NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

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Abstract: We call a Cayley graph $\Gamma = \text{Cay}(G, S)$ normal for $G$, if the right regular representation $R(G)$ of $G$ is normal in the full automorphism group of $\text{Aut}(\Gamma)$. In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.

Keywords: Cayley graph, normal Cayley graph, automorphism group.

1. Introduction

Let $G$ be a finite group, and $S$ be a subset of $G$ not containing the identity element $1_G$. The Cayley digraph $\Gamma = \text{Cay}(G, S)$ of $G$ relative to $S$ is defined as the graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma)$ consisting of those ordered pairs $(x, y)$ from $G$ for which $yx^{-1} \in S$. Immediately from the definition we find that, there are three obvious facts: (1) $\text{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of $G$ and so $\Gamma$ is vertex-transitive. (2) $\Gamma$ is connected if and only if $G = \langle S \rangle$. (3) $\Gamma$ is an undirected if and only if $S^{-1} = S$.

A Cayley (di)graph $\Gamma = \text{Cay}(G, S)$ is called normal if the right regular representation $R(G)$ of $G$ is a normal subgroup of the automorphism group of $\Gamma$.

The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see [1, 2]). Given a finite group $G$, a natural problem is to determine all normal or non-normal Cayley (di)graphs of $G$. This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley’s graphs of finite groups.

Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let $\Gamma = \text{Cay}(G, S)$ be a connected undirected Cayley graph of a finite abelian group $G$ on $S$ with valency 6. Then $\Gamma$ is normal except when one of the following cases happens:

(1): $G = \mathbb{Z}_2^5 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S = \{a, b, c, abc, d, e\}$.

(2): $G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S = \{a, b, c, abc, d, e^{-1}\}$.

(3): $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, c, ab, c^{-1}\}$.

(4): $G = \mathbb{Z}_2^4 \times \mathbb{Z}_d = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S = \{a, b, c, d, e, e^{-1}\}$.

(5): $G = \mathbb{Z}_2^3 \times \mathbb{Z}_2^m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S_1 = \{a, b, c, d^2, d, d^{-1}\}$, $S_2 = \{a, b, ab, c, d, d^2, d^{-1}\}$, $S_3 = \{a, b, c, ad^2, d, d^{-1}\}$.

(6): $G = \mathbb{Z}_2^2 \times \mathbb{Z}_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, ab, c, c^{-1}\}$.

(7): $G = \mathbb{Z}_2^3 \times \mathbb{Z}_6 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle$, $S = \{a, b, c, d, d^2, d^{-1}\}$.

(8): $G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S = \{a, b, c, d^2, d, d^{-1}\}$.

(9): $G = \mathbb{Z}_2^5 \times \mathbb{Z}_2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle \times \langle f\rangle$, $S = \{a, b, c, d, e, f\}$.

(10): $G = \mathbb{Z}_2^4 \times \mathbb{Z}_m = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle$, $S = \{a, b, c, d, e, e^{-1}\}$.

(11): $G = \mathbb{Z}_2^4 \times \mathbb{Z}_m \times \mathbb{Z}_n = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle \times \langle f\rangle \times \langle g\rangle$, $S = \{a, b, c, ab, c, d, d^{-1}\}$.

(12): $G = \mathbb{Z}_2^4 \times \mathbb{Z}_m \times \mathbb{Z}_n = \{a, b, c, d, e, f\}$.

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(14): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ ($m \geq 3$), $S = \{ a, b, c, d, c^{-1}, a^0, a^{m^0}, d, d^{-1} \}$.

(15): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m = 5, 10$), $S = \{ a, b, c, c^{-1}, c^3, c^{-3} \}$.

(16): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b, c, c^{-1}, c^{2m}, c^{2m-1} \}$.

(17): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$, $m$ is odd), $S = \{ a, a^3, b, b^3, c, c^{-1} \}$.

(18): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$), $S = \{ a, b, a^3, c, c^{-1}, c^2 \}$.

(19): $G = Z_{2m} \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$, $n \geq 3$), $S = \{ a, a^3, a^{2m}, a^{2m-1}, b, b^3 \}$.

(20): $G = Z_2^2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$, $n \geq 3$), $S = \{ a, ab, b^3, b^5, b^7, b^9 \}$.

(21): $G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m = 5, 10, n \geq 3$), $S = \{ a, a^3, b, b^3, b^5, b^7 \}$.

(22): $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, a^3, b^3, c, c^{-1} \}$.

(23): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{ a, b, a^3, b^3, c, c^{-1} \}$.

(24): $G = Z_2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, a^3, b^3, c, c^{-1} \}$.

(25): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$), $S = \{ a, b, a^3, b^3, c, c^{-1} \}$.

