M \) if and only if (a)
Proposition 2.4. Let $M$ be an $R$-module.

(i) If $N$ is a prime submodule of $M$, then $N$ is semiprime.

(ii) If $N$ is a semiprime submodule of $M$, then $(N:M)$ is semiprime ideal of $R$.

Proof. (i) Let $I \subseteq R$ and $D \subseteq M$ be such that $ID \subseteq N$ and $D \not\subseteq N$. So there exists an element $x \in D \setminus N$. Let $r$ be an element of $I$. Then $rx \in N$ and hence $r \in (N:M)$. Therefore $I \subseteq (N:M)$.

(ii) Let $r \in R$, $a \in M$ be such that $ra \in N$ and $a \notin N$. By taking $I = (r)$ and $D = Ra$ we see that $ID \subseteq N$. But $D \not\subseteq N$ and hence $I \subseteq (N:M)$, which implies that $r \in (N:M)$. Therefore $N$ is a prime submodule of $M$.

(iii) If $N$ is not a subset of $M$, then there exists a positive integer $n$ such that $I^n \subseteq (N:M)$. This implies that $r^n \in (N:M)$ and hence $N$ is a primary submodule of $M$.

Definition 2.3. A proper submodule $N$ of an $R$-module $M$ is said to be semiprime in $M$, if for every ideal $I$ of $R$ and every submodule $K$ of $M$, $I^2K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring $R$ is an $R$-module by itself, a proper ideal $I$ of $R$ is semiprime if for every ideals $J$ and $K$ of $R$, $J^2K \subseteq I$ implies that $JK \subseteq I$.

Example 2.5. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $B = \langle (9,0) \rangle$. Then it is clear that $(B:M) = \langle 0 \rangle$. Since $Z$ is an integral domain, $(B:M) = \langle 0 \rangle$ is a prime ideal and hence a semiprime ideal of $Z$. But $B$ is not a semiprime submodule of $M$; because if we take $I = \langle 3 \rangle$ and $K = \langle 2,0 \rangle$, then:

$I^2K = \langle (18q,0) \mid q \in \mathbb{Z} \rangle$ (1)

But:

$IK = \langle (6q,0) \mid q \in \mathbb{Z} \rangle$ (2)

is not a subset of $B$.

It is clear that if $N$ is a semiprime submodule of an $R$-module $M$ and $I \subseteq R$, $K \subseteq M$ be such that $I^nK \subseteq N$ for some positive integer $n$, then $IK \subseteq N$.

Theorem 2.6. Let $N$ be a proper submodule of an $R$-module $M$. Then the following statements are equivalent:

(i) $N$ is semiprime.

(ii) Whenever $r'm \in N$ for some $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$, then $r'm \in N$. 
Proof. (i) $\Rightarrow$ (ii). Let $r'm \in N$ where $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$. Taking $I = (r)$ and $K = (m)$ we have $I^tK \subseteq N$ and so $IK \subseteq N$ which implies that $rm \in N$.

(ii) $\Rightarrow$ (i). Let $I \leq R$ and $K \leq M$ be such that $I^2K \subseteq N$. Consider the set:

$$S = \{ra \mid r \in I, a \in K\} \quad (3)$$

Then for every $r \in I, a \in K$ we have $r^2a \in I^2K \subseteq N$ and hence $ra \in N$. This implies that $S \subseteq N$ and since $IK$ is the submodule of $M$ generated by $S$, we must have $IK \subseteq N$. Therefore $N$ is semiprime.

Definition 2.7. An $R$-module $M$ is said to be a multiplication module if for each submodule $N$ of $M$, $N = IM$ for some ideal $I$ of $R$.

It can be easily shown that, an $R$-module $M$ is a multiplication module if and only if $N = (N : M)M$ for every submodule $N$ of $M$.

Lemma 2.8. Let $M$ be a multiplication $R$-module. Then a submodule $N$ of $M$ is semiprime if and only if $(N : M)$ is a semiprime ideal of $R$.

