SEMIRADICALS OF SUB MODULES IN MODULES

Hamid A. Tavallaee and Rezvan. Varmazyar

Abstract: Let $R$ be a commutative ring and $M$ be a unitary $R$-module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula (s.t.s.r.f) and present the existence of such a module.

Keywords: Prime sub module, semiprime sub module, radical and semi-radical of a module, modules satisfying the semi-radical formula.

1. Introduction
In this paper all the rings are commutative with identity and all the modules are unitary. Let $R$ be a ring and $M$ be an $R$-module. If $N$ is a submodule of $M$ we use the notation $N \subseteq M$. If the submodule $N$ is generated by a subset $S$ of $M$, we write $N = \langle S \rangle$. If $N$ and $K$ are sub modules of $M$, then the set $\{ r \in R | rK \subseteq N \}$ is denoted by $\langle N : K \rangle$ or simply by $(N : K)$ which is clearly an ideal of $R$. If $I$ is an ideal of the ring $R$, we write $I \subseteq R$. In Section 2 we define prime and primary sub modules of an $R$-module $M$ and in Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if $N$ is a semiprime submodule of an $R$-module $M$, then $(N : M)$ is a semiprime ideal of $R$ but not conversely in general. In Lemma 2.8 we prove that the converse is also true if $M$ is a multiplication module. In Section 3 we define radical of an $R$-module $M$ and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule and also modules satisfying the semi-radical formula which is abbreviated as (s.t.s.r.f) and in Theorem 3.9 we show that such a module does exist. Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a $P$-semiprime sub module we consider some of its properties.

2. Some Elementary Results
We begin this section with the following definitions:

Definition 2.1. Let $N$ be a proper submodule of an $R$-module $M$.
(a) $N$ is called a prime submodule of $M$ if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r \in (N : M)$.
(b) $N$ is called a primary submodule of $M$ if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r^n \in (N : M)$ for some positive integer $n$.

In (a) it can easily be shown that $P = (N : M)$ is a prime ideal of $R$ and we say that $N$ is $P$-prime. We recall that if $I$ is an ideal of a ring $R$, then the radical of $I$, denoted by $\sqrt{I}$, is defined as the intersection of all prime ideals containing $I$. Alternatively, we define the radical of $I$ as:

$\sqrt{I} = \{ r \in R | r^n \in I$ for some positive integer $n \}$.

Also if $N$ is a primary submodule of an $R$-module $M$, then $(N : M)$ is a primary ideal of $R$ and $P = \sqrt{(N : M)}$ is a prime ideal. We describe this situation by saying that $N$ is $P$-primary.

Lemma 2.2. Let $N$ be a proper submodule of an $R$-module $M$.
(i) $N$ is a prime submodule of $M$ if and only if $ID \subseteq N$ (with $I$ an ideal of $R$ and $D$ a submodule of $M$) implies that $D \subseteq N$ or $I \subseteq (N : M)$.
(ii) $N$ is a primary submodule of $M$ if and only if for every finitely generated ideal $I$ of $R$ and any submodule $D$ of $M$, $ID \subseteq N$ implies that $D \subseteq N$ or $I^n \subseteq (N : M)$ for some positive integer $n$.
(iii) Let $P$ be a prime ideal of $R$, then $N$ is a $P$-primary submodule of $M$ if and only if (a)
$P \subseteq \sqrt{(N : M)}$, and (b) $cm \notin N$ for all $c \in R \setminus P$, $m \in M \setminus N$.

**Proof.** (i) $\Rightarrow$: Let $I \subseteq R$ and $D \subseteq M$ be such that $ID \subseteq N$ and let $D \nsubseteq N$. So there exists an element $x \in D \setminus N$. Let $r$ be any element of $I$. Then $rx \in N$ and hence $r \in (N : M)$, Therefore $I \subseteq (N : M)$. 

(ii) $\Rightarrow$: Let $r \in R$, $a \in M$ be such that $ra \in N$ and let $a \notin N$. By taking: 

$I = (r)$ and $D = Ra$ we see that $ID \subseteq N$. But $D \nsubseteq N$ and hence $I \subseteq (N : M)$, 

which implies that $r \in (N : M)$. Therefore $N$ is a prime submodule of $M$.

