LEAST – SQUARES METHOD FOR ESTIMATING DIFFUSION COEFFICIENT

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Abstract: *Determination of the diffusion coefficient on the base of solution of a linear inverse problem of the parameter estimation using the Least-square method is presented in this research. For this propose a set of temperature measurements at a single sensor location inside the heat conducting body was considered. The corresponding direct problem was then solved by the application of the heat fundamental solution.*

Key Words: *inverse heat conduction problems, direct problems, diffusion coefficient*

1. Introduction

Inverse problems are encountered in various branches of science and engineering. Mechanical, aerospace and chemical engineers, mathematicians, astrophysicists, statisticians and specialist of many other disciplines are all interested in inverse problems, each with different applications in mind. Many theoretical and experimental methods for measuring the thermal properties are developed in the literature[1]. Most of these methods only enable measurements at constant temperature. This paper deals with the method for determining space – dependent diffusion coefficient, which is based on the solution of the inverse problem of the identification of thermal parameters of a heat conducting body. The problem of parameter identification is solved by linear Least – Square method. The solution of this inverse problem requires a finite set of temperature measurements taken inside the body and assume that the diffusion coefficient belong to a set of polynomial. Significant contributions made in the field of inverse heat conduction problems are published by Beck [1] and of parameter identification by Cannon [2]. In this paper, we use fundamental solution for corresponding direct problem then by Least - Square, we estimate the diffusion coefficient.

2. Formulation of the Problem

In a one – dimensional formulation with moisture moving in the direction normal to a specimen of slice of wood of thickness L, the diffusion equation can be written as [3, 4].

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(D(x)\frac{\partial T(x,t)}{\partial x}), \qquad 0 < x < L, t > 0, \qquad (1)$$

Where T is moisture content, t is time, $D(x)$ is diffusion coefficient, and x is space coordinate measured from the center of the specimen.
The initial condition be:

$$T(x,0) = T_0, \qquad 0 \le x \le L, \qquad (2)$$

and the boundary conditions be:

$$D\frac{\partial T}{\partial x}(0,t) = S(T_e - T(0,t)), \qquad 0 \le t, \qquad (3)$$

$$\frac{\partial T}{\partial x}(L,t) = 0, \qquad 0 \le t, \qquad (4)$$

where T_0 is a constant moisture content in the specimen, S is surface emission coefficient and T_e is equilibrium moisture content corresponding to the vapor pressure in the environment remote from the surface of specimen. For the inverse problem, the diffusion coefficient and moisture content are unknown. To determine $D(x)$ and $T(x,t)$ from boundary and initial data we need addition temperature measurements $T^M(t)$ at an arbitrary spatial position $x = x^* \in (0, L)$, which are recorded in time, namely:

$$T(x^*,t) = T^M(t_j) = T_j^M, \qquad j = 1 \; to \; M, \qquad (5)$$

where M is total temperature measurement.

3. Solution of the Direct Problem

The first step of inverse technique is to develop the corresponding direct solution for the problem (1)-(4).
The mathematical formulation of the direct problem is the same as that given by problem (1)-(4), in the case of known function $D(x)$ on their domain.

If $D(x) = D_0$ (D_0 is constant) . Then the problem (1)-(4) has the solution [2, 5].

$$T(x,t) = W(x,t) - 2\int_0^t \beta(x,t-\tau)S(T_e - T(0,D\tau)) \; d\tau, \qquad (6)$$
Where:

$$W(x,t) = \int_0^L \{\beta(x-\xi,t) + \beta(x+\xi,t)\}T_o(\xi)\,d\xi$$

$$\beta(x,t) = \sum_{p=-\infty}^{+\infty} K(x+2p,t),$$

and the fundamental solution $K(x,t)$ is defined by:

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\{\frac{-x^2}{4t}\},$$

4. The Inverse Technique

For the inverse problem, the diffusion coefficient $D(x)$ and $T(x,t)$ are unknown but T_0, S, T_e are known. In addition temperature readings $T^M(t)$ taken at arbitrary spatial position $x = x^* \in (0,L)$ are considered available. The $D(x)$ is obtained as the solution of the minimization problem of the Least-Squares norm $\|T^D - T^M\|^2$, where $T^D = T(x,t,D)$ is solution of (1)-(4) at $x = x^*$.

