SEMI-RADICALS OF SUB MODULES IN MODULES

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Abstract: Let $R$ be a commutative ring and $M$ be a unitary $R$-module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula $(s.t.s.r.f)$ and present the existence of such a module.

Keywords: Prime sub module, semiprime sub module, radical and semi-radical of a module, modules satisfying the semi-radical formula.

1. Introduction

In this paper all the rings are commutative with identity and all the modules are unitary. Let $R$ be a ring and $M$ be an $R$-module. If $N$ is a submodule of $M$ we use the notation $N \subseteq M$. If the submodule $N$ is generated by a subset $S$ of $M$, we write $N = \langle S \rangle$.

If $N$ and $K$ are submodules of $M$, then the set $\{ r \in R | rK \subseteq N \}$ is denoted by $(N :_R K)$ or simply by $(N : K)$ which is clearly an ideal of $R$. If $I$ is an ideal of the ring $R$, we write $I \subseteq R$. In Section 2 we define prime and primary sub modules of an $R$-module $M$ and in Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if $N$ is a semiprime submodule of an $R$-module $M$, then $(N : M)$ is a semiprime ideal of $R$ but not conversely in general. In Lemma 2.8 we prove that the converse is also true if $M$ is a multiplication module. In Section 3 we define radical of an $R$-module $M$ and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule of a module and also modules satisfying the semi-radical formula which is abbreviated as $(s.t.s.r.f)$ and in Theorem 3.9 we show that such a module does exist.

Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a $P$-semiprime submodule we consider some of its properties.

2. Some Elementary Results

We begin this section with the following definitions:

Definition 2.1. Let $N$ be a proper submodule of an $R$-module $M$.

(a) $N$ is called a prime submodule of $M$ if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r \in (N : M)$.

(b) $N$ is called a primary submodule of $M$ if for any $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r^n \in (N : M)$ for some positive integer $n$.

In (a) it can easily be shown that $P = (N : M)$ is a prime ideal of $R$ and we say that $N$ is $P$-prime.

We recall that if $I$ is an ideal of a ring $R$, then the radical of $I$, denoted by $\sqrt{I}$, is defined as the intersection of all prime ideals containing $I$. Alternatively, we define the radical of $I$ as:

$\sqrt{I} = \{ r \in R | r^n \in I$ for some positive integer $n \}$.

Also if $N$ is a primary submodule of an $R$-module $M$, then $(N : M)$ is a primary ideal of $R$ and $P = \sqrt{(N : M)}$ is a prime ideal. We describe this situation by saying that $N$ is $P$-primary.

Lemma 2.2. Let $N$ be a proper submodule of an $R$-module $M$.

(i) $N$ is a prime submodule of $M$ if and only if $ID \subseteq N$ (with $I$ an ideal of $R$ and $D$ a submodule of $M$) implies that $D \subseteq N$ or $I \subseteq (N : M)$.

(ii) $N$ is a primary submodule of $M$ if and only if for every finitely generated ideal $I$ of $R$ and any submodule $D$ of $M$, $ID \subseteq N$ implies that $D \subseteq N$ or $I^n \subseteq (N : M)$ for some positive integer $n$.

(iii) Let $P$ be a prime ideal of $R$, then $N$ is a $P$-primary submodule of $M$ if and only if (a)
and (b) $cm \notin N$ for all $c \in R \setminus P$, $m \in M \setminus N$.

**Proof.** (i) $(\Rightarrow)$: Let $I \subseteq R$ and $D \subseteq M$ be such that $ID \subseteq N$ and let $D \subseteq N$. So there exists an element $x \in D \cap N$. Let $r$ be any element of $I$. Then $rx \in N$ and hence $r \in (N : M)$. Therefore $I \subseteq (N : M)$.

$(\Leftarrow)$: Let $r \in R$, $a \in M$ be such that $ra \in N$ and let $a \notin N$. By taking: $I = (r)$ and $D = Ra$ we see that $ID \subseteq N$. But $D \subseteq N$ and hence $I \subseteq (N : M)$.

which implies that $r \in (N : M)$. Therefore $N$ is a prime submodule of $M$.

(ii) $(\Rightarrow)$: Let $D \subseteq M$ and $I$ be a finitely generated ideal of $R$ such that $ID \subseteq N$.

