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On the Bullwhip Effect Measure in Supply Chains with VAR (1) Demand Process

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ABSTRACT

In this paper, a two-echelon supply chain, which includes two products based on the following considerations, has been studied and the bullwhip effect is quantified. Providing a measure for bullwhip effect that enables us to analyze and reduce this phenomenon in supply chains with two products is the basic purpose of this paper. Demand of products is presented by the first order vector autoregressive time series and ordering system is established according to order up to policy. Moreover, lead-time demand forecasting is based on moving average method because this forecasting method is used widely in real world. Based on these assumptions, a general equation for bullwhip effect measure is derived and there is a discussion about non-existence of an explicit expression for bullwhip effect measure according to the present approach on the bullwhip effect measure. However, bullwhip effect equation is presented for some limited cases. Finally, bullwhip effect in a two-product supply chain is analyzed by a numerical example.

1. Introduction

Today, outsourcing has an important role in industrial environments. Therefore, manufacturers are settling in supply chains and hence these supply chains grows up quickly. Raw material suppliers, part manufacturers, final product assemblers, distributors, retailers, and final customers are various sectors of supply chains. Coordination among channel members results in effective supply chain. Nowadays, the fundamental challenge is to reach coordination in spite of multiple ownerships and product variety. Demand amplification is a major obstacle to achieve coordination and creation harmony within different stages of supply chains. Many companies have observed increasing fluctuation in orders while moving up from downstream site to upstream site. The result is a loss of supply chain profitability. The first recorded documentation of this status is due to Forrester [1]. He used industrial dynamics approach to show amplification of demand variability among supply chain. After that time many researchers such as Goodman [2], Blinder [3], [4], Blanchard [5], Kahn [6], Baganha and Cohen [7], Metters [8] continued investigation about ordering variation. Sterman [9] developed Beer game at MIT. He proposed it as an evidence for existence demand amplification in supply chains. Now Beer game is an important source in teaching inventory management at the universities. Procter and Gamble (P&G) called demand fluctuation phenomenon as Bullwhip Effect. Lee et al. introduced five main causes of this phenomenon i.e. demand forecast updating, order batching, price fluctuation, rationing and non-zero lead-time [10]. Understanding these causes of the bullwhip effect can be useful for managers to find suitable solutions for haltering and controlling it.

The main aim of the earlier research on the bullwhip effect had focused on proving its existence, its causes, and remedies. In the last decade, papers have provided issues for modeling and quantifying the bullwhip effect and its solutions. In addition, investigations on the role of the forecasting method, ordering policy, information sharing, lot sizing rules, and so on are conducted in different statues. Chen et al. [11] quantified bullwhip effect in a simple supply chain and derived a lower

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bound for it. Dejonckheere et al. [12] proposed a control theory approach for measuring bullwhip effect and suggested a new general replenishment rule that can reduce variance amplification significantly. Disney and Towill [13] introduced an ordering policy that results in taming bullwhip effect. Zhang [14] considered three forecasting methods for a simple inventory control system. The results showed that forecasting methods affect bullwhip effect. He also presented three measures for bullwhip effect based on three forecasting methods. Kim et al. [15] investigated stochastic instead of deterministic lead-time and investigated role of information sharing in the bullwhip effect. Chandra and Grabis [16] measured the bullwhip effect when order size is calculatied according to multiple step forecasts using autoregressive models. Luong [17] investigated the effects of the autoregressive coefficient and lead-time on the bullwhip effect when the MMSE forecasting method is used. Luong and Phien [18] research was based on order of autoregressive demand pattern. They got an interesting result and found that the bullwhip effect is not always an increasing function of lead-time. They showed that in high order of demand pattern, the bullwhip effect could be reduced when lead-time increases. Makui and Madadi [19] utilized the Lyapunov exponent and provided a measure for bullwhip effect. They presented useful results on the behavior of the bullwhip effect by investigating the mathematical relationships. Gaalman and Disney [20] investigated the behaviour of the proportional order up to policy for ARMA (2,2) demand with arbitrary lead-times. They proposed a replenishment rule that accounts for the characteristics of the demand in a superior manner in order to compensate for possible weaknesses of the proportional OUT policy. Jaksic and Rusjan [21] demonstrated that certain replenishment policies can be inducers of the bullwhip effect and suggested that through appropriate selection and use of certain replenishment rules, the bullwhip effect can be avoided. Su and Wong [22] studied a stochastic dynamic lost-sizing problem under the bullwhip effect. They proposed a solution of two-stage ant colony optimization (TACO) and added a mutation operation in the second-stage ACO.

Although many studies are curried out on the bullwhip effect but more investigations are needed for study of this phenomenon, quantifying it and proposing the solutions in complex supply chains. Herein in this research, the researchers considered a two-echelon supply chain consisting of one retailer and one supplier. This supply chain produces two products in which demand pattern follows the first order vector autoregressive model. Moreover, utilized ordering policy for each product is order up to policy and lead-time demand forecasting is based on moving average method. According to these assumptions, we quantify the bullwhip effect in a two-product supply chain and then we analyze it using a numerical example.

The framework of the paper is as follows. After the introduction and the literature review, a two-echelon supply chain with two products is taken into consideration. In the second section and its demand pattern, ordering policy and forecasting method is explained.

In the third section of the paper, a mathematical equation is derived for the bullwhip effect. It is shown that there is no an explicit expression on the bullwhip effect measure in supply chain with two products based on our assumption about ordering policy and forecasting methods. For better perception, detailed statistical relations are presented in the paper. In section four, analytical conclusions on the bullwhip effect behavior are described, an example is presented, and after that, results of the bullwhip effect measurement are mentioned. In the last section of the paper, conclusions and future studies are suggested.

2. A Two Echelon and Two-Product Supply Chain

In this paper, a two-echelon supply chain consists of one retailer and one supplier. The Retailer encounters market demand and orders it to the supplier according to his/her ordering policy and supplier compliances received orders. Hence, demand information flow is from retailer toward supplier and product flow is from supplier to the retailer. There are two products in the supply chain and so the retailer meets the demand of two products. Each product demand has an effect on the demand of another product. Therefore, we must consider a suitable pattern for demand modeling that includes relationships between products. In part 2.1 we explain the proposed demand model.