(26): $G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle$, $S = \{ a, b, a^3, b^3, b^5, b^7 \}$.

(27): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b, a^3, b^3, c, c^{-1} \}$.

(28): $G = Z_2 \times Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b^2, b^3, b^5, c, c^{-1} \}$.

(29): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$), $S = \{ a, b^m, b^m, b^{-1}, b^{m-1} \}$.

(30): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b, ac, ac^{-1}, c, c^{-1} \}$.

(31): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 3$, $m$ is odd), $S = \{ a, b^2, b^2, b^5, b^5, b^{m-1} \}$.

(32): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b, acm, bcm, bcm, c, c^{-1} \}$.

(33): $G = Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{ a, b, c, e, ab, ac, abc \}$.

(34): $G = Z_2^4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$.

(35): $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 2$), $S = \{ a, b, c, m, abc, bcm, c, c^{-1} \}$.

(36): $G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$.

(37): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$.

(38): $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S = \{ a, b^m, ab^m, ab^m, b, b^m \}$.

(39): $G = Z_4 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$), $S = \{ a, b, ab, abm, ab^m, b, b^m \}$.

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(41): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ ($m \geq 1$), $S = \{ a, b, c, a^2c, bc, bc^{-1}, e, c^{-1} \}$.

(42): $G = Z_2 \times Z_{10} = \langle a \rangle \times \langle b \rangle$, $S = \{ a, b, a^3, b^3, b^5, b^7 \}$.

(43): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$.

(44): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$.

(45): $G = Z_{2m} = \langle a \rangle$ ($m \geq 2$), $S = \{ a, e, a^m, a^{m-1}, a^{m+1} \}$.

(46): $G = Z_{2m} = \langle a \rangle$ ($m \geq 4$).

(47): $G = Z_2 \times Z_m \times \langle a \rangle \times \langle b \rangle$ ($m \geq 5$), $S = \{ a, b, ab^{-1}, b, b^{-1}, b^3 \}$ ($2 \leq j \leq \frac{m}{2}$), (m, j) = p > 2; m = (t + 1)p,
The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of $(n)$ from $X[Y]$. Let $\Gamma$ be a graph and $\alpha$ a permutation $V(\Gamma)$ and $C_n$ a circuit of length $n$. The twisted product $\Gamma \times_a C_n$ of $\Gamma$ by $C_n$ with respect to $\alpha$ is defined by:

$$V(\Gamma \times_a C_n) = V(\Gamma) \times V(C_n)\{(x, i) | x \in V(\Gamma), i = 0, 1, ..., n-1\},$$

$$E(\Gamma \times_a C_n) = \{(\{x, i\}, \{x, i+1\}) | x \in V(\Gamma), i = 0, 1, ..., n-2\} \cup \{(x, n-1), (x^a, 0) | x \in V(\Gamma)\} \cup \{(x, i), (y, i) | (x, y) \in E(\Gamma), i = 0, 1, ..., n-1\}.$$

The graph $Q_4^d$ denotes the graph obtained by connecting all long diagonals of 4-cube $Q_4$, that is, connecting all vertices $u$ and $v$ in $Q_4$ such that $d(u,v) = 4$. The graph $K_{2m,m} \times_c C_n$ is the twisted product of $K_{2m,m}$ by $C_n$ such that $c$ is a cycle permutation on each part of the complete bipartite graph $K_{2m,m}$. The graph $Q_3 \times d C_n$ is the twisted product of $Q_3$ by $C_n$ such that $d$ transposes each pair of elements on long diagonals of $Q_3$. The graph $C_{2m}^d[2K_1]$ is defined by:

$$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1]),$$

$$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \cup \{(x, i), (x, i+m) | x \in V(C_{2m}), i = 0, 1, ..., n-1\} \cup \{(x, i), (y, i) | (x, y) \in E(C_{2m}), i = 0, 1, ..., n-2\}.$$

Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding $nX$ in $X[Y]$, where for $1 \leq i \leq n$, the $i$th copy of $X$ is the subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of $(n)$ from $X[Y]$. Let $\Gamma$ be a graph and $\alpha$ a permutation $V(\Gamma)$ and $C_n$ a circuit of length $n$. The twisted product $\Gamma \times_a C_n$ of $\Gamma$ by $C_n$ with respect to $\alpha$ is defined by:

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2. Primary Analysis

**Proposition 2.1** [9, Proposition 1.5] Let $\Gamma_1 = \text{Cay}(G_1, S_1)$ be a Cayley graph of $G_1$ over $S_1$, and $\Gamma = \text{Aut}(\Gamma_1)$. Let $A_1$ be the stabilizer of the identity element 1 in $\Gamma$. Then $\Gamma$ is normal if and only if every element of $A_1$ is an automorphism of $G_1$.