Then by [5, Corollary 1, P.99], $D \subseteq N$ or $I \subseteq \sqrt{N : M}$. Let $D \subseteq N$, then $I \subseteq \sqrt{N : M}$ and by [5, Proposition 8, P.83], there exists a positive integer $n$ such that $I^n \subseteq (N : M)$.

$(\Leftarrow)$: Let $r \in R, x \in M$ be such that $rx \in N$ and let $x \notin N$. By taking $I = (r)$ and $D = Rx$ we see that $ID \subseteq N$ and $D \subseteq N$. So there exists a positive integer $n$ such that $I^n \subseteq (N : M)$. This implies that $r^n \in (N : M)$ and hence $N$ is a primary submodule of $M$.

(iii) $(\Rightarrow)$: If $N$ is $P$–primary, then by definition $P = \sqrt{(N : M)}$. Now let $c \in R \setminus P$ and $m \in M \setminus N$. Let $cm \in N$, then there exists a positive integer $n$ such that:

c^n \in (N : M),
	hat is, $c \in \sqrt{(N : M)} = P$ (because $m \notin N$), a contradiction. Hence $cm \notin N$.

$(\Leftarrow)$: Assume that (a), (b) hold. Let $r \in R$ and $m \in M$, $rm \in N$. Assume further that $m \notin N$, then by (b), $r$ must belong to $P$ and so $r \in \sqrt{(N : M)}$ by (a). Therefore $N$ is a primary submodule of $M$. Next we must show that $P = \sqrt{(N : M)}$.

Let $r \in \sqrt{(N : M)}$, then $r^n \in (N : M)$ for some positive integer $n$, and so $r^nM \subseteq N$. Since $N$ is proper, there exist $x \in M \setminus N$. Now $r^n x \in N$ and $x \notin N$ so by (b) we conclude that $r^n \in P$ and, as $P$ is prime, $r \in P$. We find that $\sqrt{(N : M)} = P$ and therefore $N$ is $P$–primary.

The following definition is essential in the rest of the paper.

**Definition 2.3.** A proper submodule $N$ of an $R$–module $M$ is said to be semiprime in $M$, if for every ideal $I$ of $R$ and every submodule $K$ of $M$, $I^2K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring $R$ is an $R$–module by itself, a proper ideal $I$ of $R$ is semiprime if for every ideals $J$ and $K$ of $R$, $J^2K \subseteq I$ implies that $JK \subseteq I$.

**Proposition 2.4.** Let $M$ be an $R$–module.

(i) If $N$ is a prime submodule of $M$, then $N$ is semiprime.

(ii) If $N$ is a semiprime submodule of $M$, then $(N : M)$ is semiprime ideal of $R$.

**Proof.** (i) Let $I \subseteq R$, $K \subseteq M$ and $I^2K \subseteq M$. Then $I(IK) \subseteq N$ and since $N$ is prime, $I \subseteq (N : M)$ or $IK \subseteq N$. But $(N : M) \subseteq (N : K)$ and hence $I \subseteq (N : K)$, and so $IK \subseteq N$. In any case we see that $IK \subseteq N$, and therefore $N$ is semiprime.

(ii) Let $J$ and $K$ be ideals of $R$ and $J^2K \subseteq (N : M)$. Hence $(J^2K)M \subseteq N$, and so, $J^2(KM) \subseteq N$. But $KM \subseteq M$, and $N$ is semiprime, therefore $(J(KM)) \subseteq N$, and thus, $(JK)M \subseteq N$. Hence $JK \subseteq (N : M)$ and we conclude that $(N : M)$ is a semiprime ideal of $R$.

Part (i) of the above proposition implies that if $P$ is a prime ideal of $R$ then $P$ is semiprime. In the next example we show that the converse of part (ii) of Proposition 2.1. is not valid in general.

**Example 2.5.** Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$ and $B = \langle(9,0)\rangle$.

Then it is clear that $(B : M) = \langle 0 \rangle$. Since $Z$ is an integral domain, $(B : M) = \langle 0 \rangle$ is a prime ideal and hence a semiprime ideal of $Z$. But $B$ is not a semiprime submodule of $M$; because if we take $I = \langle 3 \rangle$ and $K = \langle 2, 0 \rangle$, Then:

$I^2K = \{(18q,0) \mid q \in \mathbb{Z}\}$

But:

$I = \{(6q,0) \mid q \in \mathbb{Z}\}$

is not a subset of $B$.

