

# SEMI-RADICALS OF SUB MODULES IN MODULES

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**Abstract:** Let  $R$  be a commutative ring and  $M$  be a unitary  $R$ -module. We define a semiprime submodule of a module and consider various properties of it. Also we define semi-radical of a submodule of a module and give a number of its properties. We define modules which satisfy the semi-radical formula (s.t.s.r.f) and present the existence of such a module.

**Keywords:** Prime sub module, semiprime sub module, radical and semi-radical of a module, modules satisfying the semi-radical formula.

## 1. Introduction

In this paper all the rings are commutative with identity and all the modules are unitary. Let  $R$  be a ring and  $M$  be an  $R$ -module. If  $N$  is a submodule of  $M$  we use the notation  $N \leq M$ . If the submodule  $N$  is generated by a subset  $S$  of  $M$ , we write  $N = \langle S \rangle$ .

If  $N$  and  $K$  are sub modules of  $M$ , then the set  $\{r \in R \mid rK \subseteq N\}$  is denoted by  $(N :_R K)$  or simply by  $(N : K)$  which is clearly an ideal of  $R$ . If  $I$  is an ideal of the ring  $R$ , we write  $I \trianglelefteq R$ . In Section 2 we define prime and primary sub modules of an  $R$ -module  $M$  and in Lemma 2.2, we give equivalent definitions for prime and primary sub modules. Then we present our essential definition, that is, we define semiprime sub modules of a module. Various properties of semiprime sub modules are discussed. We have shown that if  $N$  is a semiprime submodule of an  $R$ -module  $M$ , then  $(N : M)$  is a semiprime ideal of  $R$  but not conversely in general. In Lemma 2.8 we prove that the converse is also true if  $M$  is a multiplication module. In Section 3 we define radical of an  $R$ -module  $M$  and Theorem 3.1, shows that a submodule of a finitely generated multiplication module is semiprime if and only if it is radical. Next we define semi-radical of a submodule of a module and also modules satisfying the semi-radical formula which is abbreviated as (s.t.s.r.f) and in Theorem 3.9 we show that such a module does exist.

Theorem 3.12 is concerned with a number of properties of semi-radical of sub modules. After defining a  $P$ -semiprime submodule we consider some of its properties.

## 2. Some Elementary Results

We begin this section with the following definitions:

**Definition 2.1.** Let  $N$  be a proper submodule of an  $R$ -module  $M$ .

(a)  $N$  is called a prime submodule of  $M$  if for any  $r \in R$  and  $m \in M$ ,  $rm \in N$  implies that  $m \in N$  or  $r \in (N : M)$ .

(b)  $N$  is called a primary submodule of  $M$  if for any  $r \in R$  and  $m \in M$ ,  $rm \in N$  implies that  $m \in N$  or  $r^n \in (N : M)$  for some positive integer  $n$ .

In (a) it can easily be shown that  $P = (N : M)$  is a prime ideal of  $R$  and we say that  $N$  is  $P$ -prime.

We recall that if  $I$  is an ideal of a ring  $R$ , then the radical of  $I$ , denoted by  $\sqrt{I}$ , is defined as the intersection of all prime ideals containing  $I$ . Alternatively, we define the radical of  $I$  as :

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}.$$

Also if  $N$  is a primary submodule of an  $R$ -module  $M$ , then  $(N : M)$  is a primary ideal of  $R$  and  $P = \sqrt{(N : M)}$  is a prime ideal. We describe this situation by saying that  $N$  is  $P$ -primary.

**Lemma 2.2.** Let  $N$  be a proper submodule of an  $R$ -module  $M$ .

(i)  $N$  is a prime submodule of  $M$  if and only if  $ID \subseteq N$  (with  $I$  an ideal of  $R$  and  $D$  a submodule of  $M$ ) implies that  $D \subseteq N$  or  $I \subseteq (N : M)$ .

(ii)  $N$  is a primary submodule of  $M$  if and only if for every finitely generated ideal  $I$  of  $R$  and any submodule  $D$  of  $M$ ,  $ID \subseteq N$  implies that  $D \subseteq N$  or  $I^n \subseteq (N : M)$  for some positive integer  $n$ .

(iii) Let  $P$  be a prime ideal of  $R$ , then  $N$  is a  $P$ -primary submodule of  $M$  if and only if (a)

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$P \subseteq \sqrt{(N : M)}$ , and (b)  $cm \notin N$  for all  $c \in R/P$ ,  $m \in M/N$ .