(iii) $\Rightarrow$: If $N$ is $P$–primary, then by definition $P = \sqrt{(N : M)}$. Now let $c \in R \setminus P$ and $m \in M \setminus N$. Let $cm \in N$, then there exists a positive integer $n$ such that: 

$c^n \in (N : M)$, that is, $c \in (N : M)$ (because $m \notin N$), a contradiction. Hence $cm \notin N$.

Let $r \in R$ and $m \in M$, $rm \in N$. Assume further that $m \notin N$, then by (b), $r$ must belong to $P$ and so $r \in \sqrt{(N : M)}$ by (a). Therefore $N$ is a primary submodule of $M$. Next we must show that $P = \sqrt{(N : M)}$.

Let $r \in \sqrt{(N : M)}$, then $r^n \in (N : M)$ for some positive integer $n$, and so $r^nM \subseteq N$. Since $N$ is proper, there exist $x \in M \setminus N$. Now $r^n x \in N$ and $x \notin N$ so by (b) we conclude that $r^n \in P$ and, as $P$ is prime, $r \in P$. We find that $\sqrt{(N : M)} = P$ and therefore $N$ is $P$–primary.

The following definition is essential in the rest of the paper.

**Definition 2.3.** A proper submodule $N$ of an $R$–module $M$ is said to be semiprime in $M$, if for every ideal $I$ of $R$ and every submodule $K$ of $M$, $I^2 K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring $R$ is an $R$–module by itself, a proper ideal $I$ of $R$ is semiprime if for every ideals $J$ and $K$ of $R$, $J^2 K \subseteq I$ implies that $JK \subseteq I$.

**Proposition 2.4.** Let $M$ be an $R$–module.

(i) If $N$ is a prime submodule of $M$, then $N$ is semiprime.

(ii) If $N$ is a semiprime submodule of $M$, then $(N : M)$ is semiprime ideal of $R$.

**Proof.** (i) Let $I \subseteq R$, $K \subseteq M$ and $I^2 K \subseteq M$. Then $I (IK) \subseteq N$ and since $N$ is prime, $I \subseteq (N : M)$ or $IK \subseteq N$. But $(N : M) \subseteq (N : K)$ and hence $I \subseteq (N : K)$, and so $IK \subseteq N$. In any case we see that $IK \subseteq N$, and therefore $N$ is semiprime.

(ii) Let $J$ and $K$ be ideals of $R$ and $J^2 K \subseteq (N : M)$. Hence $(J^2 K)M \subseteq N$, and so, $J^2 (KM) \subseteq N$. But $KM \subseteq M$, and $N$ is semiprime, therefore $J(KM) \subseteq N$, and thus, $(JK)M \subseteq N$. Hence $JK \subseteq (N : M)$ and we conclude that $(N : M)$ is a semiprime ideal of $R$.

Part (i) of the above proposition implies that if $P$ is a prime ideal of $R$ then $P$ is semiprime. In the next example we show that the converse of part (ii) of Proposition 2.1 is not valid in general.

**Example 2.5.** Let $R = Z$, $M = Z \oplus Z$ and $B = \langle (9, 0) \rangle$. Then it is clear that $(B : M) = (0)$ since $Z$ is an integral domain, $(B : M) = (0)$ is a prime ideal and hence a semiprime ideal of $Z$. But $B$ is not a semiprime submodule of $M$; because if we take $I = (3)$ and $K = < (2, 0) >$, then:

$I^2 K = \{ (18q, 0) | q \in Z \}$

and:

$IK = \{ (6q, 0) | q \in Z \}$

is not a subset of $B$.

It is clear that if $N$ is a semiprime submodule of an $R$–module $M$ and $I \subseteq R$, $K \subseteq M$ be such that $I^2 K \subseteq N$ for some positive integer $n$, then $IK \subseteq N$.

**Theorem 2.6.** Let $N$ be a proper submodule of an $R$–module $M$. Then the following statements are equivalent:

(i) $N$ is semiprime.

(ii) Whenever $r/m \in N$ for some $r \in R$, $m \in M$ and $t \in Z^+$, then $rm \in N$. 

...
Proof. (i) \( \Rightarrow \) (ii). Let \( r \cdot m \in N \) where \( r \in R \), \( m \in M \) and \( t \in \mathbb{Z}^+ \). Taking \( I = (r) \) and \( K = (m) \) we have \( I^t K \subseteq N \) and so \( IK \subseteq N \) winch implies that \( rm \in N \).

