NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

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Abstract: We call a Cayley graph $\Gamma = \text{Cay}(G, S)$ normal for $G$, if the right regular representation $R(G)$ of $G$ is normal in the full automorphism group of $\text{Aut}(\Gamma)$. In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.

Keywords: Cayley graph, normal Cayley graph, automorphism group.

1. Introduction

Let $G$ be a finite group, and $S$ be a subset of $G$ not containing the identity element $1_G$. The Cayley digraph $\Gamma = \text{Cay}(G, S)$ of $G$ relative to $S$ is defined as the graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma)$ consisting of those ordered pairs $(x, y)$ from $G$ for which $yx^{-1} \in S$. Immediately from the definition we find that, there are three obvious facts: (1) $\text{Aut}(\Gamma)$ contains the right regular representation $R(G)$ of $G$ and so $\Gamma$ is vertex-transitive. (2) $\Gamma$ is connected if and only if $G = < S >$. (3) $\Gamma$ is an undirected if and only if $S^{-1} = S$.

A Cayley (di)graph $\Gamma = \text{Cay}(G, S)$ is called normal if the right regular representation $R(G)$ of $G$ is a normal subgroup of the automorphism group of $\Gamma$.

The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see[1,2]). Given a finite group $G$, a natural problem is to determine all normal or non-normal Cayley (di)graphs of $G$. This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley’s graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let $\Gamma = \text{Cay}(G, S)$ be a connected undirected Cayley graph of a finite abelian group $G$ on $S$ with valency 6. Then $\Gamma$ is normal except when one of the following cases happens:

(1): $G = Z_2^5 = < a > \times < b > \times < c > \times < d > \times < e >$, $S = \{ a, b, c, abc, d, e \}$.

(2): $G = Z_2^3 \times Z_m = < a > \times < b > \times < c > \times < d >$ ($m \geq 3$), $S = \{ a, b, c, abc, d, d^{-1} \}$.

(3): $G = Z_2^2 \times Z_6 = < a > \times < b > \times < c >$, $S = \{ a, b, ab, c, c^{-1} \}$.

(4): $G = Z_2^4 \times Z_4 = < a > \times < b > \times < c > \times < d > \times < e >$, $S = \{ a, b, c, d, e, e^{-1} \}$.

(5): $G = Z_2^2 \times Z_4 = < a > \times < b > \times < c > \times < d >$, $S_1 = \{ a, b, c, d^2, d, d^{-1} \}$, $S_2 = \{ a, b, ab, c, d, d^{-1} \}$, $S_3 = \{ a, b, c, ad^2, d, d^{-1} \}$.

(6): $G = Z_2^2 \times Z_8 = < a > \times < b > \times < c >$, $S = \{ a, b, ab, c^3, c, c^{-1} \}$.

(7): $G = Z_2^2 \times Z_4 = < a > \times < b > \times < c > \times < d >$, $S = \{ a, b, c, d^3, d, d^{-1} \}$.

(8): $G = Z_2^3 \times Z_m = < a > \times < b >$ ($m \geq 2$), $S = \{ a^3, b^m, a, a^{-1}, b, b^{-1} \}$.

(9): $G = Z_2^2 \times Z_6 \times Z_m = < a > \times < b > \times < c >$ ($m \geq 3$), $S = \{ a, b, b^3, c, c^{-1} \}$.

(10): $G = Z_2^3 \times Z_2^m \times < a > \times < b >$ ($m \geq 2$), $S = \{ a^3, b^m, a, a^{-1}, b, b^{-1} \}$.

(11): $G = Z_2^2 \times Z_6 \times Z_m = < a > \times < b > \times < c >$ ($m \geq 3$), $S_1 = \{ a, b, b^3, b, c, c^{-1} \}$, $S_2 = \{ a, b, b^3, ab^2, c, c^{-1} \}$.

(12): $G = Z_2^3 \times Z_4 \times Z_2^m = < a > \times < b > \times < c >$ ($m \geq 2$), $S = \{ a, b, b^3, c, c^{-1}, d, d^{-1} \}$.

(13): $G = Z_2^2 \times Z_4 \times Z_m = < a > \times < b > \times < c > \times < d >$ ($m \geq 3$), $S = \{ a, b, c, c^{-1}, d, d^{-1} \}$.


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(14): $G = Z_2^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 3), $S = \{ a, b, cd, cd^-1, d, d^-1 \}$.