**Proof.** (i)  $(\Rightarrow)$ : Let  $I \trianglelefteq R$  and  $D \leq M$  be such that  $ID \subseteq N$  and let  $D \not\subseteq N$ . So there exists an element  $x \in D \setminus N$ . Let  $r$  be any element of  $I$ . Then  $rx \in N$  and hence  $r \in (N : M)$ . Therefore  $I \subseteq (N : M)$ .

$(\Leftarrow)$ : Let  $r \in R$ ,  $a \in M$  be such that  $ra \in N$  and let  $a \notin N$ . By taking:

$I = (r)$  and  $D = Ra$  we see that  $ID \subseteq N$ . But  $D \not\subseteq N$  and hence  $I \subseteq (N : M)$ ,

which implies that  $r \in (N : M)$ . Therefore  $N$  is a prime submodule of  $M$ .

(ii)  $(\Rightarrow)$ : Let  $D \leq M$  and  $I$  be a finitely generated ideal of  $R$  such that  $ID \subseteq N$ .

Then by [5, Corollary 1, P.99],  $D \subseteq N$  or  $I \subseteq \sqrt{(N : M)}$ . Let  $D \not\subseteq N$ , then  $I \subseteq \sqrt{(N : M)}$  and by [5, Proposition 8. P.83], there exists a positive integer  $n$  such that  $I^n \subseteq (N : M)$ .

$(\Leftarrow)$ : Let  $r \in R, x \in M$  be such that  $rx \in N$  and let  $x \notin N$ . By taking  $I = (r)$  and  $D = Rx$  we see that  $ID \subseteq N$  and  $D \not\subseteq N$ . So there exists a positive integer  $n$  such that  $I^n \subseteq (N : M)$ . This implies that  $r^n \in (N : M)$  and hence  $N$  is a primary submodule of  $M$ .

(iii)  $(\Rightarrow)$ : If  $N$  is  $P$ -primary, then by definition  $P = \sqrt{(N : M)}$ . Now let  $c \in R \setminus P$  and  $m \in M \setminus N$ . Let  $cm \in N$ , then there exists a positive integer  $n$  such that:

$c^n \in (N : M)$ , that is,  $c \in \sqrt{(N : M)} = P$  (because  $m \notin N$ ), a contradiction. Hence  $cm \notin N$ .

$(\Leftarrow)$ : Assume that (a), (b) hold. Let  $r \in R$  and  $m \in M$ ,  $rm \in N$ . Assume further that  $m \notin N$ , then by (b),  $r$  must belong to  $P$  and so  $r \in \sqrt{(N : M)}$  by (a). Therefore  $N$  is a primary submodule of  $M$ . Next we must show that  $P = \sqrt{(N : M)}$ .

Let  $r \in \sqrt{(N : M)}$ , then  $r^n \in (N : M)$  for some positive integer  $n$ , and so  $r^n M \subseteq N$ . Since  $N$  is proper, there exist  $x \in M/N$ . Now  $r^n x \in N$  and  $x \notin N$  so by (b) we conclude that  $r^n \in P$  and, as  $P$  is prime,  $r \in P$ . We find that  $\sqrt{(N : M)} = P$  and therefore  $N$  is  $P$ -primary.

The following definition is essential in the rest of the paper.

**Definition 2.3.** A proper submodule  $N$  of an  $R$ -module  $M$  is said to be semiprime in  $M$ , if

for every ideal  $I$  of  $R$  and every submodule  $K$  of  $M$ ,  $I^2 K \subseteq N$  implies that  $IK \subseteq N$ . Note that since the ring  $R$  is an  $R$ -module by itself, a proper ideal  $I$  of  $R$  is semiprime if for every ideals  $J$  and  $K$  of  $R$ ,  $J^2 K \subseteq I$  implies that  $JK \subseteq I$ .

**Proposition 2.4.** Let  $M$  be an  $R$ -module.

(i) If  $N$  is a prime submodule of  $M$ , then  $N$  is semiprime.

(ii) If  $N$  is a semiprime submodule of  $M$ , then  $(N : M)$  is semiprime ideal of  $R$ .

**Proof.** (i) Let  $I \trianglelefteq R$ ,  $K \leq M$  and  $I^2 K \subseteq M$ . Then  $I(K) \subseteq N$  and since  $N$  is prime,  $I \subseteq (N : M)$  or  $IK \subseteq N$ . But  $(N : M) \subseteq (N : K)$  and hence  $I \subseteq (N : K)$ , and so  $IK \subseteq N$ . In any case we see that  $IK \subseteq N$ , and therefore  $N$  is semiprime.