Proof. $\Rightarrow$: This is clear from Proposition 2.4 (ii).

$\Leftarrow$: Let $I \leq R$, $K \leq M$ be such that $I^2K \subseteq N$. Hence:

$$(I^2K : M) \subseteq (N : M).$$

It can be shown that:

$$I^2K : M \subseteq (I^2K : M)$$

and so we obtain:

$$I^2K : M \subseteq (N : M).$$

But $(N : M)$ is a semiprime ideal of $R$ and hence $I(K : M) \subseteq (N : M)$. Thus we conclude that:

$$I(K : M)M \subseteq (N : M)M,$$

and using the fact that $M$ is a multiplication $R$-module we have $IK \subseteq N$. Therefore $N$ is a semiprime submodule of $M$. The following lemma shows that the same situation, as above, holds for prime and primary sub modules.

Lemma 2.9. Let $M$ be a multiplication $R$-module. Then:

(a) A submodule $N$ of $M$ is prime if and only if $(N : M)$ is a prime ideal of $R$.

(b) A submodule $N$ of $M$ is primary if and only if $(N : M)$ is a primary ideal of $R$.

Proof. (a) $\Rightarrow$ : Clear.

$\Leftarrow$: Let $I \leq R$, $D \leq M$ be such that $ID \subseteq N$, then $(ID : M) \subseteq (N : M)$. But $I(D : M) \subseteq (ID : M)$ and so $I(D : M) \subseteq (N : M)$. Since $(N : M)$ is a prime ideal of $R$ we have $I \subseteq (N : M)$ or $(D : M) \subseteq (N : M)$. Suppose that $I \not\subseteq (N : M)$. Then $(D : M) \subseteq (N : M)$ and from this we have $(D : M)M \subseteq (N : M)M$, that is, $D \subseteq N$. Hence $N$ is a prime submodule of $M$ by Lemma 2.2 (i).

(b) $\Rightarrow$ : Clear.

$\Leftarrow$: Let $(N : M)$ be a primary submodule of $R$. Let $I$ be a finitely generated ideal of $R$ and $D$ be a submodule of $M$ and let $ID \subseteq N$. Suppose that for any positive integer $n$, $I^n \not\subseteq (N : M)$. We see that $ID \subseteq N$ implies $(ID : M)D \subseteq (N : M)$ and hence $I(D : M) \subseteq (N : M)$. But $I^nD \subseteq (N : M)$ for any positive integer $n$, so $(D : M) \subseteq (N : M)$, because $(N : M)$ is a primary submodule of $M$. Hence $(D : M)M \subseteq (N : M)M$, that is, $D \subseteq N$. So $N$ is a primary submodule of $M$, by Lemma 2.2 (ii), the proof is now complete.

Proposition 2.10. Let $\{P_i\}_{i \in I}$ be a non-empty family of semiprime submodules of an $R$-module $M$. Then $P = \bigcap_{i \in I} P_i$ is a semiprime submodule of $M$.

Proof. Let $I \leq R$, $K \leq M$ be such that $I^2K \subseteq P = \bigcap_{i \in I} P_i$. Then $I^2k \subseteq P_i$ for every $i \in I$, and since $P_i$ is semiprime we have $Ik \subseteq P_i$. Hence $IK \subseteq \bigcap_{i \in I} P_i = P$ and $P$ is semiprime. Next we let $T = \bigcap_{i \in I} P_i$. The fact that $\{P_i\}_{i \in I}$ is totally ordered by inclusion makes it clear that $T$ is a submodule of $M$. Let $I \leq R$ and $K \leq M$ be such that $I^2K \subseteq T$. Consider the set:

$$S = \{rk \mid r \in R, k \in K\} \quad (8)$$

Then $S$ is a generating set for the submodule $IK$. If $r \in I, k \in K$ then $r^2k \in I^2K \subseteq T$ and so for some $i \in I, r^2k \in P_i$. Since $P_i$ is semiprime this implies that $rk \in P_i$. It follows that $S \subseteq T$ and hence $IK = \langle S \rangle \subseteq T$. Therefore $T$ is also a semiprime submodule of $M$.