(15): $G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m = 5, 10), $S = \{ a, b, c, c^-1, c^3, c^{-3} \}$.

(16): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, c, c^-1, c^{2m+1}, c^{2m-1} \}$.

(17): $G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ (m ≥ 3, m is odd), $S = \{ a, a^2, b, b^3, a b, a b^-1 \}$.

(18): $G = Z_4^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 3), $S = \{ a, b, b^2, b^3, b^5, b^7 \}$.

(19): $G = Z_{4m} \times Z_{n} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2, n ≥ 3), $S = \{ a, a^2, a^3, a^{2m+1}, a^{2m-1} \}$.

(20): $G = Z_3 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ (m ≥ 3, n ≥ 3), $S = \{ a, a^2, b, b^3, c, c^{-1} \}$.

(21): $G = Z_3 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m = 5, 10, n ≥ 3), $S = \{ a, a^2, a^3, a^4, b, b^3 \}$.

(22): $G = Z_4^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, ab, a b^2 \}$.

(23): $G = Z_2 \times Z_3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 3), $S = \{ a, b, b^2, b^3, b^7 \}$.

(24): $G = Z_2^i \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, b^3, b^7 \}$.

(25): $G = Z_2^2 \times Z_3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b^2, b^3, b^7, b^9, b^{11} \}$.

(26): $G = Z_3 \times Z_4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, b^2, b^3, b^7 \}$.

(27): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, b^2, b^3, b^7, b^{11} \}$.

(28): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, b^2, b^3, b^7, b^{11} \}$.

(29): $G = Z_3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b^2, b^3, b^7, b^{11} \}$.

(30): $G = Z_2^2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, b^2, b^3, b^7, b^{11} \}$.

(31): $G = Z_3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 3, m is odd), $S = \{ a, b^2, b^3, b^7, b^{11} \}$.

(32): $G = Z_3 \times Z_{4m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b^2, b^3, b^7, b^{11} \}$.

(33): $G = Z_3^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, c, ab, ac, abc \}$.

(34): $G = Z_3^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, c, ab, abc \}$.

(35): $G = Z_3^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, c, abc, ab^2, ab^3 \}$.

(36): $G = Z_3^3 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ (m ≥ 2), $S = \{ a, b, c, abc, ab^2, ab^3 \}$.

(37): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.

(38): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.

(39): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.

(40): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.

(41): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.

(42): $G = Z_2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$, $S = \{ a, b, abc, ab^2, ab^3 \}$.
Let $G = \mathbb{Z}_{2m} = \langle a \rangle, (m \geq 5)$, and $S = \{a^i, a^{-i}, a^{m+i}, a^{m-i}, a, a^{-1}\}$, $2 \leq i < m$. Then $\Gamma = \text{Cay}(G, S)$ is normal.

Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding $nX$ in $X[Y]$ where for $1 \leq i \leq n$, the ith copy of $X$ is the subgraph induced on the vertex subset $\{(x_i, y) | x_i \in V(X)\}$ in $X[Y]$. The deleted lexigraphic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) $nX$ from $X[Y]$. Let $\Gamma$ be a graph and $\alpha$ a permutation $V(\Gamma)$ and $C_\alpha$ a circuit of length $n$. The twisted product $\Gamma \times_\alpha C_\alpha$ of $\Gamma$ by $C_\alpha$ with respect to $\alpha$ is defined by:

$V(\Gamma \times_\alpha C_\alpha) = V(\Gamma) \times V(C_\alpha) = \{(x, i) | x \in V(\Gamma), i = 0, 1, ..., n-1\}$,

$E(\Gamma \times_\alpha C_\alpha) = \{(x, i), (x, i+1) | x \in V(\Gamma), i = 0, 1, ..., n-2\} \cup \{(x_0, 0), (x^\alpha, 0) | x \in V(\Gamma)\}$

The graph $Q_4^d$ denotes the graph obtained by connecting all long diagonals of 4-cube $Q_4$, that is, connecting all vertices $u$ and $v$ in $Q_4$ such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by $C_n$ such that $c$ is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$ is the twisted product of $Q_3$ by $C_n$ such that $d$ transposes each pair of elements on long diagonals of $Q_3$. The graph $C_{2m}^d[2K_1]$ is defined by:

$V(C_{2m}^d[2K_1]) = V(C_{2m}[2K_1])$,

$E(C_{2m}^d[2K_1]) = E(C_{2m}[2K_1]) \cup \{(x_i, y_j), (x_i, y_j) | i = 0, 1, ..., m = 1, j = 1, 2\}$, where $V(C_{2m}) = \{x_0, x_1, ..., x_{2m-1}\}$ and $V(2K_2) = \{y_1, y_2\}$.