It is clear that if $N$ is a semiprime submodule of an $R$–module $M$ and $I \subseteq R$, $K \subseteq M$ be such that $I^nK \subseteq N$ for some positive integer $n$, then $IK \subseteq N$.

**Theorem 2.6.** Let $N$ be a proper submodule of an $R$–module $M$. Then the following statements are equivalent:

(i) $N$ is semiprime.

(ii) Whenever $r \neq 0$ for some $r \in R$, $m \in M$ and $t \in \mathbb{Z}^+$, then $rm \in N$. 


Proof. (i) \( \Rightarrow \) (ii). Let \( r^i m \in N \) where \( r \in R \), \( m \in M \) and \( t \in \mathbb{Z}^+ \). Taking \( I = (r) \) and \( K = (m) \) we have \( r^i K \subseteq N \) and so \( IK \subseteq N \) which implies that \( rm \in N \).

(ii) \( \Rightarrow \) (i). Let \( I \subseteq R \) and \( K \subseteq M \) be such that \( I^2 K \subseteq N \). Consider the set:

\[
S = \{ ra | r \in I, a \in K \}
\]

Then for every \( r \in I, a \in K \) we have \( r^2a \in I^2K \subseteq N \) and hence \( ra \in N \). This implies that \( S \subseteq N \) and since \( IK \) is the submodule of \( M \) generated by \( S \), we must have \( IK \subseteq N \). Therefore \( N \) is semiprime.

Definition 2.7. An \( R \)-module \( M \) is said to be a multiplication module if for each submodule \( N \) of \( M \), \( N = IM \) for some ideal \( I \) of \( R \).

It can be easily shown that, an \( R \)-module \( M \) is a multiplication module if and only if \( N = (N : M)M \) for every submodule \( N \) of \( M \).

Lemma 2.8. Let \( M \) be a multiplication \( R \)-module. Then a submodule \( N \) of \( M \) is semiprime if and only if \( (N : M)M \) for every submodule \( N \) of \( M \).

Proof. \( \Rightarrow \): This is clear from Proposition 2.4 (ii).

\( \Leftarrow \): Let \( I \subseteq R \), \( K \subseteq M \), be such that \( I^2K \subseteq N \). Hence:

\[
(I^2 K : M) \subseteq (N : M)
\]

(4)

It can be shown that:

\[
I^2K : M \subseteq I^2K : M
\]

(5)

and so we obtain:

\[
I^2K : M \subseteq (N : M).
\]

(6)

But \( (N : M) \) is a semiprime ideal of \( R \) and hence \( I(K : M) \subseteq (N : M) \). Thus we conclude that:

\[
I(K : M)M \subseteq (N : M)M,
\]

(7)

and using the fact that \( M \) is a multiplication \( R \)-module we have \( IK \subseteq N \). Therefore \( N \) is a semiprime submodule of \( M \).

The following lemma shows that the same situation, as above, holds for prime and primary submodules.

Lemma 2.9. Let \( M \) be a multiplication \( R \)-module. Then:

(a) A submodule \( N \) of \( M \) is prime if and only if \( (N : M) \) is a prime ideal of \( R \).

(b) A submodule \( N \) of \( M \) is primary if and only if \( (N : M) \) is a primary ideal of \( R \).

Proof. (a) \( \Rightarrow \) : Clear.

\( \Leftarrow \) : Let \( I \subseteq R \), \( D \subseteq M \) be such that \( ID \subseteq N \), then \( (ID : M) \subseteq (N : M) \). But \( (D : M) \subseteq (ID : M) \) and so \( (D : M) \subseteq (N : M) \). Since \( (N : M) \) is a prime ideal of \( R \) we have \( I \subseteq (N : M) \) or \( D \subseteq (N : M) \).

Suppose that \( I \subseteq (N : M) \). Then \( (D : M) \subseteq (N : M) \) and using the fact that \( (D : M)M \subseteq (N : M)M \), that is, \( D \subseteq N \). Hence \( N \) is a prime submodule of \( M \) by Lemma 2.2 (i).