(ii) Let  $J$  and  $K$  be ideals of  $R$  and  $J^2 K \subseteq (N : M)$ . Hence  $(J^2 K)M \subseteq N$ , and so,  $J^2(KM) \subseteq N$ . But  $KM \leq M$ , and  $N$  is semiprime, therefore  $J(KM) \subseteq N$ , and thus,  $(JK)M \subseteq N$ . Hence  $JK \subseteq (N : M)$  and we conclude that  $(N : M)$  is a semiprime ideal of  $R$ .

Part (i) of the above proposition implies that if  $P$  is a prime ideal of  $R$  then  $P$  is semiprime. In the next example we show that the converse of part (ii) of Proposition 2.1. is not valid in general.

**Example 2.5.** Let  $R = Z$ ,  $M = Z \oplus Z$  and  $B = \langle (9,0) \rangle$ .

Then it is clear that  $(B : M) = (0)$ . Since  $Z$  is an integral domain,  $(B : M) = (0)$  is a prime ideal and hence a semiprime ideal of  $Z$ . But  $B$  is not a semiprime submodule of  $M$ ; because if we take  $I = (3)$  and  $K = \langle (2,0) \rangle$ , Then:

$$I^2 K = \{ (18q, 0) \mid q \in Z \} \tag{1}$$

But:

$$IK = \{ (6q, 0) \mid q \in Z \} \tag{2}$$

is not a subset of  $B$ .

It is clear that if  $N$  is a semiprime submodule of an  $R$ -module  $M$  and  $I \trianglelefteq R$ ,  $K \leq M$  be such that  $I^n K \subseteq N$  for some positive integer  $n$ , then  $IK \subseteq N$ .

**Theorem 2.6.** Let  $N$  be a proper submodule of an  $R$ -module  $M$ . Then the following statements are equivalent:

(i)  $N$  is semiprime.

(ii) Whenever  $r^t m \in N$  for some  $r \in R$ ,  $m \in M$  and  $t \in Z^+$ , then  $rm \in N$ .

**Proof.** (i)  $(\Rightarrow)$  (ii). Let  $r^t m \in N$  where  $r \in R$ ,  $m \in M$  and  $t \in Z^+$ . Taking  $I = (r)$  and  $K = (m)$  we have  $I^t K \subseteq N$  and so  $IK \subseteq N$  which implies that  $rm \in N$ .

(ii)  $\Rightarrow$  (i). Let  $I \trianglelefteq R$  and  $K \leq M$  be such that  $I^2 K \subseteq N$ . Consider the set:

$$S = \{ra \mid r \in I, a \in K\} \tag{3}$$

Then for every  $r \in I, a \in K$  we have  $r^2 a \in I^2 K \subseteq N$  and hence  $ra \in N$ . This implies that  $S \subseteq N$  and since  $IK$  is the submodule of  $M$  generated by  $S$ , we must have  $IK \subseteq N$ . Therefore  $N$  is semiprime.

**Definition 2.7.** An  $R$ -module  $M$  is said to be a multiplication module if for each submodule  $N$  of  $M$ ,  $N = IM$  for some ideal  $I$  of  $R$ .

It can be easily shown that, an  $R$ -module  $M$  is a multiplication module if and only if  $N = (N : M)M$  for every submodule  $N$  of  $M$ .

**Lemma 2.8.** Let  $M$  be a multiplication  $R$ -module. Then a submodule  $N$  of  $M$  is semiprime if and only if  $(N : M)$  is a semiprime ideal of  $R$ .

**Proof.**  $(\Rightarrow)$ : This is clear from Proposition 2.4 (ii).

$(\Leftarrow)$ : Let  $I \trianglelefteq R$ ,  $K \leq M$ , be such that  $I^2 K \subseteq N$ . Hence:

$$(I^2 K : M) \subseteq (N : M). \tag{4}$$

It can be shown that:

$$I^2 (K : M) \subseteq (I^2 K : M) \tag{5}$$

and so we obtain:

$$I^2 (K : M)M \subseteq (N : M). \tag{6}$$

But  $(N : M)$  is a semiprime ideal of  $R$  and hence  $I(K : M) \subseteq (N : M)$ . Thus we conclude that:

$$I(K : M)M \subseteq (N : M)M, \tag{7}$$

and using the fact that  $M$  is a multiplication  $R$ -module we have  $IK \subseteq N$ . Therefore  $N$  is a semiprime submodule of  $M$ .

The following lemma shows that the same situation, as above, holds for prime and primary sub modules.