2.1. Demand Pattern

In the current paper, considering the relationship between products, first order vector autoregressive process, VAR (1) is taken into consideration for demand modeling. Because of VAR (1) properties, it can be used not only for demand modeling in a two product supply chain but also in multi-product supply chains. In our VAR (1) model, demand of each product affects demand of another one as follows: Demand of each product in every period depends on demand of the same product and demand of another product in the last period. For example, consider supply chain of dairy products in which two products: cheese and butter are produced. In this supply chain, the demand for cheese in period *t* is relevant to the demand for cheese in period *t*-1 as well as demand of butter in period *t-1*. This situation also exists for demand of butter in each period. Suppose that D_t^i is demand of *i* th product in period *t*, hence VAR (1) process for demand of two products can be determined by:

$$\begin{cases} D_t^1 = f_{11}D_{t-1}^1 + f_{12}D_{t-1}^2 + a_t^1 \\ D_t^2 = f_{21}D_{t-1}^1 + f_{22}D_{t-1}^2 + a_t^2 \end{cases} \quad (1)$$

where a_t^i (i=1,2 and t=1,2,…) is forecast error of i th product for period t and is i.i.d. normally distributed with mean zero and variance s_{ii}. Relation between demands of two products is clarified in (1). In order to demand process to be stationary, or

$$\begin{cases} Var(D_t^1) = Var(D_{t-1}^1) = Var(D_{t-2}^1) = ... = g_{11} \\ Var(D_t^2) = Var(D_{t-1}^2) = Var(D_{t-2}^2) = ... = g_{22} \end{cases}$$

The following relationship must hold:

$$\left| \frac{(f_{11}+f_{22}) \pm \sqrt{(f_{11}-f_{22})^2 + 4f_{12}f_{21}}}{2} \right| \mathbf{p} 1$$

It is shown in appendix A that the variance of each product (i.e. g_{11}, g_{22}) can be derived by Eq. (2) and Eq. (3):

$$g_{11} = \frac{\begin{pmatrix} s_{11}\left((1-f_{11}f_{22})(1-f_{22}^2) - f_{12}f_{21}(1+f_{22}^2)\right) + \\ 2f_{12}s_{12}\left(f_{11}(1-f_{22}^2) + f_{12}f_{21}f_{22}\right) + f_{12}^2 s_{22}(1-f_{12}f_{21}+f_{11}f_{22}) \end{pmatrix}}{1+(f_{11}f_{22}-f_{12}f_{21}-1)\left(f_{11}f_{22}+f_{12}f_{21}+f_{11}^2+f_{22}^2\right)+(f_{12}f_{21}-f_{11}f_{22})^3} \quad (2)$$

$$g_{22} = \frac{\begin{pmatrix} f_{21}^2 s_{11}[1+f_{11}f_{22}-f_{12}f_{21}] + 2f_{21}s_{12}[f_{22}(1-f_{11}^2)+f_{11}f_{12}f_{21}] \\ + s_{22}[(1-f_{11}f_{22})(1-f_{11}^2) - f_{12}f_{21}(1+f_{11}^2)] \end{pmatrix}}{1+(f_{11}f_{22}-f_{12}f_{21}-1)\left(f_{11}f_{22}+f_{12}f_{21}+f_{11}^2+f_{22}^2\right)+(f_{12}f_{21}-f_{11}f_{22})^3} \quad (3)$$

In fact Eq. (1) and Eq. (2) reflect market demand variations for each of the two products according to VAR (1) process. These equations are utilized in the next sessions for quantifying the bullwhip effect and are an essential part of it. Moreover, the covariance between two products (i.e. g_{12}) can be determined by Eq. (4) which is needed for later bullwhip effect calculations in section 3.2. It is clear that complexity of the relations is due to existence of two products in the model.

$$g_{12} = \frac{\begin{pmatrix} f_{21}s_{11}[f_{11}(1-f_{22}^2)+f_{12}f_{21}f_{22}] + \\ s_{12}[(1-f_{11}^2)(1-f_{22}^2) - f_{12}^2 f_{21}^2] + f_{12}s_{22}[f_{22}(1-f_{11}^2)+f_{11}f_{12}f_{21}] \end{pmatrix}}{1+(f_{11}f_{22}-f_{12}f_{21}-1)\left(f_{11}f_{22}+f_{12}f_{21}+f_{11}^2+f_{22}^2\right)+(f_{12}f_{21}-f_{11}f_{22})^3} \quad (4)$$

2.2. Ordering Policy

In this research, following previous researchers such as, Chen et al. [11], [23], Zhang [14], Hosoda and Disney [24], Gilbert [25], Luong [17] and Luong and Phien [18], we consider an order up to policy (OUT) for retailer inventory control system. The order up to policy (OUT) is a standard ordering algorithm in many MRP systems [25]. The OUT policy is easy to understand and is often utilized by companies to coordinate orders from suppliers where setup costs may be reasonably ignored [22]. In the OUT system, level of inventory is reviewed periodically and an order is placed to bring inventory position to a predefined level. In considered inventory control system, at the beginning of each period, inventory position is observed and in order to raise the inventory level to S_t, an order Q_t, is placed. After the order is placed, customer demand D_t occurs. This sequence is consistent with Eq. (5):

$$Q_t = S_t - S_{t-1} + D_{t-1} \quad (5)$$

Using base stock policy, order up to level, S_t at the beginning of period t can be determined by Eq. (6):

$$S_t = \hat{D}_t^L + z.\hat{s}_t^L \quad (6)$$

Where $\hat{D}_t^L$ is lead-time demand forecast and $\hat{s}_t^L$ is standard deviation of lead-time demand forecast error. Moreover, z is normal z score and can be determined by normal table based on the favorable service level of the inventory system. Replacing Eq. (6) with Eq. (5) results in Eq. (7) which is order quantity in period t:

$$Q_t = \hat{D}_t^L - \hat{D}_{t-1}^L + z(\hat{s}_t^L - \hat{s}_{t-1}^L) + D_{t-1} \quad (7)$$

We suppose that each of products is ordered independently, so Eq. (7) can be used for both of them separately according to their parameters.

2.3. Forecasting Method

Because of the lead-time between placing the order and receiving the products into stock, we need to forecast demand [20]. In past research for quantifying bullwhip effect, various forecasting methods such as moving average, exponential smoothing, minimum expected mean squares of error and so on were utilized for forecasting of lead time demand and a measure for the bullwhip effect is derived according to each of them. Among the methods, moving average and exponential smoothing have been used widely in the real world in different industrial factories, because of their ease of use, flexibility, and robustness in dealing with non-linear demand processes [26]. In this research, we suppose that the retailer uses moving average forecasting method to forecast lead-time demand. In the moving average forecasting method, the calculation is based on the last p observations. Therefore, by definition we have:

$$\overline{D_t} = \frac{\sum_{j=1}^{p} D_{t-j}}{p} \quad (8)$$

Therefore, lead-time demand estimation can be expressed as follows:

$$\hat{D}_t^L = L.\overline{D_t} \quad (9)$$

Replacing Eq. (8) in Eq. (9) results:

$$\hat{D}_t^L = L\left(\frac{\sum_{j=1}^{p} D_{t-j}}{p}\right) \quad (10)$$

Now, we consider L_i as lead-time and p_i as number of observations in forecasting method for i th product (i=1, 2). Therefore, we can rewrite Eq. (10) to achieve lead-time demand forecast for each product as follows:

$$(\hat{D}_t^{L_i})^i = L_i\left(\frac{\sum_{i=1}^{p_i} D_{t-j}^i}{p_i}\right)$$

The above expression is similar to the Chen et al. study [11] which presented a single product supply chain.

3. Quantifying the Bullwhip Effect

Many of investigations on the bullwhip effect are developed by ratio of order quantity that is ordered to supplier and variance of market demand that is seen by retailer. This definition for the bullwhip effect measurement is due to its nature: amplification of demand while moving from upstream to downstream in supply chains. Therefore, the mentioned ratio that is expressed in Eq. (11) is a reasonable statement for measuring the phenomenon:

$$BE = \frac{Var(Q_t)}{Var(D)} \quad (11)$$

We can provide a measure for the bullwhip effect based on mathematical relationships using the above relationship. Therefore, it is possible to analyze the impression of parameters such as forecasting method parameters, lead-time of demand and autoregressive coefficient on the bullwhip effect measure. Consequently, remedies for the bullwhip effect reduction can be proposed analytically and based on scientific evidence. According to Eq. (11), for quantifying the bullwhip effect, it is sufficient to determine variance of order and variance of demand. However, at first we present a proposition that is useful for determination of variance of orders.