(ii) \( \Rightarrow \) (i). Let \( I \leq R \) and \( K \leq M \) be such that \( I^2 K \subseteq N \). Consider the set:

\[
S = \left\{ ra \mid r \in I, a \in K \right\}
\]

(3)

Then for every \( r \in I, a \in K \) we have \( r^2 a \in I^2 K \subseteq N \) and hence \( ra \in N \). This implies that \( S \subseteq N \) and since \( IK \) is the submodule of \( M \) generated by \( S \), we must have \( IK \subseteq N \). Therefore \( N \) is semiprime.

**Definition 2.7.** An \( R \)-module \( M \) is said to be a multiplication module if for each submodule \( N \) of \( M \), \( N = IM \) for some ideal \( I \) of \( R \).

It can be easily shown that, an \( R \)-module \( M \) is a multiplication module if and only if \( N = (N : M)M \) for every submodule \( N \) of \( M \).

**Lemma 2.8.** Let \( M \) be a multiplication \( R \)-module. Then a submodule \( N \) of \( M \) is semiprime if and only if \( (N : M) \) is a semiprime ideal of \( R \).

Proof. \( \Rightarrow \) : This is clear from Proposition 2.4 (ii).

\( \Leftarrow \) : Let \( I \leq R \), \( K \leq M \), be such that \( I^2 K \subseteq N \). Hence:

\[
I^2 K : M \subseteq (N : M).
\]

(4)

It can be shown that:

\[
I^2 (K : M) \subseteq (I^2 K : M)
\]

(5)

and so we obtain:

\[
I^2 (K : M) M \subseteq (N : M).
\]

(6)

But \((N : M)\) is a semiprime ideal of \( R \) and hence \( I(K : M) \subseteq (N : M) \). Thus we conclude that:

\[
I(K : M) M \subseteq (N : M) M,
\]

(7)

and using the fact that \( M \) is a multiplication \( R \)-module we have \( IK \subseteq N \). Therefore \( N \) is a semiprime submodule of \( M \).

The following lemma shows that the same situation, as before, holds for prime and primary submodules.

**Lemma 2.9.** Let \( M \) be a multiplication \( R \)-module. Then:

(a) A submodule \( N \) of \( M \) is prime if and only if \((N : M)\) is a prime ideal of \( R \).

(b) A submodule \( N \) of \( M \) is primary if and only if \((N : M)\) is a primary ideal of \( R \).

Proof. (a) \( \Rightarrow \) : Clear.

\( \Leftarrow \) : Let \( I \leq R \), \( D \leq M \) be such that \( ID \subseteq N \), then \( (ID : M) \subseteq (N : M) \). But \((D : M) \subseteq (ID : M)\) and so \( I(D : M) \subseteq (N : M) \). Since \((N : M)\) is a prime:

\( R \) have \( I \subseteq (N : M) \) or \((D : M) \subseteq (N : M) \). Suppose that \( I \subseteq (N : M) \). Then \((D : M) \subseteq (N : M)\) and hence we have \((D : M) M \subseteq (N : M) M\), that is, \( D \subseteq N \). Hence \( N \) is a primary submodule of \( M \) by Lemma 2.2 (ii).

(b) \( \Rightarrow \) : Clear.

**Proposition 2.10.** Let \( \{P_i\}_{i \in I} \) be a non-empty family of semiprime sub modules of an \( R \)-module \( M \). Then \( P = \bigcap P_i \) is a semiprime submodule of \( M \).

Further if \( \{P_i\}_{i \in I} \) is totally ordered (by inclusion), then \( T = \bigcap P_i \) is also a semiprime submodule whenever \( T \neq M \).

Proof. Let \( I \leq R \) and \( K \leq M \) be such that \( I^2 k \subseteq P = \bigcap P_i \). Then \( I^2 k \subseteq P_i \) for every \( i \in I \), and since \( P_i \) is semiprime we have \( Ik \subseteq P_i \). Hence \( IK \subseteq \bigcap P_i = P \) and \( P \) is semiprime. Next we let \( T = \bigcap P_i \neq M \). The fact that \( \{P_i\}_{i \in I} \) is totally ordered by inclusion makes it clear that \( T \) is a submodule of \( M \). Let \( I \leq R \) and \( K \leq M \) be such that \( I^2 K \subseteq T \). Consider the set:

\[
S = \left\{ r k \mid r \in R, k \in K \right\}
\]

(8)

Then \( S \) is a generating set for the submodule \( IK \). If \( r \in I \), \( k \in K \) then \( r^2 k \in I^2 K \subseteq T \) and so for some \( i \in I \), \( r^2 k \in P_i \). Since \( P_i \) is semiprime this implies that \( rk \in P_i \). It follows that \( S \subseteq T \) and hence \( IK = \langle S \rangle \subseteq T \). Therefore \( T \) is also a semiprime submodule of \( M \).