**Proposition 2.2** [6, Theorem 1.1] Let $G$ be a finite abelian group and $S$ be a generating subset of $G - \{1\}$. Assume $S$ satisfies the condition that, if $s, t, u, v \in S$ with $1 \neq st = uv$, implies $[s, t] = [u, v]$. Then the Cayley graph $\text{Cay}(G, S)$ is normal.

Let $X$ and $Y$ be two graphs. The direct product $X \times Y$ is defined as the graph with vertex set $(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2] \in V(X \times Y)$, $[u, v]$ is an edge in $X \times Y$ whenever $[x_1, x_2] \in E(X)$ or $[y_1, y_2] \in E(Y)$.

Let $X_1$ be a Cayley graph of $G$ over $S$, and $A = \text{Aut}(\Gamma)$. Let $A_1$ be the stabilizer of the identity element 1 in $A$. Then $\Gamma$ is normal if and only if every element of $A_1$ is an automorphism of $G$.

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Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding $nX$ in $X[Y]$, where for $1 \leq i \leq n$, the $i$th copy of $X$ is the subgraph induced on the vertex subset $\{(x, y_i) | x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of $(n)$ from $X[Y]$. Let $\Gamma$ be a graph and $\alpha$ a permutation $V(\Gamma)$ and $C_n$ a circuit of length $n$. The twisted product $\Gamma \times_a C_n$ of $\Gamma$ by $C_n$ with respect to $\alpha$ is defined by:

$$V(\Gamma \times_a C_n) = V(\Gamma) \times V(C_n)\{(x, i) | x \in V(\Gamma), i = 0, 1, ..., n-1\},$$

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Proof Let $\Gamma_1$ be the subgraph of $\Gamma$ with vertex set $\{1\} \cup S \cup S^2$ and edge set $\{(i, s), (s, st) | s, t \in S\}$. By observing the subgraph $\Gamma_1$, it is easy to prove that $\Gamma_1$ fixes $S$ pointwise, which implies that $\Gamma_1 = 1$. Thus $\Gamma_1$ acts faithfully on $S$. Observing the subgraph $\Gamma_2$ again, $\Gamma_1$, as a permutation group on $S$, is generated by $(a, a^{-1}, a^m, a^{m+1})$. So $|\Gamma_1| = 2$ and $\Gamma = \text{Cay}(G, S)$ is normal.

Lemma 3.2: Let $G = \mathbb{Z}_2^2 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $m = 4k, k \geq 2$ and $S = \{a, b, c, a^m, a^{m+1}, b, b^{-1}\}$. Then $\Gamma = \text{Cay}(G, S)$ is normal.

Proof Set $G_1 = <a, b>, G_2 = <c>, S_1 = \{a, b\}, S_2 = \{a^m, c, c^m, c^m\}$. Then $\Gamma_1 = \text{Cay}(G_1, S_1) \cong K_2 \times K_2$. Note that $\Gamma_1$ and $\Gamma_2$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], $\Gamma_1$ and $\Gamma_2$ are normal and by Lemma 2.3, $\Gamma = \text{Cay}(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let $G$ and $S$ be as the following. Then the Cayley graphs $\Gamma = \text{Cay}(G, S)$ are normal:

1. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_6^* = <a> \times <b> \times <c>$, $S = \{a, b, c, c^m, c^3m, c^3\}$.
2. $G = \mathbb{Z}_2^2 \times \mathbb{Z}_m^* = <a> \times <b>$, $m = 2$, $S = \{a, b, c, c^2, a^2m, a^2m+1, b, b^{-1}\}$.
3. $G = \mathbb{Z}_2 \times \mathbb{Z}_6^* = <a> \times <b> \times <c>$, $m = 2$, $S = \{a, b, c, c^2, a^2m, a^2m+1, b, b^{-1}\}$.
4. $G = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2^* = <a> \times <b> \times <c>$, $S = \{a, b, c, c^2, a^2m, a^2m+1, b, b^{-1}\}$.
5. $G = \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $m = 2$, $S = \{a, b, c, c^2, a^2m, a^2m+1, b, b^{-1}\}$.
6. $G = \mathbb{Z}_3 \times \mathbb{Z}_2^* = <a> \times <b>$, $S = \{a, b, c, c^2, a^2, a^3, a^3m, a^3m+1, b, b^{-1}\}$.
7. $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $S = \{a, b, c, a^m, b, b^{-1}\}$.
8. $G = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $S = \{a, b, c, a^2m, a^2m+1, a^m, b, b^{-1}\}$, $m = 2$.
9. $G = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $S = \{a, b, c, a^2m, a^2m+1, a^m, b, b^{-1}\}$, $m = 3$.
10. $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $S = \{a, b, c, a^2, a^3, a^3m, a^3m+1, b, b^{-1}\}$.
11. $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_m^* = <a> \times <b> \times <c>$, $S = \{a, b, c, a^2, a^3, a^3m, a^3m+1, b, b^{-1}\}$.
(30): $G = Z_2 \times Z_{2m+1} = \langle a, b, c \rangle$ (m, n $\geq 1$), $S = \{ab, ab^i, a^{-1}, b, c, c^{-1}\}$.