Proposition 1.
Standard deviation of lead-time demand forecast error for each product is constant during periods and does not depend on t. It can be represented by Eq. (12):

$$(\hat{s}_t^L)^2 = (L + \frac{L^2}{p})g + 2\sum_{i=1}^{L-1}(L-i)g(i) + 2(\frac{L}{p})[(\frac{L}{p})(\sum_{i=1}^{p-1}(p-i)g(i)) - \sum_{j=0}^{p-1}\sum_{i=j+1}^{j+L} g(i)] \quad (12)$$

In which g is the covariance of each product and $g(i) = Cov(D_t, D_{t+i})$ In fact, the above proposition implies that $\hat{s}_t^L = \hat{s}_{t-1}^L$.

Proof: See Appendix B.
Proposition 1 shows that standard deviation of lead time demand for each product does not influence the bullwhip effect and only is needed for determination of S_t in order up to policy for each product in every period.
According to proposition 1 result, Eq. (7) can be reduced to Eq. (13) that is used to determine variance of orders. After summarizing order quantity relationship, we have:

$$Q_t = \hat{D}_t^L - \hat{D}_{t-1}^L + D_{t-1} \quad (13)$$

3.1. Variance of Order Quantity

To provide variance of order we should have an explicit expression for order measure. So substituting Eq. (10) in Eq. (13) concludes:

$$Q_t = L[\frac{\sum_{i=1}^{p} D_{t-i}}{p} - \frac{\sum_{i=1}^{p} D_{t-i-1}}{p}] + D_{t-1} =$$
$$L[\frac{\sum_{i=1}^{p} D_{t-i}}{p} - \frac{\sum_{i=1}^{p} D_{t-i-1}}{p}] + D_{t-1} =$$
$$\frac{L}{p}[\sum_{i=1}^{p} D_{t-i} - \sum_{i=1}^{p} D_{t-i-1}] + D_{t-1} =$$
$$\frac{L}{p}[D_{t-1} + D_{t-2} + ... + D_{t-p} - (D_{t-2} + D_{t-3} + ... + D_{t-p-1})] + D_{t-1} =$$
$$\frac{L}{p}[D_{t-1} - D_{t-p-1}] + D_{t-1}$$

Therefore, we have:

$$Q_t = (1 + \frac{L}{p})D_{t-1} - (\frac{L}{p})D_{t-p-1} \quad (14)$$

Equation (14) provides order quantity of a product based on order up to policy when the retailer uses moving average forecasting method for lead-time demand estimation. The above relationship can be used for both products if we replace relevant parameters in Eq. (14) as follows:

$$Q_t^i = (1+\frac{L_i}{p_i})D_{t-1}^i - (\frac{L_i}{p_i})D_{t-p-1}^i \qquad i = 1,2$$

To specify variance of order quantity we have:

$$Var(Q_t) = Var[(1+\frac{L}{p})D_{t-1} - (\frac{L}{p})D_{t-p-1}]$$

$$Var(Q_t) = (1+\frac{L}{p})^2 Var(D_{t-1}) + (\frac{L}{p})^2 Var(D_{t-p-1}) -$$

$$2(1+\frac{L}{p})(\frac{L}{p})Cov(D_{t-1}, D_{t-p-1})$$

Because of:

$$Var(D_{t-1}) = Var(D_{t-p-1}) = Var(D) = g(0) = g$$

and

$$Cov(D_t, D_{t+k}) = g(k)$$

we have:

$$Var(Q_t) = (1+\frac{L}{p})^2 g + (\frac{L}{p})^2 g - 2(1+\frac{L}{p})(\frac{L}{p})g(p) \qquad (15)$$

Thus, to determine $Var(Q_t)$ it is necessary to calculate $g(p)$. Replacing Eq. (15) with Eq. (11) concludes the relationship for the bullwhip effect measurement that can be used for both of the products. This general form of the bullwhip effect measure is defined by Eq. (16):

$$BE = \frac{Var(Q_t)}{Var(D)} = \frac{(1+\frac{L}{p})^2 g + (\frac{L}{p})^2 g - 2(1+\frac{L}{p})(\frac{L}{p})g(p)}{g}$$

in which it can be concluded:

$$BE = [(1+\frac{L}{p})^2 + (\frac{L}{p})^2] - 2(1+\frac{L}{p})(\frac{L}{p})[\frac{g(p)}{g}] \qquad (16)$$

Summarizing Eq. (16) yields Eq. (17) that is similar to Chen et al. (2000 a) in a single product supply chain:

$$BE = 1 + (\frac{2L}{p} + \frac{2L^2}{p^2})(1 - \frac{g(p)}{g}) \qquad (17)$$

We take into account Eq. (16) again to quantify the bullwhip effect in proposed supply chain. Obviously, Eq. (16) does not submit an explicit expression of the bullwhip effect because there is no specific relationship for $g(p)$. However, knowing *p*, we can compute $g(p)$ and so the bullwhip effect measures can be achieved. In this research, we determine $g(p)$ for some specific values of p because we need the bullwhip effect relationships for analytical purposes in the fourth section of the paper. To provide a relationship for the bullwhip effect measure for both of the products, we consider $BE_{p_i}^i$ as the bullwhip effect of the *i* th product (*i*=1, 2). In addition, we use L_i and p_i to show lead-time and number of observations in the forecasting method for each of products (i.e. *L* and *p*). In addition, we consider $g_{ii}(p_i)$ as a covariance between two measures of *i* th product demand at lag p_i. Moreover, g_{ii} represents demand variance of *i* th product. Accordingly, substituting mentioned terms in Eq. (16) concludes Eq. (18) that is bullwhip effect for each product:

$$BE_{p_i}^i = [(1+\frac{L_i}{p_i})^2 + (\frac{L_i}{p_i})^2] - 2(1+\frac{L_i}{p_i})(\frac{L_i}{p_i})[\frac{g_{ii}(p_i)}{g_{ii}}] \quad i=1, 2 \qquad (18)$$

3.2. Determination of $g_{ii}(p_i)$

We know covariance matrix function for VAR (1) process is as follows (see appendix A):

$$\Gamma(p_i) = \begin{cases} \Gamma(-1)f_1' + \Sigma & p_i = 0 \\ \Gamma(p_i - 1)f_1' = \Gamma(0)(f_1')^{p_i} & p_i \geq 1 \end{cases} \qquad (19)$$

It is clear that we need $\Gamma(0)$ and $f_1'^{p_i}$ to calculate $g_{ii}(p_i)$ with $p_i \geq 1$. In the last section, we described that we do not have any explicit relationship for $g_{ii}(p_i)$. Indeed, Eq. (19) shows that due to non-existence of any determined form of $f_1'^{p_i}$ then f_1' is a 2x2 matrix and p_i is greater than one we cannot provide a general formula for $g_{ii}(p_i)$. Now consider $\Gamma(0)$ as follows:

$$\Gamma(0) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$$

in which g_{11} and g_{22} are demand variance of the first and second product respectively and g_{12} is covariance between demand of two products. Calculations for determining g_{11}, g_{22} and g_{12} are mentioned in appendix A. Coefficient matrix in a VAR (1) process for a two-product supply chain such as:

$$\begin{cases} D_t^1 = f_{11}D_{t-1}^1 + f_{12}D_{t-1}^2 + a_t^1 \\ D_t^2 = f_{21}D_{t-1}^1 + f_{22}D_{t-1}^2 + a_t^2 \end{cases}$$

is specified by $f_1 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$. So if we know p_i then we can obtain $g_{ii}(p_i)$ by Eq. (19). Detailed descriptions of $g_{ii}(p_i)$ are mentioned in appendix A. In the next section at first, $g_{ii}(p_i)$ is determined for $p_i = 1,2,3$ and *i*=1, 2 and after that bullwhip effect measures for both of products are calculated.