**Remark.** Some authors define a semiprime submodule as an intersection of prime submodules. But by our
of a semiprime submodule of a module which we can find a semiprime submodule of a module which is not an intersection of prime sub modules, for example look at J. Jenkins and P.F. Smith in the proof of [1].

3. Radicals and Semi-Radicals
Let $M$ be an $R$-module and $N$ a submodule of $M$. If there exists a prime submodule of $M$ which contain $N$, then the intersection of all prime sub modules containing $N$, is called the $M$-radical of $M$ and is denoted by $\text{rad}_M N$, or simply by $\text{rad} N$. If there is no prime submodule containing $N$, then we define $\text{rad}_M N = M$ . An ideal $I$ of a ring $R$ is called a radical ideal if $\sqrt{I} = I$. Similarly, we say that a submodule $B$ of an $R$-module $M$ is a radical submodule if $\text{rad} B = B$. It is easy to see that an ideal $I$ of a ring $R$ is semiprime if and only if it is radical. Because, let $I$ be semiprime, and let $x \in \sqrt{I}$. Then $x^k \in I$ for some positive integer $k$. So $x^k I \subseteq I$, and since $I$ is semiprime we have $x.1 = x \in I$. Therefore $I = \sqrt{I}$.

On the other hand, if $I = \sqrt{I}$ then by definition of $\sqrt{I}$ and Propositions 2.4 (i) and 2.10, $I$ is semiprime. Finally by Propositions 2.4 (i) and 2.10 we see that for any submodule $B$ of an $R$-module $M$, $\text{rad} B$ is a semiprime submodule whenever $\text{rad} B \neq M$.

Theorem 3.1. Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if it is radical.

**Proof.** Since $\text{ann}_R (M) \subseteq (N : M)$, by [2, Theorem 3, P.216],

$$\sqrt{(N : M)M} = \text{rad} (N : M)M .$$

(9)

As $M$ is a multiplication module we have $(N : M)M = M$, and if $N$ is semiprime, $(N : M)$ is a radical ideal. Therefore $
\sqrt{(N : M)M} = \text{rad} (N : M)M$ if and only if $(N : M)M = \text{rad} (N : M)M$. If $N = \text{rad} N$

this implies that $N$ is a radical submodule of $M$, that is, $N = \text{rad} N = \cap P (P$ is a prime submodule of $M$ containing $N$). Hence by Propositions 2.4 (1) and 2.10 $N$ is a semiprime submodule of $M$. The proof is now complete.

After Remark 2.11 we may ask under what condition a semiprime submodule is the intersection of prime submodules containing it. The following corollary can be considered as an answer.

**Corollary 3.2.** Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if $N = \cap P (P$ is a prime submodule of $M$ containing $N$).

**Proof.** (⇒) If $N$ is semiprime then by Theorem 3.1, it is radical, that is, $N = \cap P (P$ is a prime submodule of $M$ containing $N$).

(⇐): By Propositions 2.4 (i) and 2.10, $N$ is semiprime.

**Proposition 3.3.** If $M$ is a finitely generated $R$-module, then every proper submodule of $M$ is contained in a semiprime sub module.

**Proof.** By Corollary of [3, Proposition 4, P.63]. every proper submodule of $M$ is contained in a prime submodule. So by Proposition 2.4 (i), we have the result.

**Definition 3.4.** (1) A semiprime submodule $P$ of an $R$-module $M$ is called a minimal semiprime of a proper submodule $N$ if $N \subseteq P$ and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of $0 =< 0_M >$ is called a minimal semiprime submodule of $M$.

**Theorem 3.5.** Let $M$ be an $R$-module. If a submodule $N$ of $M$ is contained in a semiprime submodule $P$, then $P$ contains a minimal semiprime submodule of $N$.

**Proof.** It is similar to the proof of [5, Theorem 4, P.84].

**Proposition 3.6.** Every proper submodule of a finitely generated $R$-module $M$ possesses at least one minimal semiprime submodule of $M$.

**Proof.** Let $N$ be a proper submodule of $M$, then by Proposition 3.3, $N$ is contained in a semiprime submodule of $M$.