(b) \( \Rightarrow \) : Clear.

\( \Leftarrow \) : Let \( (N : M) \) be a primary submodule of \( R \). Let \( I \) be a finitely generated ideal of \( R \) and \( D \) be a submodule of \( M \) and let \( ID \subseteq N \). Suppose that for any positive integer \( n \), \( I^2 \subseteq (N : M) \). We see that \( ID \subseteq N \) implies \( (ID : M) \subseteq (N : M) \) and hence \( I(D : M) \subseteq (N : M) \). But \( I^2 \subseteq (N : M) \) for any positive integer \( n \), so \( (D : M) \subseteq (N : M) \), because \( (N : M) \) is a primary. Hence \( (D : M)M \subseteq (N : M)M \), that is, \( D \subseteq N \). So \( N \) is a primary submodule of \( M \), by Lemma 2.2 (ii), the proof is now complete.

Proposition 2.10. Let \( \{ P_i \} \subseteq I \) be a non-empty family of semiprime submodules of an \( R \)-module \( M \). Then \( P = \bigcap P_i \) is a semiprime submodule of \( M \).

Further if \( \{ P_i \} \subseteq I \) is totally ordered (by inclusion), then \( T = \bigcap P_i \) is also a semiprime submodule whenever \( T \neq M \).

Proof. Let \( I \subseteq R \) and \( K \subseteq M \) be such that \( I^2k \subseteq P = \bigcap P_i \). Then \( I^2k \subseteq P_i \) for every \( i \in I \), and since \( P_i \) is semiprime, we have \( Ik \subseteq P_i \). Hence \( IK \subseteq \bigcap P_i = P \) and \( P \) is semiprime. Next we let \( T = \bigcap P_i \neq M \). The fact that \( \{ P_i \} \subseteq I \) is totally ordered by inclusion makes it clear that \( T \) is a submodule of \( M \). Let \( I \subseteq R \) and \( K \subseteq M \) be such that \( I^2K \subseteq T \). Consider the set:

\[
S = \{ rk | r \in R, k \in K \}
\]

(8)

Then \( S \) is a generating set for the submodule \( IK \). If \( r \in I \), \( k \in K \) then \( r^2k \in I^2K \subseteq T \) and so for some \( i \in I \), \( r^2k \in P_i \). Since \( P_i \) is semiprime it implies that \( rk \in P_i \). It follows that \( S \subseteq T \) and hence \( IK = \langle S \rangle \subseteq T \). Therefore \( T \) is also a semiprime submodule of \( M \).

Remark. Some authors define a semiprime submodule as an intersection of prime submodules. But by our
definition of a semiprime submodule of a module we can find a semiprime submodule of a module which is not an intersection of prime sub modules, for example look at J. Jenkins and P.F. Smith in the proof of [1].

3. Radicals and Semi-Radicals

Let $M$ be an $R$-module and $N$ a submodule of $M$. If there exists a prime submodule of $M$ which contain $N$, then the intersection of all prime sub modules containing $N$, is called the $M$-radical of $M$ and is denoted by $\text{rad}_M N$, or simply by $\text{rad}_N M$. If there is no prime submodule containing $N$, then we define $\text{rad}_M N = M$; in particular $\text{rad}_M M = M$. An ideal $I$ of a ring $R$ is called a radical ideal if $\sqrt{I} = I$. Similarly, we say that a submodule $B$ of an $R$-module $M$ is a radical submodule if $\text{rad} B = B$. It is easy to see that an ideal $I$ of a ring $R$ is semiprime if and only if it is radical. Because, let $I$ be semiprime, and let $x \in \sqrt{I}$. Then $x^k \in I$ for some positive integer $k$. So $x^k 1 \in I$, and since $I$ is semiprime we have $x1 = x \in I$. Therefore $I = \sqrt{I}$.

On the other hand, if $I = \sqrt{I}$ then by definition of $\sqrt{I}$ and Propositions 2.4 (i) and 2.10, $I$ is semiprime. Finally by Propositions 2.4 (i) and 2.10 we see that for any submodule $B$ of an $R$-module $M$, $\text{rad} B$ is a semiprime submodule whenever $\text{rad} B \neq M$.

Theorem 3.1. Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if it is radical.