3.2.1. Determination of BE_1^1 and BE_1^2

When we use only the last observation in moving average forecasting method (i.e. $p_i = 1$), matrices $\Gamma(1)$ and f_1' are needed for specifying the bullwhip effect. Detailed calculations are mentioned below:

$$\Gamma(1) = \Gamma(0) f_1'$$

$$\Gamma(0) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$$

Hence,

$$\Gamma(1) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix}$$

$$\Gamma(1) = \begin{bmatrix} g_{11}f_{11} + g_{12}f_{12} & g_{11}f_{21} + g_{12}f_{22} \\ g_{12}f_{11} + g_{22}f_{12} & g_{12}f_{21} + g_{22}f_{22} \end{bmatrix}$$

Hence

$$g_{11}(1) = g_{11}f_{11} + g_{12}f_{12}$$

$$g_{22}(1) = g_{12}f_{21} + g_{22}f_{22}$$

Therefore using Eq. (18) the bullwhip effect for the first product can be provided:

$$BE_1^1 = [(1+L_1)^2 + (L_1)^2] - 2(1+L_1)(L_1)[\frac{g_{11}(1)}{g_{11}}]$$

$$= [(1+L_1)^2 + (L_1)^2] - 2(1+L_1)(L_1)[\frac{g_{11}f_{11} + g_{12}f_{12}}{g_{11}}]$$

Consequently

$$BE_1^1 = [(1+L_1)^2 + (L_1)^2] - 2(1+L_1)(L_1)[f_{11} + (\frac{g_{12}}{g_{11}})f_{12}] \quad (20)$$

Moreover, for the second product we have:

$$BE_1^2 = [(1+L_2)^2 + (L_2)^2] - 2(1+L_2)(L_2)[\frac{g_{12}f_{21} + g_{22}f_{22}}{g_{22}}]$$

or

$$BE_1^2 = [(1+L_2)^2 + (L_2)^2] - 2(1+L_2)(L_2)[f_{22} + (\frac{g_{12}}{g_{22}})f_{21}] \quad (21)$$

3.2.2. Determination of BE_2^1 and BE_2^2

To determine $g_{11}(2)$ and $g_{22}(2)$ we need to square of transposed coefficient matrix, f_1'. According to matrices rules we have:

$$f_1'^2 = \begin{bmatrix} f_{11}^2 + f_{21}f_{12} & f_{21}(f_{11} + f_{22}) \\ f_{12}(f_{11} + f_{22}) & f_{22}^2 + f_{21}f_{12} \end{bmatrix}$$

Hence based on Eq. (19): $\Gamma(2) = \Gamma(0)(f_1')^2$.

Consequently:

$$\Gamma(2) = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} f_{11}^2 + f_{21}f_{12} & f_{21}(f_{11} + f_{22}) \\ f_{12}(f_{11} + f_{22}) & f_{22}^2 + f_{21}f_{12} \end{bmatrix}$$

$$\Gamma(2) = \begin{bmatrix} g_{11}(f_{11}^2 + f_{21}f_{12}) + g_{12}f_{12}(f_{11} + f_{22}) & g_{11}f_{21}(f_{11} + f_{22}) + g_{12}(f_{22}^2 + f_{21}f_{12}) \\ g_{12}(f_{11}^2 + f_{21}f_{12}) + g_{22}f_{12}(f_{11} + f_{22}) & g_{12}f_{21}(f_{11} + f_{22}) + g_{22}(f_{22}^2 + f_{21}f_{12}) \end{bmatrix}$$

$$g_{11}(2) = g_{11}(f_{11}^2 + f_{21}f_{12}) + g_{12}f_{12}(f_{11} + f_{22})$$

$$g_{22}(2) = g_{12}f_{21}(f_{11} + f_{22}) + g_{22}(f_{22}^2 + f_{21}f_{12})$$

Substituting $g_{11}(2)$ and $g_{22}(2)$ in Eq. (18) results the bullwhip effect relationship for the first product:

$$BE_2^1 = [(1+\frac{L_1}{2})^2 + (\frac{L_1}{2})^2] - 2(1+\frac{L_1}{2})(\frac{L_1}{2})[\frac{g_{11}(2)}{g_{11}}] = [(1+\frac{L_1}{2})^2 + (\frac{L_1}{2})^2]$$

$$-2(1+\frac{L_1}{2})(\frac{L_1}{2})[\frac{g_{11}(f_{11}^2 + f_{21}f_{12}) + g_{12}f_{12}(f_{11} + f_{22})}{g_{11}}]$$

$$BE_2^1 = [(1+\frac{L_1}{2})^2 + (\frac{L_1}{2})^2] - 2(1+\frac{L_1}{2})(\frac{L_1}{2})[(f_{11}^2 + f_{21}f_{12}) + f_{12}(f_{11} + f_{22})(\frac{g_{12}}{g_{11}})] \quad (22)$$

The bullwhip effect measure for the second product can be provided too:

$$BE_2^2 = [(1+\frac{L_2}{2})^2 + (\frac{L_2}{2})^2] - 2(1+\frac{L_2}{2})(\frac{L_2}{2})[(f_{22}^2 + f_{21}f_{12}) + f_{21}(f_{11} + f_{22})(\frac{g_{12}}{g_{22}})] \quad (23)$$

In fact, Eq. (22) and Eq. (23) shows relationships for quantifying the bullwhip effect when the retailer uses only the last two observations for lead-time demand forecasting for both of products.