**Corollary 3.7.** Every semiprime submodule of an $R$-module $M$ contains at least one minimal semiprime submodule of $M$.

**Proof.** Let $P$ be a semiprime submodule of $M$ and take $N =< 0 >$ in the Theorem 3.5. Then $P$ contains a minimal semiprime submodule of $< 0 >$, and so a minimal semiprime submodule of $M$.

**Definition 3.8.** Let $M$ be an $R$-module and $N \leq M$. If there exists a semiprime submodule of $M$ which contains $N$, then the intersection of all semiprime sub modules containing $N$ is called the semi-radical of $N$ and is denoted by $S - \text{rad}_M N$, or simply by $S - \text{rad} N$. If there is no semiprime submodule containing $N$, then we define...
We conclude that:
\[
\text{PradCSradBSradSCradS} \subseteq \text{rad} \quad \text{(16)}
\]

We say that \(M\) satisfies the semi-radical formula, \(M\) (s.t.s.r.f) if for any \(N \leq M\), the semi-radical of \(N\) is equal to the submodule generated by its envelope, that is, \(S - \text{rad}N = \{E(N)\}\). We already know that \(\langle E(N) \rangle \subseteq \text{rad}N\), by [4, P.1815]. Now let \(x \in E(N)\) and \(P\) be a semiprime submodule of \(M\) containing \(N\). Then \(x = ra\) for some \(r \in R, a \in M\) and for positive integer \(n, r^n a \in N\). But \(r^n a \in P\) and since \(P\) is semiprime we have \(ra \in P\). Hence \(E(N) \subseteq P\). We conclude that \(E(N) \subseteq \bigcap \langle P \rangle\) (\(P\) is a semiprime submodule containing \(N\)). So \(E(N) \subseteq S - \text{rad}N\). On the other hand, since every prime submodule of \(M\) is clearly semiprime, we have \(S - \text{rad}N \subseteq \text{rad}N\). We see that:
\[
\langle E(N) \rangle \subseteq S - \text{rad}N \subseteq \text{rad}N \quad \text{(11)}
\]

Now we present an \(R\)-module which satisfies the semi-radical formula.

**Theorem 3.9.** Let \(M\) be a finitely generated multiplication \(R\)-module. Then \(M\) satisfies the semi-radical formula.

**Proof.** Let \(N \leq M\), then by [4, Theorem 4.4], we have \(\langle E(N) : M \rangle = (\text{rad}N : M)\).

Hence \(\langle E(N) : M \rangle M = (\text{rad}N : M)M\) and since \(M\) is a multiplication \(R\)-module, \(\langle E(N) \rangle = \text{rad}N\).

Next from (4) we have:
\[
\langle E(N) \rangle M \subseteq (S - \text{rad}N : M)M \subseteq (\text{rad}N : M)M
\]
that is,
\[
\langle E(N) \rangle \subseteq S - \text{rad}N \subseteq \text{rad}N.
\]

Thus we find that \(S - \text{rad}N = \{E(N)\}\).

**Remark.** Under the conditions of Theorem 3.9, we see that for any submodule \(N \neq M\) of \(M\) we always have \(\text{Rad}N = S - \text{Rad}N\).

**Proposition 3.10.** Let \(M\) be a finitely generated \(R\)-module. Then the semi-radical of a proper submodule \(N\) of \(M\) is the intersection of its minimal semi-prime sub modules.

**Proof.** This is clear by using Theorem 3.5 and Proposition 3.6. For the rest of this section we state and prove some properties of semi-radical of sub modules.

**Theorem 3.11.** Let \(B\) and \(C\) be sub modules of an \(R\)-module \(M\). Then,
1. \(B \subseteq S - \text{rad}B\),
2. \(S - \text{rad}(S - \text{rad}B) = S - \text{rad}B\),
3. \(S - \text{rad}(B \cap C) \subseteq S - \text{rad}B \cap S - \text{rad}C\), and we have the equality when for every semiprime submodule \(P\), \(B \cap C \subseteq P\) implies that \(B \subseteq \text{Por}C \subseteq P\),
4. \(S - \text{rad}(B + C) = S - \text{rad}(S - \text{rad}B + S - \text{rad}C)\),
5. \((B : M) \subseteq (S - \text{rad}B : M)\),
6. If \(M\) is finitely generated, then \(S - \text{rad}B = M\) if and only if \(B = M\),
7. If \(M\) is finitely generated, then \(B + C = M\) if and only if \(S - \text{rad}B + S - \text{rad}C = M\),
8. \(S - \text{rad}I M = S - \text{rad}M^I\) for every ideal \(I\) of \(R\).