Remark. Some authors define a semiprime submodule as an intersection of prime sub modules. But by our
3. Radicals and Semi-Radicals

Let $M$ be an $R-$module and $N$ a submodule of $M$. If there exists a prime submodule of $M$ which contain $N$, then the intersection of all prime sub modules containing $N$, is called the $M-$radical of $M$ and is denoted by $\text{rad}_M N$, or simply by $\text{rad}N$. If there is no semiprime submodule containing $N$, then we define the $N-$radical of $M$ to be $N$. If there exists a semiprime submodule of $M$, then we define the $N-$radical of $M$ to be the intersection of all prime sub modules containing $N$. The following corollary can be considered as an answer.

**Definition 3.4.** (1) A semiprime submodule $P$ of an $R-$module $M$ is called a minimal semiprime submodule if $N \subseteq P$ and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of $0 = \langle 0 \rangle$ is called a minimal semiprime submodule of $M$.

**Theorem 3.5.** Let $M$ be an $R-$module. If a submodule $N$ of $M$ contains a semiprime submodule $P$, then $P$ contains a minimal semiprime submodule of $N$.

**Proof.** It is similar to the proof of [5, Theorem 4, P.84].

**Proposition 3.6.** Every proper submodule of a finitely generated $R-$module $M$ contains at least one minimal semiprime submodule of $M$.

**Proof.** Let $N$ be a proper submodule of $M$, then by Proposition 3.3, $N$ contains a semiprime submodule of $M$.

**Corollary 3.7.** Every semiprime submodule of an $R-$module $M$ contains at least one minimal semiprime submodule of $M$.

**Proof.** Let $P$ be a semiprime submodule of $M$ and take $N = \langle 0 \rangle$ in the Theorem 3.5. Then $P$ contains a minimal semiprime submodule of $\langle 0 \rangle$, and so a minimal semiprime submodule of $M$.

**Definition 3.8.** Let $M$ be an $R-$module and $N \subseteq M$. If there exists a semiprime submodule of $M$ which contains $N$, then the intersection of all semiprime submodules containing $N$ is called the semi-radical of $N$ and is denoted by $S-\text{rad}_M N$, or simply by $S-\text{rad}N$. If there is no semiprime submodule containing $N$, then we define...
\[ S - \text{rad} N = M \text{, in particular } S - \text{rad} M = M \text{. We call } S - \text{rad}\{0\} \text{ the semiprime radical of } M \text{.} \]

If \( N \leq M \text{, then the envelope of } N \text{, denoted by } E(N) \text{, is defined as:} \]

\[ E(N) = \left\{ x \in M \mid x = ra \text{ for some } r \in R, a \in M \text{ and } r^n a \in N \text{ for some } n \in \mathbb{Z^+} \right\} \tag{10} \]

We say that \( M \) satisfies the semi-radical formula, \( M \) (s.t.s.r.f) if for any \( N \leq M \text{, the semi-radical of } N \text{ is equal to the submodule generated by its envelope, that is, } S - \text{rad} N = \{E(N)\} \). We already know that \( \langle E(N) \rangle \subseteq \text{rad} N \text{, by [4, P.1815]. Now let } x \in E(N) \text{ and } P \text{ be a semiprime submodule of } M \text{ containing } N \text{. Then } x = ra \text{ for some } r \in R, a \in M \text{ and for positive integer } n, r^n a \in N \text{. But } r^n a \in P \text{ and since } P \text{ is semiprime we have } ra \in P \text{. Hence } E(N) \subseteq P \text{. Hence we conclude that } E(N) \subseteq \text{rad} N \text{.} \]

On the other hand, since every prime submodule of \( M \) is clearly semiprime, we have \( S - \text{rad} N \subseteq \text{rad} N \). We see that:

\[ \langle E(N) \rangle \subseteq S - \text{rad} N \subseteq \text{rad} N \tag{11} \]

Now we present an \( R \)-module which satisfies the semi-radical formula.