3.2.3. Determination of BE_3^1 and BE_3^2

To provide $g_{11}(3)$ and $g_{22}(3)$ we need to $f_1'^3$. The steps are similar to previous sections, so we present only results without any expression:

$$f_1'^3 = \begin{bmatrix} f_{11}^3 + f_{21}f_{12}(2f_{11} + f_{22}) & f_{21}(f_{11}^2 + f_{21}f_{12} + f_{11}f_{22} + f_{22}^2) \\ f_{12}(f_{11}^2 + f_{11}f_{22} + f_{21}f_{12} + f_{22}^2) & f_{22}^3 + f_{21}f_{12}(f_{11} + 2f_{22}) \end{bmatrix}$$

$$BE_3^1 = (1+\frac{L_1}{3})^2 + (\frac{L_1}{3})^2 - 2(1+\frac{L_1}{3})(\frac{L_1}{3})\begin{pmatrix} f_{11}^3 + f_{21}f_{12}(2f_{11} + f_{22}) + \\ f_{12}(f_{11}^2 + f_{11}f_{22} + f_{21}f_{12} + f_{22}^2)(\frac{g_{12}}{g_{11}}) \end{pmatrix} \quad (24)$$

$$BE_3^2 = (1+\frac{L_2}{3})^2 + (\frac{L_2}{3})^2 - 2(1+\frac{L_2}{3})(\frac{L_2}{3})\begin{pmatrix} f_{22}^3 + f_{21}f_{12}(f_{11} + 2f_{22}) + \\ f_{21}(f_{11}^2 + f_{21}f_{12} + f_{11}f_{22} + f_{22}^2)(\frac{g_{12}}{g_{22}}) \end{pmatrix} \quad (25)$$

It is necessary to determine BE_4^1, BE_4^2, BE_5^1 and BE_5^2 for our analytical approach in the next section, so we have provided and submited them in appendix C.

4. Numerical Analysis

In this section, an analytical discussion about the bullwhip effect behavior in a two-product supply chain is represented, using a numerical example. Consider that demand process of the two products in a supply chain is defined by:

$$\begin{cases} D_t^1 = 0.7D_{t-1}^1 + 0.6D_{t-1}^2 \\ D_t^2 = 0.2D_{t-1}^1 + 0.5D_{t-1}^2 \end{cases}$$

In the above demand process, we have:

$$f_1 = \begin{bmatrix} 0.7 & 0.6 \\ 0.2 & 0.5 \end{bmatrix}$$

According to stationary condition, we must have:

$$\left| \frac{(f_{11} + f_{22}) \pm \sqrt{(f_{11} - f_{22})^2 + 4f_{12}f_{21}}}{2} \right| \mathbf{p}\, 1$$

Substituting coefficients concludes:

$$\left| \frac{(0.7 + 0.5) + \sqrt{(0.7 - 0.5)^2 + 4(0.6)(0.2)}}{2} \right| = 0.96$$

$$\left| \frac{(0.7 + 0.5) - \sqrt{(0.7 - 0.5)^2 + 4(0.6)(0.2)}}{2} \right| = 0.23$$

Therefore, stationary condition is satisfied. Now we can obtain the bullwhip effect measure for different values of L and limited measures of p (p=1,2,3,4,5) for two products by previously determined relationships. For simplicity in mathematical calculations, we assume that the error terms have standard normal distribution and are uncorrelated. This hypothesis does not affect the generality of the problem. Table 1 and Table 2 contain various measures of the bullwhip effect for the first and second product, respectively.

The following figures depict the bullwhip effect behavior while L and p vary for each of products separately. It is clear that the bullwhip effect is related to lead-time directly and is relevant to the number of observations in moving average calculations (p) reversely. Fig. 1 (as well as Table 1) shows the relationship between the bullwhip effect and the number of periods in the moving average method forecasting and lead-time simultaneously.

Tab. 1. Bullwhip effect measures for the first product

		p1				
		1	2	3	4	5
L1	1	1.215	1.142	1.116	1.103	1.095
	2	1.644	1.377	1.291	1.248	1.222
	3	2.287	1.708	1.524	1.434	1.381
	4	3.145	2.132	1.814	1.661	1.571
	5	4.218	2.651	2.164	1.93	1.793
	6	5.505	3.265	2.571	2.24	2.047

Tab. 2. Bullwhip effect measures for the second product

		p2				
		1	2	3	4	5
L2	1	1.73	1.374	1.255	1.198	1.165
	2	3.191	1.997	1.638	1.476	1.386
	3	5.383	2.869	2.148	1.832	1.661
	4	8.305	3.99	2.786	2.268	1.992
	5	11.96	5.36	3.551	2.783	2.378
	6	16.34	6.979	4.444	3.378	2.819

Surface plane is bullwhip measure according to relevant p and L. It is clear that bullwhip effect increases when period number decreases and lead-time moves up, i.e. more observations and less lead time result in better bullwhip effect measure. A steady rise can be seen for p=2,3,4,5 but when p=1 bullwhip effect curve increases dramatically. The difference between the bullwhip effects when p changes from 1 to 2 indicates that only the last observation is not sufficient for lead-time demand forecasting and causes huge demand fluctuation.

Moreover, Fig. 1 shows that slope of the bullwhip effect curve decreases significantly when lead-time increases from minimum value to maximum measure. Therefore,

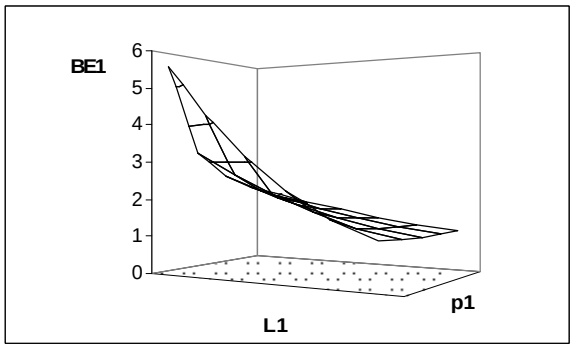

Fig. 1. Bullwhip effect variation with respect to L1 and p1

decreasing lead-time as well as increasing number of utilized data in forecasting method based on moving average could be useful in the bullwhip effect reduction.

A similar situation can be found in Fig. 2 for the second product and all above notes about the bullwhip effect of the first product are valid for the second product but the difference between bullwhip effects for p=1 and p=2 is more than the first product. Anyway, a considerable rise in the bullwhip effect curve is evident while we are using the last observation alone for lead-time demand forecasting.

As a result, we can conclude the according to the above graphs as well as Table 1 and Table 2, that decreasing lead-time along with increasing number of observations in lead-time demand forecasting based on moving average method can reduce the bullwhip effect

of each product. Accordingly, in our example, minimum measure of bullwhip effect occurs when L=1 and p=5 and its measure is 1.095 for the first product and is equal to 1.165 for the second one. In addition, when lead-time is long and the number of observations is small, we encounter maximum value of the bullwhip effect. In fact, longer L and smaller p results in larger bullwhip effect for both of products. Therefore, maximum measure of bullwhip effect for two products is 5.505 and 16.34 respectively when L=6 and p=1.

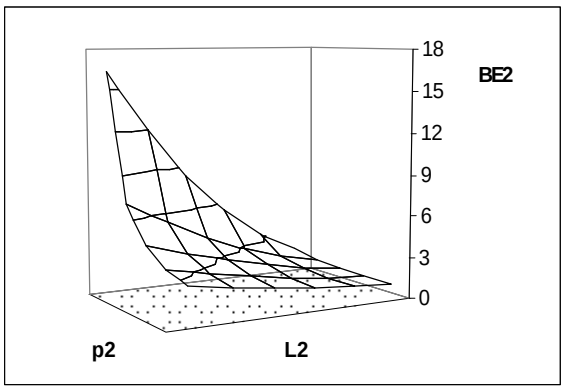

Fig. 2. Bullwhip effect variation with respect to L2 and p2

Studying the relationship between p and L results in another interpretation of the bullwhip effect measure while these parameters change simultaneously. Fig. 1 and Fig. 2 show that we can move the effects of lead lime in bullwhip effect value by increasing p, because bullwhip effect is direct function of L and reverse function of p.