(31): $G = Z_{4m} = \langle a \rangle$ (m $\geq 2$), $S = \{a, a^{-1}, a^2, a^3, a^m, a^{2m}\}$, (1 $< k < 2m$, k $\neq m$, 2m - 1).

(32): $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ (m $\geq 3$), $S = \{a, a^{-1}, b, b^{-1}, a^2b, b^3a\}$, (1 $< j < m$ (for every j $\neq 1, m - 1$).

(33): $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ (m $\geq 5$), $S = \{a, a^{-1}, b, b^{-1}, b^2, b^3\}$ (1 $< j < m$ (when m $\neq 2k$, 5 or m = 2k, j $\neq k$).

(34): $G = Z_4 \times Z_{2m} = \langle a \rangle \times \langle b \rangle$ (m $\geq 4$), $S = \{a, a^{-1}, b, b^{-1}, a^2b, a^3b\}$ (2 $\leq j < m - 2$).

(35): $G = Z_2 \times Z_{m} = \langle a \rangle \times \langle b \rangle$ (m $\geq 5$), $S = \{a, a^{-1}, b, b^{-1}, b^2, b^3\}$ (1 $< j < m$ (when m $\neq 3k$ for every j and when m = 3k, for k odd; j $\neq k - 1$ and for k even; j $\neq k - 1, 3k - 3$.

(36): $G = Z_{2m} = \langle a \rangle$ (m $\geq 4$), $S = \{a, a^{-1}, a^2, a^{2m-1}, a^{m-1}\}$ (2 $\leq j < m - 2$), when (m, j) = 1 or (m, j) = 2m $\neq 4i + 2$ (i $\geq 1$).

(37): $G = Z_2 \times Z_{m} = \langle a \rangle \times \langle b \rangle$ (m $\geq 5$, m $\neq 8$), $S_i = \{abc, a b^i, b, b^i, b^2, b^3\}$, $S_2 = \{ab, ab^i, b, b^i, b^2, b^3\}$ (2 $\leq j < m$ (when m $\neq 2k$, 5 or m = 2k, k $\neq j$).

(38): $G = Z_8 \times Z_8 = \langle a \rangle \times \langle b \rangle$, $S_1 = \{abc, a b^i, b, b^i, b^2, b^3\}$, $S_2 = \{abc, a b^i, b, b^i, b^2, b^3\}$.

(39): $G = Z_m = \langle a \rangle$ (m $\geq 9$, m $\neq 14$), $S = \{a, a^2, a^3, a^5, a^7, a^9\}$ (3 $\leq j < m$ (when m $\neq 6k$, $\forall j$ or m = 6k, j $\neq 3k - 1$.

(40): $G = Z_{14} = \langle a \rangle$, $S = \{abc, a b^i, b, b^i\}$ for j = 2, 4, 6.

(41): $G = Z_{m} = \langle a \rangle$ (m $\geq 7$), $S = \{a, a^2, a^3, a^5, a^7\}$, (2 $\leq j < m$ (when m $\neq 6k$, $\forall j$ or m = 6k, j $\neq 3k - 1$.

(42): $G = Z_{m} = \langle a \rangle$ (m $\geq 8$, m $\neq 14$), $S = \{a, a^2, a^3, a^5, a^7\}$ (if m = 2k then 2 $\leq j < m$ / 2 and if m = 2k + 1 then 2 $\leq j < (m + 3) / 2$). When m $\neq 3k$ for every j and when m = 3k, for k odd; j $\neq k - 1$ and for k even; j $\neq k - 1, 3k - 3$.

(43): $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle$ (j = 2, 4), $S = \{a, a^2j, a^{-2}j, a^j, a^{-j}\}$ for j = 2, 4.

(44): $G = Z_2 \times Z_4 \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ (m $\geq 3$), $S = \{abc, b, b^{-1}, c, c^{-1}\}$.

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and [6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that $\Gamma$ is not normal. In view of Proposition 2.2, we have the following assumption: $\exists s, t, u, v \in S$ such that st = ub $\neq 1$ but $\{s, t\} \neq \{u, v\}$. (*)

We divide S into four cases:

**Case 1:** $S = \{a, b, c, d, e, f\}$, where a, b, c, d, e, f are involutions. In this case G is an elementary abelian 2-group and a, b, c, d, e, f are not independent by the assumption (*). Consequently $G = Z_2^2$ or $G = Z_2^2$ or $G = Z_2^5$. If $G = Z_2^5$, we can let $S = \{a, b, c, d, e, f\}$. We have $\sigma = (a, b, c, d), (a, b, c, d) \notin \text{Aut}(G, S)$; and by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (33) of Theorem 1.1. If $G = Z_2^5$, we can let $S = \{a, b, c, d, e, f\}$ and hence $\Gamma = \text{Cay}(G, S)$ is non-normal, the Case (1) of Theorem 1.1.