**Theorem 3.9.** Let \( M \) be a finitely generated multiplication \( R \)-module. Then \( M \) satisfies the semi-radical formula.

**Proof.** Let \( N \leq M \text{, then by [4, Theorem 4.4], we have } \langle E(N) \rangle : M = \langle \text{rad} N : M \rangle \text{.} \]

Hence \( \langle E(N) \rangle : M = \langle \text{rad} N : M \rangle M \text{ and since } M \text{ is a multiplication } R \text{-module, } \langle E(N) \rangle = \text{rad} N \text{.} \]

Next from (*) we have:

\[ \langle E(N) \rangle : M \subseteq (S - \text{rad} N : M) M \subseteq \langle \text{rad} N : M \rangle M \tag{12} \]

that is,

\[ \langle E(N) \rangle \subseteq S - \text{rad} N \subseteq \text{rad} N \text{.} \tag{13} \]

**Remark.** Under the conditions of Theorem 3.9, we see that for any submodule \( N \neq M \text{ of } M \text{ we always have } \text{Rad} N = S - \text{Rad} N \text{.} \]

**Proposition 3.10.** Let \( M \) be a finitely generated \( R \)-module. Then the semi-radical of a proper submodule \( N \) of \( M \) is the intersection of its minimal semiprime sub modules.

**Proof.** This is clear by using Theorem 3.5 and Proposition 3.6.

For the rest of this section we state and prove some properties of semi-radical of sub modules.

**Theorem 3.11.** Let \( B \) and \( C \) be sub modules of an \( R \)-module \( M \). Then,

1. \( B \subseteq S - \text{rad} B \).
2. \( S - \text{rad}(S - \text{rad} B) = S - \text{rad} B \).
3. \( S - \text{rad}(B \cap C) \subseteq S - \text{rad} B \cap S - \text{rad} C \), and we have the equality when for every semiprime submodule \( P \), \( B \cap C \subseteq P \) implies that \( B \subseteq \text{Por} C \subseteq P \).
4. \( S - \text{rad}(B + C) = S - \text{rad}(S - \text{rad} B + S - \text{rad} C) \).
5. \( \sqrt{B : M} \subseteq (S - \text{rad} B : M) \).
6. If \( M \) is finitely generated, then \( S - \text{rad} B = M \) if and only if \( B \subseteq M \).
7. If \( M \) is finitely generated, then \( B + C = M \) if and only if \( S - \text{rad} B + S - \text{rad} C = M \).
8. \( S - \text{rad} M = S - \text{rad} \sqrt{IM} \) for every ideal \( I \) of \( R \).

**Proof.** (1) clear.

(2) Since \( S - \text{Rad} B \) is semiprime by Proposition 2.10, we have:

\[ S - \text{Rad} (S - \text{Rad} B) = S - \text{Rad} B \text{.} \tag{14} \]

(3) Let \( P \) be a semiprime submodule of \( M \) such that \( B \subseteq P \text{, so } B \cap C \subseteq P \text{ and hence } S - \text{rad} (B \cap C) \subseteq S - \text{rad} B \). By a similar argument we have \( S - \text{rad} (B \cap C) \subseteq S - \text{rad} C \). Now let \( P \) be a semiprime submodule of \( M \) such that \( B \cap C \subseteq P \) and assume that \( B \subseteq P \). Then \( S - \text{rad} B \subseteq P \text{ and so } S - \text{rad} B \cap S - \text{rad} C \subseteq P \text{. Since } P \text{ is arbitrary this implies that } S - \text{rad} B \cap S - \text{rad} C \subseteq S - \text{rad} (B \cap C) \text{ and hence we have the equality.} \]