**Proof.** (1) clear.

(2) Since \(S - \text{Rad}B\) is semiprime by Proposition 2.10, we have:
\[
S - \text{Rad}(S - \text{Rad}B) = S - \text{Rad}B. \quad \text{(14)}
\]

(3) Let \(P\) be a semiprime submodule of \(M\) such that \(B \subseteq P\), so \(B \cap C \subseteq P\) and hence \(S - \text{rad}(B \cap C) \subseteq S - \text{rad}B\). By a similar argument we have \(S - \text{rad}(B \cap C) \subseteq S - \text{rad}C\). Now let \(P\) be a semiprime submodule of \(M\) such that \(B \cap C \subseteq P\) and assume that \(B \subseteq P\). Then \(S - \text{rad}B \subseteq P\) and so \(S - \text{rad}B \cap S - \text{rad}C \subseteq P\). Since \(P\) is arbitrary this implies that \(S - \text{rad}B \cap S - \text{rad}C \subseteq S - \text{rad}(B \cap C)\) and hence we have the equality.

(4) Let \(P\) be a semiprime submodule of \(M\) such that \((S - \text{rad}B + S - \text{rad}C) \subseteq P\). So \(S - \text{rad}B \subseteq P\) and \(S - \text{rad}C \subseteq P\). Hence \(B \subseteq P\) and \(C \subseteq P\) which implies \(B + C \subseteq P\). Therefore \(S - \text{rad}(B + C) \subseteq P\). But \(P\) is chosen arbitrary, so:
\[
S - \text{rad}(B + C) \subseteq S - \text{rad}(S - \text{rad}B + S - \text{rad}C). \quad \text{(15)}
\]

Now suppose that \(P\) be a semiprime submodule such that \(B + C \subseteq P\). So \(B \subseteq P\) and \(C \subseteq P\). Hence \(S - \text{rad}B \subseteq P\) and \(S - \text{rad}C \subseteq P\) and therefore \(S - \text{rad}B + S - \text{rad}C \subseteq P\). But \(S - \text{rad}(S - \text{rad}B + S - \text{rad}C) \subseteq P\) and we conclude that:
\[
S - \text{rad}(S - \text{rad}B + S - \text{rad}C) \subseteq S - \text{rad}(B + C). \quad \text{(16)}
\]
(5) If \( S - \text{rad}B = M \), then we have the result. So let \( P \) be a semiprime submodule of \( M \) such that \( B \subseteq P \). So \( (B : M) \subseteq (P : M) \). We know that \( (P : M) \) is a semiprime ideal of \( R \) and we have shown that \( \sqrt{(P : M)} = (P : M) \). Hence 
\[
\sqrt{(B : M)} \subseteq (P : M) \implies (P : M) P \subseteq P,
\]
and since \( P \) can be any semiprime submodule of \( M \) containing \( B \), we have \( (B : M) M \subseteq S - \text{rad}B \), that is, \( \sqrt{(B : M)} M \subseteq (S - \text{rad}M) : M \).

(6) If \( B = M \), then \( S - \text{rad}B = S - \text{rad}M = M \). Conversely, let \( S - \text{rad}B = M \), but \( B \neq M \). Since \( M \) is finitely generated, it contains a prime and so a semiprime submodule \( P \) containing \( B \), by Corollary after Proposition 4 of [3]. Hence \( S - \text{rad}B \neq M \), a contradiction.

(7) Using parts (4) and (6) we have:
\[
(B : M) M \subseteq S - \text{rad}B correlation.
\]
(8) If \( M \) has no semiprime submodule containing \( IM \), then \( S - \text{rad}IM = M \) and we have:
\[
I \subseteq S - \text{rad}IM \implies I M \subseteq S - \text{rad}IM \implies S - \text{rad}IM = M \implies S - \text{rad}IM.
\]

Now let \( P \) be a semiprime submodule of \( M \) such that \( IM \subseteq P \), so \( I \subseteq (M : P) \subseteq (P : M) \) and since \( (P : M) \) is semiprime \( \sqrt{I} \subseteq \sqrt{(P : M)} = (P : M) \). So \( \sqrt{I} \subseteq P \) and hence \( S - \text{rad}\sqrt{I} = P \). Since \( P \) is arbitrary we have:
\[
S - \text{rad}\sqrt{I} M = S - \text{rad}\sqrt{I} M.
\]