Let $G = G_1 \times G_2$ be the direct product of two finite groups $G_1$ and $G_2$, let $S_1$ and $S_2$ be subsets of $G_1$ and $G_2$, respectively, and let $S = S_1 \cup S_2$ be the disjoint union of two subsets $S_1$ and $S_2$. Then we have,

**Lemma 2.3**

1. Cay $(G, S) \cong$ Cay $(G_1, S_1) \times$ Cay $(G_2, S_2)$.
2. If Cay $(G, S)$ is normal, then Cay $(G_1, S_1)$ is also normal.
3. If both of Cay $(G_1, S_1)$ and Cay $(G_2, S_2)$ are normal and relatively prime, then Cay $(G, S)$ is normal.

### 3. Proof of the Main Theorem

In this section, $\Gamma$ always denotes the Cayley graph Cay$(G, S)$ of an abelian group $G$ on $S$ with valency 6. Let $A = \text{Aut}(\Gamma)$. Then $A_1$ and $A_1^*$ denote the stabilizer of $1$ in $A$ and the subgroup of $A$ which fixes $\{1\}$, pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

**Lemma 3.1**

Let $G = \mathbb{Z}_{2m} = \langle a \rangle, (m \geq 5)$, and $S = \{a^i, a^{-i}, a^{mi}, a^{-mi}, a, a^{-1}\}$, $2 \leq i < m$. Then $\Gamma = \text{Cay}(G, S)$ is normal.
Proof Let $\Gamma_G(1)$ be the subgraph of $\Gamma_G$ with vertex set $\{1\} \cup \{S \cup S^2 \}$ and edge set $\{(1,s), (s, t) | s, t \in S\}$. By observing the subgraph $\Gamma_G(1)$, it is easy to prove that $A_1$ fixes $S^1$ pointwise, which implies that $A_1 = 1$. Thus $A_1$ acts faithfully on $S$. Observing the subgraph $\Gamma_G(1)$ again, $A_1$, as a permutation group on $S$, is generated by $(a, a^m)(b^m, a^{m+1}b^m, b^-1)$. So $|A_1| = 2$ and $\Gamma_G = Cay(G, S)$ is normal.

Lemma 3.2: Let $G = Z_2^2 \times Z_m = <a> \times <b> \times <c>$, $m = 4k$, $k \geq 2$ and $S = \{a, b, c, b^m, c^3, c, c^{-1}\}$. Then $\Gamma_G = Cay(G, S)$ is normal.

Proof Set $G_1 = <a, b>$, $G_2 = <c>$, $S_1 = \{a, b\}$, $S_2 = \{c^k, c^{3k}, c, c^{-1}\}$. Then $\Gamma_1 = Cay(G_1, S_1) \cong K_2 \times K_2$. Note that $\Gamma_1$ and $\Gamma_2 = Cay(G_2, S_2)$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], $\Gamma_1$ and $\Gamma_2$ are normal and by Lemma 2.3, $\Gamma_G = Cay(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let $G$ and $S$ be as the following. Then the Cayley graphs $\Gamma = Cay(G, S)$ are normal.

(1) $G = Z_2^2 = <a> \times <b> \times <c> \times <d>$, $S = \{a, b, c, d, ad, abc\}$.

(2) $G = Z_2^2 \times Z_n = <a> \times <b> \times <c>$, $S = \{a, b, ac^j, c^3, c, c^{-1}\}$.

(3) $G = Z_2^2 \times Z_{2m} = <a> \times <b> \times <c> (m \geq 2)$, $S = \{a, b, ab^m, c, c^{-1}\}$.

(4) $G = Z_2 \times Z_{6m} = <a> \times <b> (m \geq 2)$, $S = \{a, b, ab, ab^m, b, b^{-1}\}$.

(5) $G = Z_2 \times Z_{6m} = <a> \times <b> (m \geq 2)$, $S = \{a, b, b^m, b^m, b, b^{-1}\}$.

(6) $G = Z_2 \times Z_{2m} = <a> \times <b> (m \geq 3)$, $S = \{a, a^m, a^m, b, b^{-1}\}$.