(4) Let \( P \) be a semiprime submodule of \( M \) such that \( S - \text{rad} B + S - \text{rad} C \subseteq P \text{.} \text{ So } S - \text{rad} B \subseteq P \text{ and } S - \text{rad} C \subseteq P \text{. Hence } B \subseteq C \text{ and } C \subseteq P \text{ which implies } B + C \subseteq P \text{. Therefore } S - \text{rad} (B + C) \subseteq P \text{. But } P \text{ is chosen arbitrary, so:} \]

\[ S - \text{rad} (B + C) \subseteq S - \text{rad} (S - \text{rad} B + S - \text{rad} C) \text{.} \tag{15} \]

Now suppose that \( P \) be a semiprime submodule such that \( B + C \subseteq P \text{.} \text{ So } B \subseteq P \text{, and } C \subseteq P \text{. Hence } S - \text{rad} B \subseteq P \text{ and } S - \text{rad} C \subseteq P \text{ and therefore } S - \text{rad} B + S - \text{rad} C \subseteq P \text{. But } S - \text{rad} (S - \text{rad} B + S - \text{rad} C) \subseteq P \text{ and we conclude that:} \]

\[ S - \text{rad} (S - \text{rad} B + S - \text{rad} C) \subseteq S - \text{rad} (B + C) \text{.} \tag{16} \]
(5) If \( S - \text{rad} B = M \), then we have the result. So let \( P \) be a semiprime submodule of \( M \) such that \( B \subseteq P \). So \( (B : M) \subseteq (P : M) \). We know that \((P : M)\) is a semiprime ideal of \( R \) and we have shown that \( (P : M) = (P : M) \). Hence \( \sqrt{(B : M)} \subseteq (P : M) \) implies that: 
\( (B : M) \subseteq (P : M) \) 
and since \( P \) can be any semiprime submodule of \( M \), that is, \( L K M \subseteq \). But 
\( (P : M) \) implies that \( K M P \subseteq \). Therefore \( L K MP \subseteq \subseteq \).

(6) If \( B = M \), then \( S - \text{rad} B = S - \text{rad} M = M \). Conversely, let \( S - \text{rad} B = M \), but \( B \neq M \). Since \( M \) is finitely generated, it contains a prime and so a semiprime submodule \( P \) containing \( B \), by Corollary after Proposition 4 of [3]. Hence \( S - \text{rad} B \neq M \), a contradiction.

(7) Using parts (4) and (6) we have:
\[ (S - \text{rad} B + S - \text{rad} C) = M \]
 iff \( S - \text{rad} B = M \).

(8) If \( M \) has no semiprime submodule containing \( IM \), then \( S - \text{rad} IM = M \) and we have:
\[ I \subseteq \sqrt{I} \Rightarrow IM \subseteq \sqrt{IM} \Rightarrow S - \text{rad} IM \subseteq S - \text{rad} \sqrt{IM} \]
\[ \Rightarrow M \subseteq S - \text{rad} \sqrt{IM} \Rightarrow M = S - \text{rad} \sqrt{IM} : \] (17)
\[ = S - \text{rad} IM \].

Now let \( P \) be a semiprime submodule of \( M \) such that \( IM \subseteq P \), so \( I \subseteq (IM : M) \subseteq (P : M) \) and since \((P : M)\) is semiprime \( \sqrt{I} \subseteq (P : M) \). So \( \sqrt{IM} \subseteq P \) and hence \( S - \text{rad} IM \subseteq P \). Since \( P \) is arbitrary we have:
\( S - \text{rad} IM \subseteq S - \text{rad} M \).

Therefore \( S - \text{rad} IM = S - \text{rad} IM \). The proof is now complete.

**Corollary 3.12.** Let \( M \) be an \( R \)-module and \( I \) an ideal of \( R \). Then \( S - \text{rad} I^n M = S - \text{rad} M \) for every positive integer \( n \).

**Proof.** We know that \( \sqrt{I^n} = \sqrt{I} \) so by part (8) of Theorem 3.11:
\[ S - \text{rad} I^n M = S - \text{rad} \sqrt{I^n} M = \]
\[ S - \text{rad} \sqrt{IM} = S - \text{rad} IM \].