**Case 2:** $S = \{a, b, c, d, e, f\}$, where a, b, c, d, e are involutions but e is not. In this case, $S^*_2 = \{abc, abf, adf, bcf, bdf, cdf\}$, (m $\geq 2$) or $S^*_2 = \{abc, abc, adf, bcf, bdf, cdf\}$. By the assumption (*) $d = abc$, $o(e) = 4$ or $d = e$. Suppose $d = abc$. Then $G = Z_2^2 \times Z_{2m} \ (m \geq 2)$ or $G = Z_2^2 \times Z_{m} \ (m \geq 3)$.

If $G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, (m $\geq 2$), we can let $S = \{a, b, c, abc, c, c^{-1}\}$ or $S = \{a, b, c, abc, c, c^{-1}\}$. When $S = \{a, b, abc, c, c^{-1}\}$, $\sigma = (ab, abc), (ab, abc), (ab, abc), (abc, abc, c, c^{-1}) \notin \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (35) of the main theorem.

When $S = \{a, b, e, ab, abc, c, c^{-1}\}$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3. If $G = Z_2^5$, we can let $S = \{a, b, c, d, e\}$ and hence $\Gamma = \text{Cay}(G, S)$ is non-normal, the Case (1) of Theorem 1.1.
In the Case (1), when $m = 3$, $\sigma = (b^2, b^4) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (36 – $S_1$) of Theorem 1.1. When $S = S_2$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (35, $m = 2$) of Theorem 1.1. When $S = S_1$, $\sigma = (a, c)(bc)(ac^2, c^2)(bc, abc^2)(abc, bc^2) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.4, $\Gamma = \text{Cay}(G, S)$ is not normal the Case (36 – $S_1$) of Theorem 1.1. When $S = S_3$, we have the Case (3) of Theorem 1.1. When $S = S_5, S_6, \Gamma$ is normal by Lemma 3.3 (3, $m=2$). If $G = Z_2^3 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle < G >$, we have $S = \langle a, b, c, d, d^{-1} \rangle$, where $u = d^2, a, ad, abd$ or $abd$ with $u = d^2$. When $u = d$, we have the Case (5) of Theorem 1.1. When $u = ab$, we have the Case $S = \langle S_2 \rangle$ of Theorem 1.1. When $u = ad$, we have the Case (2) of Theorem 1.1. When $u = a$, we have the Case (24) of Theorem 1.1. When $u = abc$, we have the Case (6) of Theorem 1.1. For $S_2$ and $S_3$, we have the Cases (2) and (3, $m = 3$) of Lemma 3.3 respectively. If $G = Z_2^3 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, we have the Case (4) of Theorem 1.1. Now suppose $d = e$. Then $G = Z_2^3 \times Z_4$ or $G = Z_2^3 \times Z_6$. If $G = Z_2^3 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, we see that $S$ is one of the following cases: $S_1 = \langle a, b, c, e, c^2 \rangle$, $S_2 = \langle a, b, ac, c, c^2 \rangle$, $S_3 = \langle a, b, ac, c^2, c, c^2 \rangle$. When $S = S_1$, we have the Case (2) of Theorem 1.1. For $S_2$ and $S_3$, we have the Cases (2) and (3, $m = 3$) of Lemma 3.3 respectively. If $G = Z_2^2 \times Z_6 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, then $S = \langle a, b, b^2, b^3, b^4 \rangle$, the Case (7) of Theorem 1.1.

**Case 3:** $S = \langle a, b, c, d, d^{-1} \rangle$, where $a, b$ are involutions but $c, d$ are not. By the assumption (*) and the symmetry of $c, c^2, d$ and $d^{-1}$, we have five subcases (I) $a = c$, (II) $a = c^2d$, (III) $a = c, (IV) c^2 = d$ and (V) $c^2 = d$. Suppose $a = c$, then $\Gamma$ is isomorphic to one of the following: $Z_2 \times Z_{6m}$ ($m \geq 2$), $Z_2 \times Z_{6n}$, $Z_2 \times Z_{2m}$ ($m \geq 2$), $Z_2 \times Z_{3n}$ ($m \geq 1$), $Z_2 \times Z_{3m}$ ($m \geq 3$). If $Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, (I) we see that $S$ is one of the following cases:

- $S_1 = \langle a, b, c, b^2, b^{-1} \rangle$, $S_2 = \langle a, ab, a^2, b^{-1} \rangle$, $S_3 = \langle a, b, b^2, b^3, b^4, b^{-1} \rangle$. When $S = S_1$, $\sigma = (a, b^2, a^2, b^{-1})(ab, ab^{-1})(a^{-1}, b^{-1}) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (38) of the main theorem. For the Cases $S_2$ and $S_3$, we have the Cases (4) and (5) of Lemma 3.3. If $G = Z_2 \times Z_{6m} = \langle a \rangle \times \langle b \rangle$, we see that $S$ is one of the following cases:

- $S_1 = \langle a, b, b^2, a^2, b^{-1}, b^{-1} \rangle$, $S_2 = \langle a, b^2, a^2, b^{-1} \rangle$, $S_3 = \langle a, b^2, b^{-1}, ab^{-1} \rangle$.

When $S = S_1$, $\sigma = (a, ab, a^2, a^{-1}) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 – $S_1$) of Theorem 1.1. When $S = S_2$, we have the Case (29, $m=3$) of Theorem 1.1. When $S = S_3$, $\sigma = (b^2, b^4)(ab, ab^{-1}) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43 – $S_1$) of Theorem 1.1. If $G = Z_2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle$, we see that $S$ is one of the following cases:

- $S_1 = \langle a, b^m, a, b^{-1}, b^{-1} \rangle$, $S_2 = \langle a, b^m, a, b^{-1} \rangle$. When $S_3$, by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (8) of Theorem 1.1. For $S = S_5$, when $m = 2$, $\sigma = (b^2, b^4)(ab, ab^{-1})(a^2, ab^2)(a^2, ab^{-1})(ab, ab^{-1}) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; $\Gamma = \text{Cay}(G, S)$ is normal, the Case (40, $m=3$) of Theorem 1.1 and when $m \geq 3$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3(6). If $G = Z_2^3 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, (I) we obtain the Case (25) of Theorem 1.1. If $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, (I) we obtain the Case (9) of Theorem 1.1. Suppose $a = c^2d$. Then we have one of the following cases:

- (1): $G = Z_2^2 \times Z_{3m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a, b, b^2, b^3, b^4 \rangle$. Then we obtain one of the following cases:

- (2): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a, b^m, a^{-1}, b^{-1} \rangle$, $S_2 = \langle a, b^m, a^{-1}, b^{-1} \rangle$.

- (3): $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a^2, b, b^2, b^4 \rangle$, $S_2 = \langle a, b^2, b^4 \rangle$.

- (4): $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a^2, b, b^2, b^4 \rangle$, $S_2 = \langle a, b^2, b^4 \rangle$.

- (5): $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a, b, c^2, a^{-2}, a^2 \rangle$.

- (6): $G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle$, $S = \langle a, b, c^2, a^{-2}, a^2 \rangle$.
normal, the Case (43–$S_5$, $m = 3$) of Theorem 1.1. When $m \not\equiv 4$, $\Gamma$ is normal by Lemma 3.3 (7–$S_5$).

In the Case (2), $S = S_1$ when $m = 3$, $\sigma = (b^5, ab^3)(b^3, ab^5) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1.

When $m = 4$, $\sigma = (b, b^5)(b^5, b^9)(b^9, b^3) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1. In the Case (2), $S = S_2$, when $m = 5$, we have the Case (26) of Theorem 1.1. When $m \geq 6$, $\Gamma$ is normal by Lemma 3.3 (7–$S_5$).

In the Case (3), $S = S_1$, when $m = 1$, we have the Case (43–$S_5$) of Theorem 1.1. When $m \geq 2$, $\Gamma$ is normal by Lemma 3.3 (8–$S_5$). In the Case (3), $S = S_2$, $\Gamma$ is normal by Lemma 3.3 (8–$S_5$). In the Case (3), $S = S_1$, when $m = 1$, 2, we have the Cases (29, $m = 3, 5$) of Theorem 1.1 respectively. When $m \geq 3$, $\Gamma$ is normal by Lemma 3.3 (8–$S_5$). In the Case (3), $S = S_4$, when $m = 1$, $\sigma = (ab, ab^3) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1.

When $m = 4$, $\sigma = (b, b^5)(b^5, b^9)(b^9, b^3) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1. In the Case (II), $S = S_1$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (11–$S_5$). In the Case (II), $S = S_2$, $\sigma = (b, b^5)(b^5, b^9)(b^9, b^3) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1.

When $m = 5$, $\sigma = (b, b^5)(b^5, b^9)(b^9, b^3) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1. In the Case (II), $S = S_4$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (11–$S_5$).