However, it is clear that the relationship between L and p is not one to one, i.e. when lead time the producer's processes grows up one period then retailer must increase number of p in his forecasting more than one period, therefore bullwhip effect measure reminds constant. Obviously increasing p has cost for the retailer and decreasing L has benefits for producer, so balancing of this cost and benefit is important in profit of the supply chain and decision-making. Suppose that we want to adjust our previous supply chain on $BE \approx 2.38$, so different values for L and p can provide this bullwhip effect measure. Fig. 3 depicts the relationship between L and p for the first product. In fact, when $p_1 = 5$ and $L_1 = 7$ the bullwhip effect of the first product in supply chain is the similar to the bullwhip effect situation when $p_1 = 8$ and $L_1 = 9$ and cost benefit analysis has an important role in determination of the best combination of L and p.

This situation can be found for the second product and producer and retailer cooperation would be inducers of the bullwhip effect for both of products. Other analytical interpretations

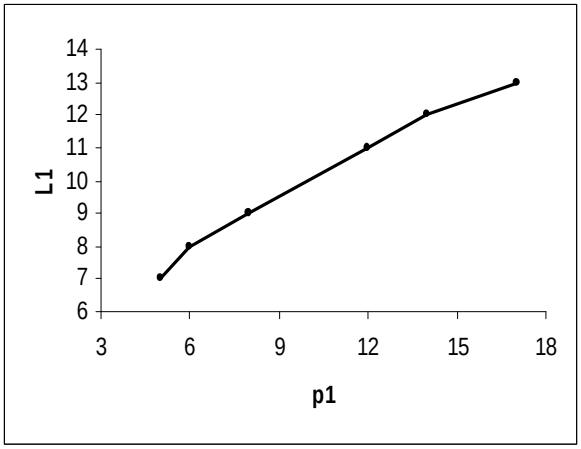

Fig. 3. Relationship between L and p when $BE \approx 2.38$

can be presented according to previous equations.

5. Conclusion

In this research we have investigated bullwhip effect in a two echelon supply chain consist of two products. The demand of each product was releving to the demand of another one. This relationship is described by VAR (1) model. The retailer used OUT policy for ordering of products for both of the products. Order of each product does not depend on order of another one. We assumed retailer uses moving average method to forecast lead-time demand of each product independently.

After description of model, we derived a general expression for the bullwhip effect and mentioned that it is not possible to provide an explicit equation for the bullwhip effect of two products when we use the covariance function of VAR (1) process for quantifying the bullwhip effect. Then we provide bullwhip effect measure of each product for limited cases for better analyzing it. Finally, in the last section we analyzed behavior of the bullwhip effect by a numerical example. This research would be incomplete if we did not mention its drawbacks.

Our aim was to provide the mathematical relationships for quantifying the bullwhip effect and a more analytical approach is needed. In this paper, our assumptions is based on Chen et al. [11], therefore disadvantages of that paper (such as cost considerations, ordering policy and forecasting method) exist in the current article. Also similar to Zhang [14], a study on different forecasting methods as well as more analytical approaches on conditions that the bullwhip effect exists in our supply the chain can be accomplished. It is more interesting when retailer uses different forecasting methods for two products.

Appendix A [27]:

Vector Time Series Process

Suppose vector $Z_t = [Z_{1,t}, Z_{2,t}, \ldots, Z_{m,t}]'$ where $t = 0, \pm 1, \pm 2, \ldots$ is a stationary vector process with real value and dimension m. For mean of the process we have:

$$E(Z_t) = m = \begin{bmatrix} m_1 \\ m_2 \\ . \\ . \\ . \\ m_m \end{bmatrix}$$

In addition, covariance matrix is:

$$\Gamma(k) = Cov\{Z_t, Z_{t+k}\} = E[(Z_t - m)(Z_{t+k} - m)']$$

$$= E\begin{bmatrix} Z_{1,t} - m_1 \\ Z_{2,t} - m_2 \\ . \\ . \\ . \\ Z_{m,t} - m_m \end{bmatrix} [Z_{1,t+k} - m_1, Z_{2,t+k} - m_2, ..., Z_{m,t+k} - m_m]$$

$$= \begin{bmatrix} g_{11}(k) & g_{12}(k) & . & . & g_{1m}(k) \\ g_{21}(k) & g_{22}(k) & . & . & g_{2m}(k) \\ . & . & . & . & . \\ . & . & . & . & . \\ g_{m1}(t,t+k) & g_{m2}(k) & . & . & g_{mm}(t,t+k) \end{bmatrix} = Cov\{Z_{t-k}, Z_t\}$$

In which for $k = 0, \pm1, \pm2, ...$ and $i = 1,2,...,m$ we have:

$$g_{ij}(k) = E(Z_{i,t} - m_i)(Z_{j,t+k} - m_j) = E(Z_{i,t-k} - m_i)(Z_{j,t} - m_j)$$

so covariance matrix of the vector process, $\Gamma(k)$, is a function of *k*. It is clear that $\Gamma(0)$ is variance-covariance matrix of the process.

Covariance Function for VAR (1) Model

For the first order vector autoregressive model, we have:

$$\Gamma(k) = E[Z_{t-k} Z_t']$$
$$= E[Z_{t-k}(f_1 Z_{t-1} + a_t)']$$
$$= \begin{cases} \Gamma(-1) f_1' + \Sigma & k = 0 \\ \Gamma(k-1) f_1' = \Gamma(0)(f_1')^k & k \geq 1 \end{cases}$$

Notice that for $k \geq 1$ we have $E(Z_{t-k} a_t') = 0$. If $k = 1$ then $\Gamma(1) = \Gamma(0) f_1'$. Therefore $f_1 = \Gamma'(1)\Gamma^{-1}(0)$ and

$$\Sigma = \Gamma(0) - \Gamma(-1)\Gamma^{-1}(0)\Gamma(1)$$
$$= \Gamma(0) - \Gamma'(1)\Gamma^{-1}(0)\Gamma(1)$$

If *m*=2 we can achieve $\Gamma(0)$ as follows:

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} - \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

solving this matrix equation yields Eq. (2) and Eq. (3) and Eq. (4).

Appendix B:

Proof: To simplify calculations, we did not consider each product separately in (12), but the result can be used for both of them. By variance definition:

$$(\hat{s}_t^L)^2 = Var(D_t^L - \hat{D}_t^L) = Var(D_t^L) + Var(\hat{D}_t^L) - 2Cov(D_t^L, \hat{D}_t^L)$$

in which

$$D_t^L = D_t + D_{t+1} + ... + D_{t+L-1} = \sum_{i=0}^{L-1} D_{t+i} \qquad (26)$$

and $\hat{D}_t^L$ is derived before in Eq. (10). For the appointment of $(\hat{s}_t^L)^2$ we must determine three relationships for $Var(D_t^L)$ and $Var(\hat{D}_t^L)$ and $Cov(D_t^L, \hat{D}_t^L)$.