Proof. We know that \( \sqrt{I^n} \subseteq \sqrt{I} \) by part (8) of Theorem 3.11:
\[
S - \text{rad}\sqrt{I^n} M = S - \text{rad}\sqrt{I} M = S - \text{rad}\sqrt{I} M.
\]

Proposition 3.13. Let \( Q \) be a \( P \)-primary submodule of an \( R \)-module \( A \). Then \( S - \text{rad}Q = S - \text{rad}\langle Q + PA \rangle \).

Proof. We have \( Q \subseteq Q + PA \), so \( S - \text{rad}Q \subseteq S - \text{rad}\langle Q + PA \rangle \). Let \( S - \text{rad}Q = \bigcap_{i \in I} P_i \), where any \( P_i \) is a semiprime submodule of \( A \) containing \( Q \). We see that
\[
P = \sqrt{(Q : A)} \subseteq \sqrt{(P_i : A)} = (P_i : A) \]
implies \( PA \subseteq P_i \). So \( (Q + PA) \subseteq P_i \), for every \( i \in I \) and hence \( S - \text{rad}(Q + PA) \subseteq P_i \). Therefore \( S - \text{rad}(Q + PA) \subseteq \bigcap_{i} P_i \) and so \( S - \text{rad}Q = \bigcap_{i} P_i \).

Definition 3.14. Let \( N \) be a semiprime submodule of an \( R \)-module \( M \), and let \( P = \sqrt{(N : M)} = (N : M) \). We call \( N \) a \( P \)-semiprime submodule of \( M \), if \( P \) is prime ideal of \( R \).

Lemma 3.15. Let \( M \) be a finitely generated \( R \)-module and let \( K \) be a maximal ideal of \( R \). If \( Q \) is a \( K \)-primary submodule of \( M \), then \( S - \text{rad}Q \) is a \( K \)-semiprime sub module.

Proof. By Theorem 3.11, part (5), we have \( K = \sqrt{(Q : M)} \subseteq (S - \text{rad}Q : M) \).

But \( K \) is a maximal ideal of \( R \), so \( (S - \text{rad}Q : M) = R \) or \( (S - \text{rad}Q : M) = K \). If \( (S - \text{rad}Q : M) = R \) then \( S - \text{rad}Q = M \) and by Theorem 3.11, part (6) we have \( Q = M \) which is a contradiction since \( Q \) is primary. Hence \( (S - \text{rad}Q : M) = K \) and since \( S - \text{rad}Q \) is an intersection of semiprime sub modules containing \( Q \) it is semiprime and in fact \( K \)-semiprime.

Proposition 3.16. Let \( N_1, N_2, \ldots, N_I \) be \( P \)-semiprime sub modules of an \( R \)-module \( M \). Then \( N = N_1 \cap N_2 \cap \cdots \cap N_I \) is also \( P \)-semiprime.

Proof. By Proposition 2.10, \( N \) is semiprime and we have:
\[
(N : M) = (N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_I : M) = \bigcap_{i} (N_i : M).
\]

Therefore \( N = P \cap \cdots \cap P = P \). Therefore \( N \) is \( P \)-semiprime.

Lemma 3.17. Let \( M \) be a multiplication \( R \)-module and \( L \), \( N \) be submodules of \( M \). Also let \( K \) be a prime ideal of \( R \) and \( P \) be a \( K \)-semiprime submodule of \( M \) such that \( N \cap L \subseteq P \). If \( (N : M) \subseteq K \) then \( L \subseteq P \).

Proof. We have \( N \cup L \subseteq P \Rightarrow (N \cap L : M) \subseteq (P : M) = K \Rightarrow (N : M) \cap (L : M) \subseteq K \).

and since \( K \) is a prime ideal of \( R \), \( (N : M) \subseteq K \) or \( (L : M) \subseteq K \). Since \( (N : M) \subseteq K \), we find that \( (L : M) \subseteq K \). From this we conclude that \( (L : M) M \subseteq KM \), that is, \( L \subseteq KM \). But \( (P : M) = K \) implies that \( KM \subseteq P \). Therefore \( L \subseteq KM \subseteq P \).
4. Conclusion

In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

References