**Lemma 2.9.** Let  $M$  be a multiplication  $R$ -module. Then:

(a) A submodule  $N$  of  $M$  is prime if and only if  $(N : M)$  is a prime ideal of  $R$ .

(b) A submodule  $N$  of  $M$  is primary if and only if  $(N : M)$  is a primary ideal of  $R$ .

**Proof.** (a)  $(\Rightarrow)$ : Clear.

$(\Leftarrow)$ : Let  $I \trianglelefteq R$ ,  $D \leq M$  be such that  $ID \subseteq N$ , then  $(ID : M) \subseteq (N : M)$ . But  $I(D : M) \subseteq (ID : M)$  and so  $I(D : M) \subseteq (N : M)$ . Since  $(N : M)$  is a prime:

ideal of  $R$  we have  $I \subseteq (N : M)$  or  $(D : M) \subseteq (N : M)$ .

Suppose that  $I \not\subseteq (N : M)$ . Then  $(D : M) \subseteq (N : M)$  and from this we have  $(D : M)M \subseteq (N : M)M$ , that is,  $D \subseteq N$ . Hence  $N$  is a prime submodule of  $M$  by Lemma 2.2 (i).

(b)  $(\Rightarrow)$ : Clear.

$(\Leftarrow)$ : Let  $(N : M)$  be a primary ideal of  $R$ . Let  $I$  be a finitely generated ideal of  $R$  and  $D$  be a submodule of  $M$  and let  $ID \subseteq N$ . Suppose that for any positive integer  $n$ ,  $I^n \not\subseteq (N : M)$ . We see that  $ID \subseteq N$  implies

$(ID : M) \subseteq (N : M)$  and hence  $I(D : M) \subseteq (N : M)$ . But

$I^n \not\subseteq (N : M)$  for any positive integer  $n$ , so

$(D : M) \subseteq (N : M)$ , because  $(N : M)$  is a primary.

Hence  $(D : M)M \subseteq (N : M)M$ , that is,  $D \subseteq N$ . So  $N$  is a primary submodule of  $M$ , by Lemma 2.2 (ii). the proof is now complete.

**Proposition 2.10.** Let  $\{P_i\}_{i \in I}$  be a non-empty family of semiprime sub modules of an  $R$ -module  $M$ . Then  $P = \bigcap P_i$  is a semiprime submodule of  $M$ . Further if  $\{P_i\}_{i \in I}$  is totally ordered (by inclusion), then  $T = \bigcap P_i$  is also a semiprime submodule whenever  $T \neq M$ .

**Proof.** Let  $I \trianglelefteq R$  and  $K \leq M$  be such that  $I^2 k \subseteq P = \bigcap P_i$ . Then  $I^2 k \subseteq P_i$  for every  $i \in I$ , and since  $P_i$  is semiprime we have  $Ik \subseteq P_i$ . Hence  $IK \subseteq \bigcap P_i = P$  and  $P$  is semiprime. Next we let  $T = \bigcap P_i \neq M$ . The fact that  $\{P_i\}_{i \in I}$  is totally ordered by inclusion makes it clear that  $T$  is a submodule of  $M$ . Let  $I \trianglelefteq R$  and  $K \leq M$  be such that  $I^2 K \subseteq T$ . Consider the set:

$$S = \{rk \mid r \in R, k \in K\} \tag{8}$$

Then  $S$  is a generating set for the submodule  $IK$ . If  $r \in I$ ,  $k \in K$  then  $r^2 k \in I^2 K \subseteq T$  and so for some  $i \in I, r^2 k \in P_i$ . Since  $P_i$  is semiprime this implies that  $rk \in P_i$ . It follows that  $S \subseteq T$  and hence  $IK = \langle S \rangle \subseteq T$ . Therefore  $T$  is also a semiprime submodule of  $M$ .

**Remark.** Some authors define a semiprime submodule as an intersection of prime sub modules. But by our

definition of a semiprime submodule of a module we can find a semiprime submodule of a module which is not an intersection of prime sub modules. for example look at .J. Jenkins and P.F. Smith in the proof of [1].