Proof. Since $\text{ann}_R(M) \subseteq (N : M)$, by [2, Theorem 3, P.216],
\[
\sqrt{(N : M)M} = \text{rad}(N : M)M.
\]

As $M$ is a multiplication module we have $(N : M)M = M$, and if $N$ is semiprime, $(N : M)$ is a radical ideal. Therefore $\sqrt{(N : M)M} = \text{rad}(N : M)M$ if and only if $(N : M)M = \text{rad}(N : M)M$. If $N = \text{rad} N$ this implies that $N$ is a radical submodule of $M$, that is, $N = \text{rad} N = \bigcap P (P$ is a prime submodule of $M$ containing $N$). Hence by Propositions 2.4 (1) and 2.10 $N$ is a semiprime submodule of $M$. The proof is now complete.

After Remark 2.11 we may ask under what condition a semiprime submodule is the intersection of prime submodules containing it. The following corollary can be considered as an answer.

Corollary 3.2. Let $M$ be a finitely generated multiplication $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is semiprime if and only if $N = \bigcap P (P$ is a prime submodule of $M$ containing $N$).

Proof. ($\Rightarrow$): If $N$ is semiprime then by Theorem 3.1, it is radical, that is, $N = \bigcap P (P$ a prime submodule of $M$ containing $N$).

($\Leftarrow$): By Propositions 2.4 (i) and 2.10, $N$ is semiprime.

Proposition 3.3. If $M$ is a finitely generated $R$-module, then every proper submodule of $M$ is contained in a semiprime submodule.

Proof. By Corollary of [3, Proposition 4, P.63], every proper submodule of $M$ is contained in a prime submodule. So by Proposition 2.4 (i), we have the result.

Definition 3.4. (1) A semiprime submodule $P$ of an $R$-module $M$ is called a minimal semiprime submodule of a proper submodule $N$ if $N \subseteq P$ and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of $0 = < 0_M >$ is called a minimal semiprime submodule of $M$.

Theorem 3.5. Let $M$ be an $R$-module. If a submodule $N$ of $M$ is contained in a semiprime submodule $P$, then $P$ contains a minimal semiprime submodule of $N$.

Proof. It is similar to the proof of [5, Theorem 4, P.84].

Proposition 3.6. Every proper submodule of a finitely generated $R$-module $M$ possesses at least one minimal semiprime submodule of $M$.

Proof. Let $N$ be a proper submodule of $M$, then by Proposition 3.3, $N$ is contained in a semiprime submodule of $M$.

Corollary 3.7. Every semiprime submodule of an $R$-module $M$ contains at least one minimal semiprime submodule of $M$.

Proof. Let $P$ be a semiprime submodule of $M$ and take $N = < 0 >$ in the Theorem 3.5. Then $P$ contains a minimal semiprime submodule of $< 0 >$, and so a minimal semiprime submodule of $M$.

Definition 3.8. Let $M$ be an $R$-module and $N \leq M$. If there exists a semiprime submodule of $M$ which contains $N$, then the intersection of all semiprime sub modules containing $N$ is called the semi-radical of $N$ and is denoted by $S_{\text{rad}} M N$, or simply by $S_{\text{rad}} N$. If there is no semiprime submodule containing $N$, then we define

\[
S_{\text{rad}} N = N.
\]
$S - \operatorname{rad}N = M$, in particular $S - \operatorname{rad}M = M$. We call $S - \operatorname{rad}\{0\}$ the semi-prime radical of $M$.

If $N \leq M$, then the envelope of $N$, denoted by $E(N)$, is defined as:

$$E(N) = \left\{ x \in M \mid x = ra \text{ for some } r \in R, a \in M \text{ and } r^n a \in N \text{ for some } n \in \mathbb{Z}^+ \right\}.$$  (10)