In the Case (III), $S = S_5$, we have the Case (10) of Theorem 1.1. When $S = S_2$, $m = 2$, $\sigma = (a^2b^3, ab^5)(ab^5, ab^7)(ab^7, ab^9) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1.

In the Case (IV), when $S = S_1$, $\sigma = (c^2, ac^2)(bc^2, abc^2) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (43–$S_5$) of Theorem 1.1.

When $m = m_1$, with an argument similar to the Case (IV–$S_2$) we obtain the same result. When $m \geq 2$, $\Gamma$ is normal by Lemma 3.3 (13–$S_5$).

In the Case (V), $S = S_5$, when $m = 1$, with an argument similar to the Case (IV–$S_5$), we obtain the same result.

When $m \geq 2$, $\Gamma$ is normal by Lemma 3.3 (13–$S_5$).

In the Case (V), $S = S_5$, $\Gamma$ is normal by Lemma 3.3 (13–$S_5$).

In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), $S = S_1$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (13–$S_2$). In the Case (VII), $S = S_5$, $\Gamma = \text{Cay}(G, S)$ is normal by Lemma 3.3 (13–$S_2$).

In the Case (VIII), $S = S_2$, when $m = 1$, we have the Case (21, $m = 2$) of Theorem 1.1. If $m \geq 2$, $\Gamma$ is normal by Lemma 3.3 (13–$S_2$).

In the Case (VIII), $S = S_2$, $\sigma = (ab, abc^2)(abc^2, abc^4)$ $\in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (57) of Theorem 1.1.

When $m \geq 3$, $\Gamma$ is normal by Lemma 3.3 (44–$S_5$). In the Case (V), $S = S_5$, $\Gamma$ is normal by Lemma 3.3 (44–$S_5$).

In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), $S = S_1$, $S = S_2$ and $S = S_5$, $m = 2$, we have the Cases (12), (28) and (11–$S_5$, $m = 4$) of Theorem 1.1 respectively. In the Case (VII), $S = S_5$, $m \geq 3$, $\Gamma$ is normal by Lemma 3.3 (18). In the Case (VII), $S = S_2$, for $m = 2$, $\sigma = (b^5, c)(ab^5, ac)(ab^5, abc^2)(bc^2)(b^5, c)(bc^2, ab^5, abc^2) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (57) of Theorem 1.1.

When $m \geq 3$, $\Gamma$ is normal by Lemma 3.3 (44–$S_5$). In the Case (V), $S = S_5$, $\Gamma$ is normal by Lemma 3.3 (44–$S_5$).

In the Case (VI), we have the Case (11) of Theorem 1.1. In the Case (VII), $S = S_1$, $S = S_2$ and $S = S_5$, $m = 2$, we have the Cases (12), (28) and (11–$S_5$, $m = 4$) of Theorem 1.1 respectively. In the Case (VII), $S = S_5$, $m \geq 3$, $\Gamma$ is normal by Lemma 3.3 (18). In the Case (VII), $S = S_2$, for $m = 2$, $\sigma = (b^5, c)(ab^5, ac)(ab^5, abc^2)(bc^2)(b^5, c)(bc^2, ab^5, abc^2) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (57) of Theorem 1.1.
(IX), we have the Case (13) of Theorem 1.1. In the Case (X), m = 1, we have the Case (14) of Theorem 1.1, and for m ≥ 2, Γ = Cay(G, S) is normal by Lemma 3.3(14). Suppose c^3 = d, then G = Z_2^x × zm (m ≥ 4) or G = Z_2^y × zm (m ≥ 5, m ≠ 6). If G = Z_2 × zm = <a> × <b> (m ≥ 4), we can let S to be S_1 = {a, b^m, b, b^1, b, b^3, b^1} or S_2 = {a, ab^m, b, b^1, b^3, b^1}. Let S = S_1, for m = 4, 5, we have the Cases (29), (26) of Theorem 1.1 respectively, and for m ≥ 6, Γ is normal by Lemma 3.3(19 – S). Let S = S_2 when m = 4, σ = (ab^1, ab^3) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44–S7) of the main theorem.

Proposition 2.1, one of the following cases: (1): S_1 = {a, b, c, c^1, ac, ac^{-1}}, m ≥ 2, (2): S_2 = {a, b, c, c^1, abc, abc^{-1}}, m ≥ 2, (3): S_3 = {a, b, c, c^1, ac^m, ac^{-m}}, m ≥ 3, (4): S_4 = {a, b, c, c^1, ac^{m+1}, abc^{m+1}}, m ≥ 2, (5): S_5 = {a, b, c, c^1, ac^{m+1}, abc^{m+1}}, m ≥ 2, (6): S_6 = {a, cm, c, c^1, bc, bc^{-1}}, m ≥ 2, (7): S_7 = {a, ac, c, c^1, bc, bc^{-1}}, m ≥ 2, (8): S_8 = {a, c^m, c, c^1, bc, bc^{-1}}, m ≥ 2, (9): S_9 = {a, ac, c, c^1, bc, bc^{-1}}, m ≥ 2.