### 3. Radicals and Semi-Radicals

Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . If there exists a prime submodule of  $M$  which contain  $N$ , then the intersection of all prime sub modules containing  $N$ , is called the  $M$ -radical of  $M$  and is denoted by  $rad_M N$ , or simply by  $radN$ . If there is no prime submodule containing  $N$ , then we define  $rad_M N = M$ ; in particular  $rad_M M = M$ . An ideal  $I$  of a ring  $R$  is called a radical ideal if  $\sqrt{I} = I$ . Similarly, we say that a submodule  $B$  of an  $R$ -module  $M$  is a radical submodule if  $radB = B$ . It is easy to see that an ideal  $I$  of a ring  $R$  is semiprime if and only if it is radical. Because, let  $I$  be semiprime, and let  $x \in \sqrt{I}$ . Then  $x^k \in I$  for some positive integer  $k$ . So  $x^k \cdot 1 \in I$ , and since  $I$  is semiprime we have  $x \cdot 1 = x \in I$ . Therefore  $I = \sqrt{I}$ .

On the other hand, if  $I = \sqrt{I}$  then by definition of  $\sqrt{I}$  and Propositions 2.4 (i) and 2.10,  $I$  is semiprime. Finally by Propositions 2.4 (i) and 2.10 we see that for any submodule  $B$  of an  $R$ -module  $M$ ,  $radB$  is a semiprime submodule whenever  $radB \neq M$ .

**Theorem 3.1.** Let  $M$  be a finitely generated multiplication  $R$ -module and let  $N$  be a proper submodule of  $M$ . Then  $N$  is semiprime if and only if it is radical.

**Proof.** Since  $ann_R(M) \subseteq (N : M)$ , by [2, Theorem 3, P.216],

$$\sqrt{(N : M)M} = rad(N : M)M. \quad (9)$$

As  $M$  is a multiplication module we have  $(N : M)M = M$ , and if  $N$  is semiprime,  $(N : M)$  is a radical ideal. Therefore  $\sqrt{(N : M)M} = rad(N : M)M$  iff

$$(N : M)M = rad(N : M)M. \text{ If } N = radN$$

this implies that  $N$  is a radical submodule of  $M$ , that is,  $N = radN = \bigcap P$  ( $P$  is a prime submodule of  $M$  containing  $N$ ). Hence by Propositions 2.4 (1) and 2.10  $N$  is a semiprime submodule of  $M$ . The proof is now complete.

After Remark 2.11 we may ask under what condition a semiprime submodule is the intersection of prime submodules containing it. The following corollary can be considered as an answer.

**Corollary 3.2.** Let  $M$  be a finitely generated multiplication  $R$ -module and let  $N$  be a proper submodule of  $M$ . Then  $N$  is semiprime if and only if  $N = \bigcap P$  ( $P_i$  a prime submodule of  $M$  containing  $N$ ).

**Proof.** ( $\Rightarrow$ ): If  $N$  is semiprime then by Theorem 3.1, it is radical, that is,  $N = \bigcap P$  ( $P_i$  a prime submodule of  $M$  containing  $N$ ).

( $\Leftarrow$ ): By Propositions 2.4 (i) and 2.10,  $N$  is semiprime.

**Proposition 3.3.** If  $M$  is a finitely generated  $R$ -module, then every proper submodule of  $M$  is contained in a semiprime sub module.

**Proof.** By Corollary of [3, Proposition 4, P.63], every proper submodule of  $M$  is contained in a prime submodule. So by Proposition 2.4 (i), we have the result.

**Definition 3.4.** (1) A semiprime submodule  $P$  of an  $R$ -module  $M$  is called a minimal semiprime of a proper submodule  $N$  if  $N \subseteq P$  and there is no smaller semiprime submodule with this property.

(2) A minimal semiprime of  $0 = \langle 0_M \rangle$  is called a minimal semiprime submodule of  $M$ .

**Theorem 3.5.** Let  $M$  be an  $R$ -module. If a submodule  $N$  of  $M$  is contained in a semiprime submodule  $P$ , then  $P$  contains a minimal semiprime submodule of  $N$ .

**Proof.** It is similar to the proof of [5, Theorem 4. P.84].

**Proposition 3.6.** Every proper submodule of a finitely generated  $R$ -module  $M$  possesses at least one minimal semiprime submodule of  $M$ .

**Proof.** Let  $N$  be a proper submodule of  $M$ , then by Proposition 3.3,  $N$  is contained in a semiprime submodule of  $M$ .

**Corollary 3.7.** Every semiprime submodule of an  $R$ -module  $M$  contains at least one minimal semiprime submodule of  $M$ .

**Proof.** Let  $P$  be a semiprime submodule of  $M$  and take  $N = \langle 0 \rangle$  in the Theorem 3.5. Then  $P$  contains a minimal semiprime submodule of  $\langle 0 \rangle$ , and so a minimal semiprime submodule of  $M$ .