In order to achieve a unique solution problem, the unknown function $D(x)$ is parameterized by assuming that the $D(x)$ is taken as a polynomial [4,-6],

$$D(x) = D_\circ + D_1(x-x_\circ) + D_2(x-x_\circ)(x-x_1)$$
$$+ \ldots + D_N(x-x_\circ)(x-x_1)\ldots(x-x_{N-1}) \quad (7)$$

where D_i, is the value of $D(x)$, at $x = x_i$ and N, is the total number of space segments. Then for the duration $x \in [x_{i-1}, x]$, the diffusion coefficient function $D(x)$, can be represented by $D(x) = D_i$, is a parameter to be estimated based on temperature measurements during $x \in [x_{i-1}, x]$. We may rewrite the equation (6), for $x \in (x_{i-1}, x_i)$ as follows[7]:

$$T(x,t,D_i) = W(x,t)$$

$$-2\int_0^t \beta(x,t-\tau)S(T_e - T(0,D_i\tau))d\tau, \quad t>0. \quad (8)$$

At $x = x^*$, $t = t_j$, we obtain:

$$T(x^*,t_j,D_i) = W(x,t) - 2\int_0^{t_j}\beta(x^*,t_j-\tau)S(T_e-T(0,D_i\tau))$$
$$d\tau. \quad (9)$$

dure for estimation D_i, with temperature measurements $T_1^M, T_2^M, \ldots, T_M^M$, minimization:

$$F(D_i) = \sum_{j=1}^{M}[T_j^M - T(x^*,t_j,D_i)]^2, \quad i = 1\ to\ N \quad (10)$$

Equation (10) is minimized by differentiating with respect to unknown parameter D_i, and then setting the resuting expression equal to zero, i.e.,

$$\frac{\partial F(D_i)}{\partial D_i} = -2\sum_{j=1}^{M}\frac{\partial T}{\partial D_i}[T_j^M - T(x^*,t_j,D_i)] = 0, \quad (11)$$
$$i = 1\ to\ N$$

The system of equations (11) are nonlinear, an iterative technique is necessary for its solution. The Newton-Raphson method, is used to solve the non-linear Least-Squares equations (11) by iteration.

Consequently, substituting D_i in (7) we can approximate $D(x)$ for $0 < x < L$, and by solved the direct problem, we problem, we approximate $T(x,t)$ for $0 < x < L$, $t > 0$.

5. Numerical Results

To examine the accuracy of predictions by the inverse technique, we consider the inverse problem of estimating the unknown diffusion coefficient $D(x)$ in the problem defined by [8]:

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(D(x)\frac{\partial T(x,t)}{\partial x}), \quad 0 < x < 14,\ 0 < t < 3.6,$$

$$T(x,0) = 35.9, \quad 0 \le x \le 14,$$

$$D\frac{\partial T}{\partial x}(0,t) = 5.5 - T(0,t), \quad 0 \le t \le 3.6,$$

$$\frac{\partial T}{\partial x}(14,t) = 0, \quad 0 \le t \le 3.6.$$

Temperature readings are considered taken at the $x = x^* = 0.5$, with small dimensionless time step $\delta t = 0.04$ over whole time domain $0 \le t \le 3.6$. thus we have $N = 90$ measurements. To simulate the measured temperature containing measurement error Err are introduced to the exact temperature as:

$$T^M = T_{exact} + Err,$$

where the exact temperature T_{exact} is determined from the solution of the direct problem by using the exact value of the diffusion coefficient.
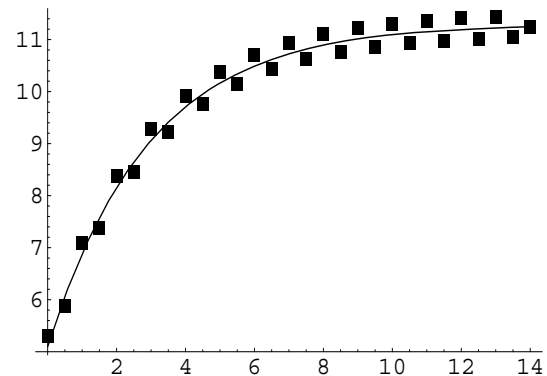

Fig 1. Estimating and exact solution $D(x)$

6. Conclusion

According to the inverse method which has been discussed in this paper, we can conclude that for estimating a two dimensional problem, the proposed model can be used offectively.

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