We say that $M$ satisfies the semi-radical formula, $M$ (s.t.s.r.f) if for any $N \leq M$, the semi-radical of $N$ is equal to the submodule generated by its envelope, that is, $S - \operatorname{rad}N = \langle E(N) \rangle$. We already know that $\langle E(N) \rangle \subseteq \operatorname{rad}N$, by [4, P.1815]. Now let $x \in E(N)$ and $P$ be a semi-prime submodule of $M$ containing $N$. Then $x = ra$ for some $r \in R, a \in M$ and for positive integer $n, r^n a \in N$. But $r^n a \in P$ and since $P$ is semiprime we have $r a \in P$. Hence $E(N) \subseteq P$. We conclude that $E(N) \subseteq \bigcap P$ ($P$ is a semiprime submodule containing $N$). So $E(N) \subseteq S - \operatorname{rad}N$. On the other hand, since every prime submodule of $M$ is clearly semiprime, we have $S - \operatorname{rad}N \subseteq radN$. We see that:

$$\langle E(N) \rangle \subseteq S - \operatorname{rad}N \subseteq radN$$  (11)

Now we present an $R$-module which satisfies the semi-radical formula.

**Theorem 3.9.** Let $M$ be a finitely generated multiplication $R$-module. Then $M$ satisfies the semi-radical formula.

**Proof.** Let $N \leq M$, then by [4, Theorem 4.4], we have $\langle (E(N)) : M \rangle = (\operatorname{rad}N : M)$.

Hence $\langle (E(N)) : M \rangle M = (\operatorname{rad}N : M)M$ and since $M$ is a multiplication $R$-module, $\langle E(N) \rangle = \operatorname{rad}N$.

Next from (*) we have:

$$\langle E(N) \rangle : M \subseteq S - \operatorname{rad}N : M \subseteq \operatorname{rad}N : M$$  (12)

that is,

$$\langle E(N) \rangle \subseteq S - \operatorname{rad}N \subseteq radN.$$  (13)

Thus we find that $S - \operatorname{rad}N = \langle E(N) \rangle$.

**Remark.** Under the conditions of Theorem 3.9, we see that for any submodule $N \neq M$ of $M$ we always have $\operatorname{Rad}N = S - \operatorname{Rad}N$.

**Proposition 3.10.** Let $M$ be a finitely generated $R$-module. Then the semi-radical of a proper submodule $N$ of $M$ is the intersection of its minimal semi-prime submodules.

**Proof.** This is clear by using Theorem 3.5 and Proposition 3.6.

For the rest of this section we state and prove some properties of semi-radical of sub modules.

**Theorem 3.11.** Let $B$ and $C$ be sub modules of an $R$-module $M$. Then,

1. $B \subseteq S - \operatorname{rad}B$.
2. $S - \operatorname{rad}(S - \operatorname{rad}B) = S - \operatorname{rad}B$.
3. $S - \operatorname{rad}(B \cap C) = S - \operatorname{rad}B \cap S - \operatorname{rad}C$.
4. $S - \operatorname{rad}(B + C) = S - \operatorname{rad}(S - \operatorname{rad}B + S - \operatorname{rad}C)$.
5. $\sqrt{(B : M)} = (S - \operatorname{rad}B : M)$.
6. If $M$ is finitely generated, then $S - \operatorname{rad}B = M$ if and only if $B = M$.
7. If $M$ is finitely generated, then $B + C = M$ if and only if $S - \operatorname{rad}B + S - \operatorname{rad}C = M$.
8. $S - \operatorname{rad}M = S - \operatorname{rad}\sqrt{IM}$ for every ideal $I$ of $R$.

**Proof.** (1) clear.

(2) Since $S - \operatorname{Rad}B$ is semiprime by Proposition 2.10, we have:

$$S - \operatorname{Rad}(S - \operatorname{Rad}B) = S - \operatorname{Rad}B.$$  (14)

(3) Let $P$ be a semiprime submodule of $M$ such that $B \subseteq P$, so $B \cap C \subseteq P$ and hence $S - \operatorname{rad}(B \cap C) \subseteq P$. But $P$ is arbitrary, therefore $S - \operatorname{rad}(B \cap C) \subseteq \operatorname{S - rad}B$. By a similar argument we have $S - \operatorname{rad}(B \cap C) \subseteq S - \operatorname{rad}C$. Now let $P$ be a semiprime submodule of $M$ such that $B \cap C \subseteq P$ and assume that $B \subseteq P$. Then $S - \operatorname{rad}B \subseteq P$ and so $S - \operatorname{rad}(B \cap C) \subseteq P$. Since $P$ is arbitrary this implies that $S - \operatorname{rad}(B \cap C) \subseteq S - \operatorname{rad}(B \cap C)$ and hence we have the equality.