In the Case (1), Γ is not normal, the Case (30) of Theorem 1.1. In the Case (2), σ = (ac^{m-1}, bc^{m-1}) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44–S_6) of Theorem 1.1. In the Case (3), when m = 2i, Γ = Cay(G, S) is not normal, the Case (16) of Theorem 1.1.

In the Case (4), when m = 2i, i ≥ 2, m ≥ 2, σ = (c^m, ac^{m+1}) ∈ A_1, but σ ∉ Aut(G, S), and when m = 2i + 1, σ = (c^m, ac^{m+1}) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (15) of Theorem 1.1. In the Case (5), when m = 2i, i ≥ 2, m ≥ 2, σ = (c^m, ac^{m+1}) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44–S_2) of Theorem 1.1. In the Case (6), m ≥ 2, Γ is not normal, we have the Case (27) of Theorem 1.1.

In the Case (7), for m = 2i and m = 2i + 1, σ = (b^i+1, ab^{i+1}) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 – S_6) of Theorem 1.1.

In the Case (8), for m = 2i and m = 2i + 1, σ = (b^i, ab^{i+1})(ab^{i+1}, ab^i) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43 – S_6) of Theorem 1.1.
When $S = S_2$, $\sigma = (ac^{m-1}, bc^{m-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (44$-$S$_1$) of Theorem 1.1. If $G = Z_{m-n}^{2} \times Z_{m-n} = \langle a, b \rangle < \times > \times \cdots < \times \cdots >$, we can consider $m \geq 3$, $S = \{a, b, d, d^{-1}, c, d^{-1}\}$. In this case for $m = 2i$ and $m = 2i-1$, $(\geq 2)\sigma = (d, c, d')(a, d', a, d')(a, d') \in A_1$, but $\sigma \notin Aut(G, S)$ and by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal the Case (14 of Theorem 1.1).

**Case 4:** $S = \{a, a^2, b, b^{-1}, c, c^{-1}\}$, where the elements of the set $S$ are not in $Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (45 of Theorem 1.1). When $S = S_2$, $\sigma = (a, a^{-1}, a^m, b, b^{-1}) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (46 of Theorem 1.1).

When $S = S_3$, for $m = 2j$, $\sigma = (a, a^2b^2, a, a^2b^2, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (47 of Theorem 1.1). When $S = S_2$, $\sigma = (a, a^2b^2, a, a^2b^2, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (48 of Theorem 1.1). When $S = S_3$, $\sigma = (a, a^2b^2, a, a^2b^2, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (49 of Theorem 1.1). When $S = S_1$, $\sigma = (a, a^2b^2, a, a^2b^2, a^2b^2) \in A_1$, but $\sigma \notin Aut(G, S)$, by Proposition 2.1, $\Gamma = Cay(G, S)$ is not normal, the Case (50 of Theorem 1.1).
S = \{a^mb^n, a^{2m-1}b^{2n}, a^i, a^{i+1}, b, b^{-1}\}.

(4): \Gamma = \text{Aut}(G, S), by Proposition 2.1, \Gamma = \text{Cay}(G, S) is not normal, the Case (56) of Theorem 1.1. Otherwise, \Gamma = \text{Cay}(G, S) is not normal, the Case (55) of Theorem 1.1. In the Case (3), if \(m = n = 3, \sigma = (ab, a^2b^2)\) for \(k = 2, j = 1, \sigma = (a^2, a^4b^2)(a^2, a^{-2}b^2)\) for \(k = 2, j = 3, \sigma = (a^2, a^4b^2)\) for \(k = 2, j = 5, \), \(\sigma = (a^2, a^4b^2)\) for \(k = 2, j = 7, \), and \(\sigma = (a^2, a^4b^2)\) for \(k = 2, j = 9, \). Otherwise, \(\Gamma = \text{Cay}(G, S)\) is normal by Lemma 3.3(42).

In the Case (4), if \(m = 2, \), we have the Case (21) of Theorem 1.1. If \(m = 3, \), \(\Gamma = \text{Cay}(G, S)\) is normal by Lemma 3.3(28). In the Case (5), if \(m = 4, \), \(\Gamma = \text{Cay}(G, S)\) is normal by Lemma 3.3(27).

In the Case (6), \(\Gamma = \text{Cay}(G, S)\) is normal by Lemma 3.3(29).

4. Conclusion

Let \(\Gamma = \text{Cay}(G, S)\) be a connected Cayley graph of an abelian group \(G\) on \(S\). In this paper we have shown all non-normal Cayley graph \(\Gamma\) with valency 6.

References