**Proposition 3.13.** Let \( Q \) be a \( P \)-primary submodule of an \( R \)-module \( A \). Then \( S - \text{rad} Q = S - \text{rad} (Q + PA) \).

**Proof.** We have \( Q \subseteq Q + PA \), so \( S - \text{rad} Q \subseteq S - \text{rad} (Q + PA) \). Let \( S - \text{rad} Q = \cap_i P_i \), where any \( P_i \) is a semiprime submodule of \( A \) containing \( Q \). We see that

\[ P = \sqrt{(Q : A)} \subseteq \sqrt{(P_i : A)} = (P_i : A) \] (19)
implies \( PA \subseteq P \). So \( (Q + PA) \subseteq P \), for every \( i \in I \) and hence \( S - \text{rad} (Q + PA) \subseteq P \). Therefore \( S - \text{rad} (Q + PA) \subseteq \cap_i P_i \), so \( S - \text{rad} Q = S - \text{rad} (Q + PA) \).

**Definition 3.14.** Let \( N \) be a semiprime submodule of an \( R \)-module \( M \), and let \( P = \sqrt{(N : M)} = (N : M) \). We call \( N \) a \( P \)-semiprime submodule of \( M \), if \( P \) is prime ideal of \( R \).

**Lemma 3.15.** Let \( M \) be a finitely generated \( R \)-module and let \( K \) be a maximal ideal of \( R \). If \( Q \) is a \( K \)-primary submodule of \( M \), then \( S - \text{rad} Q \) is a \( K \)-semiprime submodule.

**Proof.** By Theorem 3.11, part (5), we have \( K = \sqrt{(Q : M)} \subseteq (S - \text{rad} Q : M) \).

But \( K \) is a maximal ideal of \( R \), so \( (S - \text{rad} Q : M) = R \) or \( (S - \text{rad} Q : M) = K \). If \( (S - \text{rad} Q : M) = R \) then \( S - \text{rad} Q = M \) and by Theorem 3.11, part (6) we have \( Q = M \) which is a contradiction since \( Q \) is primary. Hence \( (S - \text{rad} Q : M) = K \) and since \( S - \text{rad} Q \) is an intersection of semiprime submodules containing \( Q \) it is semiprime and in fact \( K \)-semiprime.

**Proposition 3.16.** Let \( N_1, N_2, \ldots, N_t \), be \( P \)-semiprime sub modules of an \( R \)-module \( M \). Then \( N = N_1 \cap N_2 \cap \ldots \cap N_t \) is also \( P \)-semiprime.

**Proof.** By Proposition 2.10, \( N \) is semiprime and we have:
\[ (N : M) = (N_1 \cap N_2 \cap \ldots \cap N_t : M) = \]
\[ (N_1 : M) \cap (N_2 : M) \cap \ldots \cap (N_t : M) \]
\[ = P \cap P \cap \ldots \cap P = P \]. Therefore \( N \) is \( P \)-semiprime.

**Lemma 3.17.** Let \( M \) be a multiplication \( R \)-module and \( L, N \) be sub modules of \( M \). Also let \( K \) be a prime ideal of \( R \) and \( P \) be a \( K \)-semiprime submodule of \( M \) such that \( N \cap L \subseteq P \). If \( (N : M) \subseteq K \) then \( L \subseteq P \).

**Proof.** We have \( N \cup L \subseteq P \Rightarrow (N \cap L : M) \subseteq (P : M) = K \Rightarrow (N : M) \cap (L : M) \subseteq K \).

and since \( K \) is a prime ideal of \( R \), \( (N : M) \subseteq K \) or \( (L : M) \subseteq K \). Since \( (N : M) \subseteq K \) we find that \( (L : M) \subseteq K \). From this we conclude that \( (L : M) \subseteq KM \), that is, \( L \subseteq KM \). But \( (P : M) = K \) implies that \( KM \subseteq P \). Therefore \( L \subseteq KM \subseteq P \).
4. Conclusion
In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

References