**Definition 3.8.** Let  $M$  be an  $R$ -module and  $N \leq M$ . If there exists a semiprime submodule of  $M$  which contains  $N$ , then the intersection of all semiprime sub modules containing  $N$  is called the semi-radical of  $N$  and is denoted by  $S-rad_M N$ , or simply by  $S-radN$ . If there is no semiprime submodule containing  $N$ . then we define

$S-radN = M$ , in particular  $S-radM = M$ . We call  $S-rad\langle 0 \rangle$  the semiprime radical of  $M$ .

If  $N \leq M$ , then the envelope of  $N$ , denoted by  $E(N)$ , is defined as:

$$E\langle N \rangle = \left\{ \begin{array}{l} x \in M \mid x = ra \text{ for some } r \in R, a \in M \\ \text{and } r^n a \in N \text{ for some } n \in \mathbb{Z}^+ \end{array} \right\} \quad (10)$$

We say that  $M$  satisfies the semi-radical formula,  $M$  (s.t.s.r.f) if for any  $N \leq M$ , the semi-radical of  $N$  is equal to the submodule generated by its envelope, that is,  $S-radN = \langle E(N) \rangle$ . We already know that  $\langle E(N) \rangle \subseteq radN$ , by [4, P.1815]. Now let  $x \in E(N)$  and  $P$  be a semiprime submodule of  $M$  containing  $N$ . Then  $x = ra$  for some  $r \in R, a \in M$  and for positive integer  $n, r^n a \in N$ . But  $r^n a \in P$  and since  $P$  is semiprime we have  $ra \in P$ . Hence  $E(N) \subseteq P$ . We conclude that  $E(N) \subseteq \bigcap P$  ( $P$  is a semiprime submodule containing  $N$ ). So  $E(N) \subseteq S-radN$ . On the other hand, since every prime submodule of  $M$  is clearly semiprime, we have  $S-radN \subseteq radN$ . We see that:

$$\langle E(N) \rangle \subseteq S-radN \subseteq radN \quad (11)$$

Now we present an  $R$ -module which satisfies the semi-radical formula.

**Theorem 3.9.** Let  $M$  be a finitely generated multiplication  $R$ -module. Then  $M$  satisfied the semi-radical formula.

**Proof.** Let  $N \leq M$ , then by [4. Theorem 4.4], we have  $\langle \langle E(N) \rangle : M \rangle = \langle radN : M \rangle$ .

Hence  $\langle \langle E(N) \rangle : M \rangle M = \langle radN : M \rangle M$  and since  $M$  is a multiplication  $R$ -module,  $\langle E(N) \rangle = radN$ . Next from (\*) we have:

$$\langle \langle E(N) \rangle : M \rangle M \subseteq \langle S-radN : M \rangle M \subseteq \langle radN : M \rangle M \quad (12)$$

that is,

$$\langle \langle E(N) \rangle \rangle \subseteq S-radN \subseteq radN. \quad (13)$$

Thus we find that  $S-radN = \langle E(N) \rangle$ .

**Remark.** Under the conditions of Theorem 3.9, we see that for any submodule  $N \neq M$  of  $M$  we always have  $RadN = S-RadN$ .

**Proposition 3.10.** Let  $M$  be a finitely generated  $R$ -module. Then the semi-radical of a proper

submodule  $N$  of  $M$  is the intersection of its minimal semiprime sub modules.

**Proof.** This is clear by using Theorem 3.5 and Proposition 3.6.

For the rest of this section we state and prove some properties of semi-radical of sub modules.

**Theorem 3.11.** Let  $B$  and  $C$  be sub modules of an  $R$ -module  $M$ . Then ,

- (1)  $B \subseteq S-radB$ .
- (2)  $S-rad(S-radB) = S-radB$ ,
- (3)  $S-rad(B \cap C) \subseteq S-radB \cap S-radC$ , and we have the equality when for every semiprime submodule  $P, B \cap C \subseteq P$  implies that  $B \subseteq P$  or  $C \subseteq P$ ,
- (4)  $S-rad(B+C) = S-rad(S-radB + S-radC)$ ,
- (5)  $\sqrt{(B:M)} \subseteq (S-radB : M)$ ,
- (6) If  $M$  is finitely generated, then  $S-radB = M$  if and only if  $B = M$ ,
- (7) If  $M$  is finitely generated, then  $B+C = M$  if and only if  $S-RadB + S-RadC = M$ ,
- (8)  $S-radIM = S-rad\sqrt{I}M$  for every ideal  $I$  of  $R$ .