Determination $Var(D_t^L)$

By definition and based on (26) and statistical rules we have:

$$Var(D_t^L) = Var(D_t + D_{t+1} + ... + D_{t+L-1})$$
$$= Var(D_t) + Var(D_{t+1}) + ... + Var(D_{t+L-1}) +$$
$$2Cov(D_t, D_{t+1}) + 2Cov(D_t, D_{t+2}) + ... + 2Cov(D_t, D_{t+L-1}) +$$
$$2Cov(D_{t+1}, D_{t+2}) + 2Cov(D_{t+1}, D_{t+3}) + ... + 2Cov(D_{t+1}, D_{t+L-1}) +$$
$$2Cov(D_{t+2}, D_{t+3}) + 2Cov(D_{t+2}, D_{t+4}) + ... + 2Cov(D_{t+2}, D_{t+L-1}) +$$
$$... + 2Cov(D_{t+L-2}, D_{t+L-1})$$

In addition, $Cov(D_t, D_{t+k}) = g(k)$ and demand process is stationary. Therefore, we have:

$$Var(D_t) = Var(D_{t+1}) = ... = g(0) = g$$

Thus:

$$Var(D_t^L) = g + g + ... + g + 2g(1) + 2g(2) + ... + 2g(L-1) +$$
$$2g(1) + 2g(2) + ... + 2g(L-2) + 2g(1) + ... + 2g(L-3) + ... + 2g(1)$$
$$= Lg + 2[g(1) + g(1) + ... + g(1) + g(2) + ... + g(2) + ... + g(L-1)]$$
$$= Lg + 2[(L-1)g(1) + (L-2)g(2) + ... + g(L-1)]$$
$$= Lg + 2[\sum_{i=1}^{L-1} (L-i)g(i)]$$

Determination $Var(\hat{D}_t^L)$:

We showed previously in Eq. (10)

$$\hat{D}_t^L = L\left(\frac{\sum_{i=1}^{p} D_{t-i}}{p}\right)$$

Hence

$$Var(\hat{D}_t^L) = Var\left(L\left(\frac{\sum_{i=1}^{p} D_{t-i}}{p}\right)\right)$$

According to variance calculation rules, we have:

$$Var(\hat{D}_t^L)=(\frac{L}{p})^2 Var(\sum_{i=1}^{p} D_{t-i})=(\frac{L}{p})^2 Var(D_{t-1}+D_{t-2}+...+D_{t-p})$$

$$=(\frac{L}{p})^2[Var(D_{t-1})+Var(D_{t-1})+...+Var(D_{t-p})+$$
$$2Cov(D_{t-1},D_{t-2})+2Cov(D_{t-1},D_{t-3})+...+2Cov(D_{t-1},D_{t-p})+$$
$$2Cov(D_{t-2},D_{t-3})+2Cov(D_{t-2},D_{t-4})+...+2Cov(D_{t-2},D_{t-p})+$$
$$2Cov(D_{t-3},D_{t-4})+2Cov(D_{t-3},D_{t-5})+...+2Cov(D_{t-3},D_{t-p})+...$$
$$+2Cov(D_{t-p+1},D_{t-p})]$$
$$=(\frac{L}{p})^2[g+g+...+g+2g(1)+2g(2)+...+2g(p-1)+$$
$$2g(1)+2g(2)+...+2g(p-2)+$$
$$2g(1)+2g(2)+...+2g(p-3)+...+2g(1)]$$
$$=(\frac{L}{p})^2[pg+2(g(1)+g(1)+...+g(1)+g(2)+...+g(2)+...+g(p-1))]$$
$$=(\frac{L}{p})^2[pg+2((p-1)g(1)+(p-2)g(2)+...+g(p-1))]$$
$$=(\frac{L}{p})^2[pg+2(\sum_{i=1}^{p-1}(p-i)g(i))]$$

Determination $Cov(D_t^L,\hat{D}_t^L)$:

Using definition of D_t^L and $\hat{D}_t^L$ that are mentioned before, we can write:

$$Cov(D_t^L,\hat{D}_t^L)=Cov[(D_t+D_{t+1}+...+D_{t+L-1}),(\frac{L}{p})(D_{t-1}+D_{t-2}+...+D_{t-p})]$$
$$=(\frac{L}{p})Cov[(D_t+D_{t+1}+...+D_{t+L-1}),(D_{t-1}+D_{t-2}+...+D_{t-p})]$$
$$=(\frac{L}{p})[Cov(D_t,D_{t-1})+Cov(D_t,D_{t-2})+...+Cov(D_t,D_{t-p})+$$
$$=(\frac{L}{p})[Cov(D_t,D_{t-1})+Cov(D_t,D_{t-2})+...+Cov(D_t,D_{t-p})+$$
$$Cov(D_{t+1},D_{t-1})+Cov(D_{t+1},D_{t-2})+...+Cov(D_{t+1},D_{t-p})+...+$$
$$Cov(D_{t+L-1},D_{t-1})+Cov(D_{t+L-1},D_{t-2})+...+Cov(D_{t+L-1},D_{t-p})]$$
$$=(\frac{L}{p})[g(1)+g(2)+...+g(p)+g(2)+g(3)+...+g(p+1)+...$$
$$+g(L)+g(L+1)+...+g(L+p-1)]$$
$$=(\frac{L}{p})[\sum_{i=1}^{L}g(i)+\sum_{i=2}^{L+2}g(i)+...+\sum_{i=p}^{L+p-1}g(i)]$$
$$=(\frac{L}{p})[\sum_{j=0}^{p-1}\sum_{i=j+1}^{j+L}g(i)]$$

Now consider again the previous variance relationship:

$$(\hat{s}_t^L)^2=Var(D_t^L)+Var(\hat{D}_t^L)-2Cov(D_t^L,\hat{D}_t^L).$$

Replacing the three determined measures, concludes proposition1.

Appendix C:

Providing BE_4^1 **and** BE_4^2

$$BE_4^1=\left(1+\frac{L_1}{4}\right)^2+\left(\frac{L_1}{4}\right)^2-2\left(1+\frac{L_1}{4}\right)\left(\frac{L_1}{4}\right)$$
$$\left(\begin{array}{l} f_{11}^4+f_{11}f_{21}f_{12}(3f_{11}+2f_{22})+ \\ f_{21}f_{12}\left(f_{21}f_{12}+f_{22}^2\right)+ \\ f_{12}(f_{11}+f_{22})\left(f_{11}^2+2f_{21}f_{12}+f_{22}^2\right)\left(\frac{g_{12}}{g_{11}}\right) \end{array}\right)$$

$$BE_4^2=\left(1+\frac{L_2}{4}\right)^2+\left(\frac{L_2}{4}\right)^2-2\left(1+\frac{L_2}{4}\right)\left(\frac{L_2}{4}\right)$$
$$\left(\begin{array}{l} f_{22}^4+f_{21}f_{12}f_{22}(3f_{22}+2f_{11})+ \\ f_{21}f_{12}\left(f_{21}f_{12}+f_{11}^2\right)+ \\ f_{21}(f_{11}+f_{22})\left(f_{11}^2+2f_{21}f_{12}+f_{22}^2\right)\left(\frac{g_{12}}{g_{22}}\right) \end{array}\right)$$