(4) Let $P$ be a semiprime submodule of $M$ such that $(S - \operatorname{rad}B + S - \operatorname{rad}C) \subseteq P$. So $S - \operatorname{rad}B \subseteq P$ and $S - \operatorname{rad}C \subseteq P$. Hence $B \subseteq P$ and $C \subseteq P$ which implies $B + C \subseteq P$. Therefore $S - \operatorname{rad}(B + C) \subseteq P$. But $P$ is chosen arbitrary, so:

$$S - \operatorname{rad}(B + C) \subseteq S - \operatorname{rad}(S - \operatorname{rad}B + S - \operatorname{rad}C).$$  (15)

Now suppose that $P$ be a semiprime submodule such that $B + C \subseteq P$. So $B \subseteq P$, and $C \subseteq P$. Hence $S - \operatorname{rad}B \subseteq P$ and $S - \operatorname{rad}C \subseteq P$ and therefore $S - \operatorname{rad}B + S - \operatorname{rad}C \subseteq P$.

But $S - \operatorname{rad}(S - \operatorname{rad}B + S - \operatorname{rad}C) \subseteq P$ and we conclude that:

$$S - \operatorname{rad}(S - \operatorname{rad}B + S - \operatorname{rad}C) \subseteq S - \operatorname{rad}(B + C).$$  (16)
(5) If \( S - \text{rad} B = M \), then we have the result. So let \( P \) be a semiprime submodule of \( M \) such that \( B \subseteq P \). So \( (B : M) \subseteq (P : M) \). We know that \( (P : M) \) is a semiprime ideal of \( R \) and we have shown that \( \sqrt{(P : M)} = (P : M) \). Hence

\[
\sqrt{(B : M)} \subseteq \sqrt{(P : M)} = (P : M)
\]

implies that:

\[
\sqrt{(B : M)} M \subseteq (P : M) P \subseteq P,
\]

and since \( P \) can be any semiprime submodule of \( M \) containing \( B \), we have \( \sqrt{(B : M)} M \subseteq S - \text{rad} B \), that is, \( \sqrt{(B : M)} M \subseteq (S - \text{rad} B) \).

(6) If \( S - \text{rad} B = S - \text{rad} M = M \). Conversely, let \( S - \text{rad} B = M \), but \( B \not\subseteq M \). Since \( M \) is finitely generated, it contains a prime and so a semiprime submodule \( P \) containing \( B \), by Corollary after Proposition 4 of [3]. Hence \( S - \text{rad} B \neq M \), a contradiction.

(7) Using parts (4) and (6) we have:

\[
\text{if } S - \text{rad} (B + C) = M \text{ iff } S - \text{rad} (S - \text{rad} B + S - \text{rad} C) = M
\]

iff \( S - \text{rad} B + S - \text{rad} C = M \).

(8) If \( M \) has no semiprime submodule containing \( IM \), then \( S - \text{rad} IM = M \) and we have:

\[
I \subseteq \sqrt{I} \Rightarrow IM \subseteq \sqrt{IM} \Rightarrow S - \text{rad} IM \subseteq S - \text{rad} \sqrt{IM}
\]

\[
\Rightarrow M \subseteq S - \text{rad} \sqrt{IM} \Rightarrow M = S - \text{rad} \sqrt{IM}.
\]

Now let \( P \) be a semiprime submodule of \( M \) such that \( IM \subseteq P \), so \( I \subseteq (IM : M) \subseteq (P : M) \) and since \( (P : M) \) is semiprime \( \sqrt{I} \subseteq \sqrt{(P : M)} = (P : M) \).

So \( \sqrt{IM} \subseteq P \) and hence \( S - \text{rad} \sqrt{IM} \subseteq P \). Since \( P \) is arbitrary we have:

\[
S - \text{rad} \sqrt{IM} \subseteq S - \text{rad} IM.
\]

Therefore \( S - \text{rad} IM = S - \text{rad} \sqrt{IM} \). The proof is now complete.

Corollary 3.12. Let \( M \) be an \( R \)-module and \( I \) an ideal of \( R \). Then \( S - \text{rad}^n M = S - \text{rad} M \) for every positive integer \( n \).