**Proof.** (1) clear.

(2) Since  $S-RadB$  is semiprime by Proposition 2.10, we have:

$$S-Rad(S-RadB) = S-RadB. \quad (14)$$

(3) Let  $P$  be a semiprime submodule of  $M$  such that  $B \cap C \subseteq P$ , so  $B \cap C \subseteq P$  and hence  $S-rad(B \cap C) \subseteq P$ . But  $P$  is arbitrary, therefore  $S-rad(B \cap C) \subseteq S-radB$ . By a similar argument we have  $S-rad(B \cap C) \subseteq S-radC$ . Now let  $P$  be a semiprime submodule of  $M$  such that  $B \cap C \subseteq P$  and assume that  $B \subseteq P$ . Then  $S-radB \subseteq P$  and so  $S-radB \cap S-radC \subseteq P$ . Since  $P$  is arbitrary this implies that  $S-radB \cap S-radC \subseteq S-rad(B \cap C)$  and hence we have the equality.

(4) Let  $P$  be a semiprime submodule of  $M$  such that  $(S-radB + S-radC) \subseteq P$ . So  $S-radB \subseteq P$  and  $S-radC \subseteq P$ . Hence  $B \subseteq P$  and  $C \subseteq P$  which implies  $B+C \subseteq P$ . Therefore  $S-rad(B+C) \subseteq P$ . But  $P$  is chosen arbitrary, so:

$$S-rad(B+C) \subseteq S-rad(S-radB + S-radC). \quad (15)$$

Now suppose that  $P$  be a semiprime submodule such that  $B+C \subseteq P$ . So  $B \subseteq P$ , and  $C \subseteq P$ . Hence  $S-radB \subseteq P$  and  $S-radC \subseteq P$  and therefore  $S-radB + S-radC \subseteq P$ .

But  $S-rad(S-radB + S-radC) \subseteq P$  and we conclude that:

$$S-rad(S-radB + S-radC) \subseteq S-rad(B+C). \quad (16)$$

(5) If  $S-radB = M$ , then we have the result. So let  $P$  be a semiprime submodule of  $M$  such that  $B \subseteq P$ . So  $(B:M) \subseteq (P:M)$ . We know that  $(P:M)$  is a semiprime ideal of  $R$  and we have shown that  $\sqrt{(P:M)} = (P:M)$ . Hence  $\sqrt{(B:M)} \subseteq \sqrt{(P:M)} = (P:M)$  implies that:

$$\sqrt{(B:M)}M \subseteq (P:M)P \subseteq P,$$

and since  $P$  can be any semiprime submodule of  $M$  containing  $B$ , we have  $\sqrt{(B:M)}M \subseteq S-radB$ , that is,  $\sqrt{(B:M)}M \subseteq (S-radM:M)$ .

(6) If  $B = M$ , then  $S-radB = S-radM = M$ . Conversely, let  $S-radB = M$ , but  $B \neq M$ . Since  $M$  is finitely generated, it contains a prime and so a semiprime submodule  $P$  containing  $B$ , by Corollary after Proposition 4 of [3]. Hence  $S-radB \neq M$ , a contradiction.

(7) Using parts (4) and (6) we have:

$$\begin{aligned} B+C=M \text{ iff } S-rad(B+C) &= M \\ \text{iff } S-rad(S-radB+S-radC) &= M \\ \text{iff } S-radB+S-radC &= M. \end{aligned}$$

(8) If  $M$  has no semiprime submodule containing  $IM$ , then  $S-radIM = M$  and we have:

$$\begin{aligned} I \subseteq \sqrt{I} \Rightarrow IM \subseteq \sqrt{I}M \Rightarrow S-radIM \subseteq S-rad\sqrt{I}M \\ \Rightarrow M \subseteq S-rad\sqrt{I}M \Rightarrow M = S-rad\sqrt{I}M : \\ = S-radIM. \end{aligned} \tag{17}$$

Now let  $P$  be a semiprime submodule of  $M$  such that  $IM \subseteq P$ , so  $I \subseteq (IM:M) \subseteq (P:M)$  and since  $(P:M)$  is semiprime  $\sqrt{I} \subseteq \sqrt{(P:M)} = (P:M)$ .

So  $\sqrt{I}M \subseteq P$  and hence  $S-rad\sqrt{I}M \subseteq P$ . Since  $P$  is arbitrary we have:  $S-rad\sqrt{I}M \subseteq S-radIM$ .

Therefore  $S-radIM = S-rad\sqrt{I}M$ . The proof is now complete.

**Corollary 3.12.** Let  $M$  be an  $R$ -module and  $I$  an ideal of  $R$ . Then  $S-radI^n M = S-radM$  for every positive integer  $n$ .