(7) $G = Z_2 \times Z_{2m} = <a> \times <b>$, $S_1 = \{a, ab, ab^2, b^2, b, b^{-1}\}$, $S_2 = \{a, ab^m, ab^m, b, b^{-1}\}$, $S_3 = \{a, b, b^m, b^m, b, b^{-1}\}$, $S_4 = \{a, b^m, b^m, b, b^{-1}\}$.

(8) $G = Z_2 \times Z_{2m} = <a> \times <b>$, $S_1 = \{a, ab, ab^2, b^2, b, b^{-1}\}$, $S_2 = \{a, ab^m, ab^m, b, b^{-1}\}$, $S_3 = \{a, b, b^m, b^m, b, b^{-1}\}$, $S_4 = \{a, b^m, b^m, b, b^{-1}\}$, $S_5 = \{a, b^2m, b^2m, b, b^{-1}\}$.

(9) $G = Z_2 \times Z_{2m} = <a> \times <b> (m \geq 1)$, $S_1 = \{ab^m, b^m, ab^m, b, b^{-1}\}$, $S_2 = \{a^2, a^2b^m, ab^m, ab^m, b, b^{-1}\}$.

(10) $G = Z_2 \times Z_{2m} = <a> \times <b> (m \geq 4, m \neq 6)$, $S = \{a, b, c, c^{-1}, ac^2, ac^{-2}\}$.

(11) $G = Z_2 \times Z_{4m} = <a> \times <b> (m \geq 2)$, $S_1 = \{a, ab^m, ab^m, b, b^{-1}\}$, $S_2 = \{a, b, b, b^m, b, b^{-1}\}$, $S_3 = \{a, ab^m, b^m, b, b^{-1}\}$, $S_4 = \{a, b^m, b, b^{-1}\}$.

(12) $G = Z_2 \times Z_{2m} = <a> \times <b> (m \geq 3)$, $S = \{a^2, a^3b^m, a, a^{-1}, b, b^{-1}\}$.

(13) $G = Z_2 \times Z_{4m} = <a> \times <b> \times <c> (m \geq 2)$, $S_1 = \{a, b, abc^m, c, c^{-1}\}$, $S_2 = \{a, ac^m, ac^m, c, c^{-1}\}$, $S_3 = \{a, b^m, b^m, c^m, c, c^{-1}\}$, $S_4 = \{a, c^m, b^m, c^m, c, c^{-1}\}$.

(14) $G = Z_2 \times Z_{2m} = <a> \times <b> \times <c> (m \geq 2)$, $S = \{a, b, cd^m, cd^m, d, d^{-1}\}$.

(15) $G = Z_2 \times Z_{4m} = <a> \times <b> \times <c> (m = 7, 9, m \geq 11)$, $S = \{a, b, c, c^{-1}, c^3, c^{-3}\}$.

(16) $G = Z_2 \times Z_2 \times Z_{2m} = <a> \times <b> \times <c> (m \geq 1)$, $S = \{a, c^m, b, b^m, b, b^{-1}\}$.

(17) $G = Z_2 \times Z_{4m} = <a> \times <b> \times <c> (m \geq 2)$, $S = \{a, c^2m, b^c, b^m, b^m, c, c^{-1}\}$.

(18) $G = Z_2 \times Z_2 \times Z_{2m} = <a> \times <b> \times <c> (m \geq 3)$, $S = \{a, ac^m, b, b^m, b, b^{-1}\}$.

(19) $G = Z_2 \times Z_{2m} = <a> \times <b> (m \geq 6)$, $S_1 = \{a, c^m, b, b^m, b^m, b^m, b^m\}$, $S_2 = \{a, c^m, b, b^m, b, b^{-1}\}$.

(20) $G = Z_{4m} \times Z_n = <a> \times <b> (m \geq 2, n \geq 3)$, $S = \{a, a^m, c, c^{-1}, b, b^{-1}\}$.

(21) $G = Z_{4m} \times Z_n = <a> \times <b> (m, n \neq 4)$, $S = \{a, a^m, b, b^m, c, c^{-1}\}$.

(22) $G = Z_4 \times Z_{m} \times Z_n = <a> \times <b> \times <c> (m, n \neq 3)$, $S = \{a, a^m, b, b^m, c, c^{-1}\}$.

(23) $G = Z_{2m} (m \geq 5)$, $S = \{a, a^m, a^m, c, c^{-1}, a^{-1}, b, b^{-1}\} (2 \leq j \leq \frac{m}{2})$.