Proof. We know that \( \sqrt{I^n} = \sqrt{I} \). so by part (8) of Theorem 3.11:

\[
S - \text{rad}^n M = S - \text{rad} \sqrt{I^n} M = S - \text{rad} \sqrt{I} M = S - \text{rad} IM.
\]

Proposition 3.13. Let \( Q \) be a \( P \)-primary submodule of an \( R \)-module \( A \). Then \( S - \text{rad} Q = S - \text{rad} (Q + PA) \).

Proof. We have \( Q \subseteq Q + PA \). So \( S - \text{rad} Q \subseteq S - \text{rad} (Q + PA) \). Let \( S - \text{rad} Q = \bigcap_{i \in I} P_i \). where any \( P_i \) is a semiprime submodule of \( A \) containing \( Q \). We see that

\[
P = \sqrt{(Q : A)} \subseteq \sqrt{(P_i : A)} = (P_i : A)
\]

implies \( PA \subseteq P_i \). So \( (Q + PA) \subseteq P_i \), for every \( i \in I \) and hence \( S - \text{rad} (Q + PA) \subseteq P_i \). Therefore \( S - \text{rad} (Q + PA) \subseteq \bigcap_{i \in I} P_i \) and so \( S - \text{rad} Q = S - \text{rad} (Q + PA) \).

Definition 3.14. Let \( N \) be a semiprime submodule of an \( R \)-module \( M \), and let \( P = \sqrt{(N : M)} = (N : M) \).

We call \( N \) a \( P \)-semiprime submodule of \( M \), if \( P \) is prime ideal of \( R \).

Lemma 3.15. Let \( M \) be a finitely generated \( R \)-module and let \( K \) be a maximal ideal of \( R \). If \( Q \) is a \( K \)-primary submodule of \( M \), then \( S - \text{rad} Q \) is a \( K \)-semiprime submodule.

Proof. By Theorem 3.11, part (5), we have \( K = \sqrt{(Q : M)} \subseteq (S - \text{rad} Q : M) \).

But \( K \) is a maximal ideal of \( R \), so \( (S - \text{rad} Q : M) = R \) or \( (S - \text{rad} Q : M) = K \). If \( (S - \text{rad} Q : M) = R \) then \( S - \text{rad} Q = M \) and by Theorem 3.11, part (6) we have \( Q = M \) which is a contradiction since \( Q \) is primary. Hence \( (S - \text{rad} Q : M) = K \) and since \( S - \text{rad} Q \) is an intersection of semiprime submodules containing \( Q \) it is semiprime and in fact \( K \)-semiprime.

Proposition 3.16. Let \( N_1, N_2, \ldots, N_t \), be \( P \)-semiprime submodules of an \( R \)-module \( M \). Then \( N = N_1 \cap N_2 \cap \cdots \cap N_t \) is also \( P \)-semiprime.

Proof. By Proposition 2.10, \( N \) is semiprime and we have:

\[
(N : M) = (N_1 : M) \cap (N_2 : M) \cap \cdots \cap (N_t : M) = (N_1 \cap N_2 \cap \cdots \cap N_t : M)
\]

\[
= P \cap \cdots \cap P = P.
\]

Therefore \( N \) is \( P \)-semiprime.

Lemma 3.17. Let \( M \) be a multiplication \( R \)-module and \( L, N \) be submodules of \( M \). Also let \( K \) be a \( K \)-prime ideal of \( R \) and \( P \) be a \( K \)-semiprime submodule of \( M \) such that \( N \cap L \subseteq P \). If \( (N : M) \not\subseteq K \) then \( L \subseteq P \).

Proof. We have \( N \cup L \subseteq P \Rightarrow (N \cup L : M) \subseteq (P : M) \Rightarrow (N : M) \cap (L : M) \subseteq K \).

and since \( K \) is a \( K \)-prime ideal of \( R \), \( (N : M) \subseteq K \) or \( (L : M) \subseteq K \). Since \( (N : M) \subseteq K \), we find that \( (L : M) \subseteq K \). From this we conclude that \( (L : M) M \subseteq KM \), that is, \( L \subseteq KM \). But \( (P : M) = K \) implies that \( KM \subseteq P \). Therefore \( L \subseteq KM \subseteq P \).
4. Conclusion
In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

References