**Proof.** We know that  $\sqrt{I^n} = \sqrt{I}$ . so by part (8) of Theorem 3.11:

$$\begin{aligned} S-radI^n M &= S-rad\sqrt{I^n} M = \\ S-rad\sqrt{I} M &= S-radIM. \end{aligned} \tag{18}$$

**Proposition 3.13.** Let  $Q$  be a  $P$ -primary submodule of an  $R$ -module  $A$ . Then  $S-radQ = S-rad(Q+PA)$ .

**Proof.** We have  $Q \subseteq Q+PA$ , so  $S-radQ \subseteq S-rad(Q+PA)$ . Let  $S-radQ = \bigcap_{i \in I} P_i$ , where any  $P_i$  is a semiprime submodule of  $A$  containing  $Q$ . We see that

$$P = \sqrt{(Q:A)} \subseteq \sqrt{(P_i:A)} = (P_i:A) \tag{19}$$

implies  $PA \subseteq P_i$ . So  $(Q+PA) \subseteq P_i$ , for every  $i \in I$  and hence  $S-rad(Q+PA) \subseteq P_i$ . Therefore  $S-rad(Q+PA) \subseteq \bigcap P_i$  and so  $S-radQ = S-rad(Q+PA)$ .

**Definition 3.14.** Let  $N$  be a semiprime submodule of an  $R$ -module  $M$ , and let  $P = \sqrt{(N:M)} = (N:M)$ . We call  $N$  a  $P$ -semiprime submodule of  $M$ , if  $P$  is prime ideal of  $R$ .

**Lemma 3.15.** Let  $M$  be a finitely generated  $R$ -module and let  $K$  be a maximal ideal of  $R$ . If  $Q$  is a  $K$ -primary submodule of  $M$ , then  $S-radQ$  is a  $K$ -semiprime sub module.

**Proof.** By Theorem 3.11, part (5), we have  $K = \sqrt{(Q:M)} \subseteq (S-radQ:M)$ .

But  $K$  is a maximal ideal of  $R$ , so  $(S-radQ:M) = R$  or  $(S-radQ:M) = K$ . If  $(S-radQ:M) = R$  then  $S-radQ = M$  and by Theorem 3.11, part (6) we have  $Q = M$  which is a contradiction since  $Q$  is primary. Hence  $(S-radQ:M) = K$  and since  $S-radQ$  is an intersection of semiprime sub modules containing  $Q$  it is semiprime and in fact  $K$ -semiprime.

**Proposition 3.16.** Let  $N_1, N_2, \dots, N_t$ , be  $P$ -semiprime sub modules of an  $R$ -module  $M$ . Then  $N = N_1 \cap N_2 \cap \dots \cap N_t$  is also  $P$ -semiprime.

**Proof.** By Proposition 2.10,  $N$  is semiprime and we have:

$$\begin{aligned} (N:M) &= (N_1 \cap N_2 \cap \dots \cap N_t : M) = \\ (N_1:M) \cap (N_2:M) \cap \dots \cap (N_t:M) \end{aligned} \tag{20}$$

$= P \cap P \cap \dots \cap P = P$ . Therefore  $N$  is  $P$ -semiprime.

**Lemma 3.17.** Let  $M$  be a multiplication  $R$ -module and  $L, N$  be sub modules of  $M$ . Also let  $K$  be a prime ideal of  $R$  and  $P$  be a  $K$ -semiprime submodule of  $M$  such that  $N \cap L \subseteq P$ . If  $(N:M) \not\subseteq K$  then  $L \subseteq P$ .

**Proof.** We have  $N \cup L \subseteq P \Rightarrow (N \cap L : M) \subseteq (P:M) = K \Rightarrow (N:M) \cap (L:M) \subseteq K$ .

and since  $K$  is a prime ideal of  $R$ ,  $(N:M) \subseteq K$  or  $(L:M) \subseteq K$ . Since  $(N:M) \not\subseteq K$ , we find that  $(L:M) \subseteq K$ . From this we conclude that  $(L:M)M \subseteq KM$ , that is,  $L \subseteq KM$ . But  $(P:M) = K$  implies that  $KM \subseteq P$ . Therefore  $L \subseteq KM \subseteq P$ .

#### 4. Conclusion

In this research we defined the notion of a semi-radical for sub modules of a module and find various properties for it. We also defined and investigated modules satisfying the semi-radical formula (s.t.s.r.f) and exhibited a module satisfying the above condition.

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