Providing BE_5^1 **and** BE_5^2:

$$BE_5^1=\left(1+\frac{L_1}{5}\right)^2+\left(\frac{L_1}{5}\right)^2-2\left(1+\frac{L_1}{5}\right)\left(\frac{L_1}{5}\right)$$
$$\left(\begin{array}{l} f_{11}^5+f_{21}f_{12}\left(4f_{11}^3+(f_{11}f_{22}+f_{21}f_{12})(3f_{11}+2f_{22})+f_{22}^3\right)+ \\ +f_{12}\left(\begin{array}{l} f_{22}(f_{11}+f_{22})\left(f_{11}^2+2f_{21}f_{12}+f_{22}^2\right)+ \\ f_{21}f_{12}\left(\begin{array}{l} f_{11}(3f_{11}+2f_{22})+ \\ \left(f_{21}f_{12}+f_{22}^2\right) \end{array}\right)+f_{11}^4 \end{array}\right)\left(\frac{g_{12}}{g_{11}}\right) \end{array}\right)$$

$$BE_5^2=\left(1+\frac{L_2}{5}\right)^2+\left(\frac{L_2}{5}\right)^2-2\left(1+\frac{L_2}{5}\right)\left(\frac{L_2}{5}\right)$$
$$\left(\begin{array}{l} f_{22}^5+f_{21}f_{12}\left(4f_{22}^3+(f_{11}f_{22}+f_{21}f_{12})(3f_{22}+2f_{11})+f_{11}^3\right)+ \\ +f_{21}\left(\begin{array}{l} f_{11}(f_{11}+f_{22})\left(f_{11}^2+2f_{21}f_{12}+f_{22}^2\right)+ \\ f_{21}f_{12}\left(\begin{array}{l} f_{22}(3f_{22}+2f_{11})+ \\ \left(f_{21}f_{12}+f_{11}^2\right) \end{array}\right)+f_{22}^4 \end{array}\right)\left(\frac{g_{12}}{g_{22}}\right) \end{array}\right)$$

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References

[1] Forrester, J.W., *Industrial Dynamics - a Major Break Through for Decision Making,* Harvard Business Review, 1958.

[2] Goodman, M., *"A New Look at Higher-Order Exponential Smoothing for Forecasting,"* Operational Research, Vol. 22, 1974, PP. 880-888.

[3] Blinder, A.S., *"Inventories and Sticky Process,"* American Economics Review, Vol. 72, 1982, pp. 334-349.

[4] Blinder A.S., "*Can the Production Smoothing Model of Inventory Behavior be Saved*" Quarterly Journal of Economics, Vol. 101, 1986, pp. 431-454.

[5] Blanchard, O.J., *"The Production and Inventory Behavior of the American Automobile Industry,*" Journal of Political Economy, Vol. 91, 1983, pp. 365-400.

[6] Kahn, J., "*Inventories and Volatility of Production,*" American Economics Review, Vol. 77, 1987, pp. 667-679.

[7] Bagahnha, M.P., Cohen, M.A., *"The Stabilizing Effect of Inventory in Supply Chain,"* Management Science, Vol. 46, pp. 436-443.

[8] Metters, R., "*Quantifying the Bullwhip Effect in Supply Chains,"* Journal of Operations Management, Vol. 15, 1997, pp. 89-100.

[9] Sterman, J.D., *"Optimal Policy for a Multi-Product, Dynamic, No Stationary Inventory Problem,*" Management Science, Vol. 12, 1989, pp. 206-222.

[10] Lee, H.L., Padmanabhan, V., Whang , S., *"The Bullwhip Effect in Supply Chains,"* Sloan Management Review Vol. 38, 1997, pp. 93-102.

[11] Chen, F., Drezner, Z., Ryan, J.K., Smith-Levi, D., *"Quantifying the Bullwhip Effect in a Simple Supply Chain: The Impact of Forecasting, Lead Times and Information,*" Management Science, Vol. 46, 2000, pp. 436-443.

[12] Dejonckheere, J., Disney, S.M., Lambrecht, M.R., Towill, D.R., *"Measuring and Avoiding the Bullwhip Effect: A Control Theoretic Approach,*" European Journal of Operational Research, Vol. 147, 2003, pp. 567-590.

[13] Disney, S.M., Towill, D.R., *"On the Bullwhip and Inventory Variance Produced by an Ordering Policy,*" Omega, Vol. 31, 2003, pp. 157-167.

[14] Zhang, X., "*The Impact of Forecasting Methods on the Bullwhip Effect,*" International Journal of Production Economics, Vol. 88, 2004, pp. 15-27.

[15] Kim, J.G., Chatfield, D., Harrison, T.P., Hayya, J.C., "*Quantifying the Bullwhip Effect in Supply Chain with Stochastic Lead Time,*" European Journal of the Operational Research. Vol. 173, 2006, pp. 617-636.

[16] Chandra, C., Grabis, J., *"Application of Multi-Steps Forecasting for Restraining the Bullwhip Effect and Improving Inventory Performance Under Autoregressive Demand,"* European Journal of Operational Research, Vol. 166, 2005, pp. 337-350.

[17] Luong, H.T., *"Measure of Bullwhip Effect in Supply Chains with Autoregressive Demand Process,*" European Journal of Operational Research. Vol. 180, 2007, pp. 1086-1097.

[18] Luong, H.T., Phien, N.H., *"Measure of Bullwhip Effect in Supply Chains: The Case of High Order Autoregressive Demand Process,"* European Journal of Operational Research, Vol. 183, 2007, pp. 197-209.

[19] Makui, A., Madadi, A., "*The Bullwhip Effect and Lyapunov Exponent,"* Applied Mathematics and Computation, Vol. 189, 2007, pp. 35-40.

[20] Gaalman, G., Disney, S.D., "*On Bullwhip in a Family of Order up to Policies with ARMA (2,2) Demand and Arbitrary Lead-Time*", International Journal of Production Economics, In Press, Corrected Proof, Available online 25 February 2007.

[21] Jaksic, M., Rusjan, B., "*The Effect of Replenishment Policies on the Bullwhip Effect: A Transfer Function Approach,"* European Journal of Operational Research, Vol. 184, 2008, pp. 946-961.

[22] Su, C.T., Wong, J.T., *"Design of a Replenishment System for a Stochastic Dynamic Production/Forecast lot-Sizing Problem Under Bullwhip Effect,*" Expert Systems with Applications, Vol. 34, 2008, pp. 173-180.

[23] Chen, F., Drezner, Z., Ryan, J.K., Smith-Levi, D., *"The Impact of Exponential Smoothing Forecasts on the Bullwhip Effect,"* Naval Research Logistics Vol. 47, 2000, pp. 269-286.

[24] Hosoda T. and Disney S.M., "*On Variance Amplification in a Three-Echelon Supply Chain with Minimum Mean Square Error Forecasting,*" Omega, Vol. 34, 2006, pp.344-358.

[25] Gilbert, K., "*An ARIMA Supply Chain Model,*" Management Science, Vol. 51, 2005, pp. 305-310.

[26] Silver, E., Peterson, R., Pyke, D.F., Decision Systems for Inventory Management and production, Willey, New York, 2000.

[27] Wei, W.W.S., Time Series Analysis: Univariate and Multivariate Methods, Ferdowsi University Press, 1998.