(24) $G = Z_{2m} \times Z_n = <a> \times <b> (m = 7, 9, m \geq 11, n \geq 3)$, $S = \{a, a^m, a^m, b, b^{-1}\}$.

(25) $G = Z_{4m} \times Z_{5n} = <a> \times <b> (m \geq 2, n \geq 1)$, $S = \{a, a^m, b, b^m, a^m, b^m, b^m\}$.

(26) $G = Z_{4m} \times Z_{5n} = <a> \times <b> (m \geq 1)$, $S = \{a, a^m, b, b^m, a^m, b^m, b^m\}$.

(27) $G = Z_2 \times Z_n = <a> \times <b> (m \geq 5, n \geq 3)$, $S = \{a, a^m, b, b^m, a^m, b^m, b^m\}$.

(28) $G = Z_{2m} \times Z_n = <a> \times <b> (m, n \geq 3)$, $S = \{a, a^m, b, b^m, a^m, b^m, b^{-1}\}$.

(29) $G = Z_{2m} \times Z_{2m} = <a> \times <b> (m, n \geq 2)$, $S = \{a, a^m, a^{-1}, b, b^{-1}\}$.
(30): G = Z₂ × Z₂m₁ × Z₂m₂ = <a> × <b> × <c> (m, n ≥ 1), S = {abⁿ, a⁻ⁿ, ab⁻ⁿ, c, b⁻¹, b, c⁻¹}.

(31): G = Z₄m = <a> (m ≥ 2), S = {a, a⁻¹, b, b⁻¹, a⁻²b, a⁻²b⁻¹} (1 < k < 2m, k ≠ m, 2m-1).

(32): G = Z₄ × Z₂m = <a> × <b> (m ≥ 3), S = {a, b⁻¹, b⁻¹b, a⁻¹b, b⁻¹, a⁻¹b⁻¹} (1 < j < m)
(for every j ≠ 1, m - 1).

(34): G = Z₂ × Z₄m₁ = <a> × <b> (m ≥ 2),
S = {a, a⁻¹, b, b⁻¹, a⁻¹b⁻¹, a⁻²b⁻¹} (1 < j < 2m-1).

(35): G = Z₂ × Z₄m = <a> × <b> (m ≥ 5),
S = {a, a⁻¹, b, b⁻¹, b⁻¹, b⁻¹} (1 < j ≤ 2m - 2).

(36): G = Z₂m = <a> (m ≥ 4),
S = {a, a⁻¹, a⁻², a⁻³, a⁻³, a⁻³} (2 ≤ j ≤ m - 2).

(37): G = Z₂ × Z₄ = <a> × <b> (m ≥ 5, m ≠ 8),
S₁ = {ab, ab⁻¹, b, b⁻¹, b⁻¹, b⁻¹}.
S₂ = {ab, ab⁻¹, b, b⁻¹, ab⁻¹, ab⁻¹} (2 ≤ j < m - 2),
when (m, j) = 2.

(38): G = Z₂ × Z₈ = <a> × <b>,
S₁ = {ab, ab⁻¹, b, b⁻¹, b⁻¹, b⁻¹},
S₂ = {ab, ab⁻¹, b, b⁻¹, ab⁻¹, ab⁻¹}.

(39): G = Zₘₕ = <a> (m ≥ 9, m ≠ 14),
S = {a, a⁻¹, a⁻², a⁻³, a⁻³, a⁻³} (j ≠ 3, 2 < j < m - 2) when m ≠ 6k, ∀ j or m = 6k, j ≠ 3k - 1.

(40): G = Z₄ × Z₄ = <a> × <a>,
S₁ = {a, a⁻¹, a⁻², a⁻³, a⁻³} for j = 2, 4, 6.

(41): G = Zₘₕ = <a> (m ≥ 7),
S = {a, a⁻¹, a⁻², a⁻³, a⁻³, a⁻³} (2 ≤ j < m - 2), 3j ≠ 0, 1,
m - 2, j - 2 (mod m),
when m ≠ 7, 14, 6k (k ≥ 2) and m = 7; j = 2 or m = 14; j = 2, 3, 4, 6 or m = 6k; j = 3k - 1.

(42): G = Zₘₕ = <a> (m ≥ 8, m ≠ 14),
S = {a, a⁻¹, a⁻², a⁻³, a⁻³, a⁻³} (if m = 2k then 2 ≤ j ≤ m - 2)
and if m = 2k + 1 then 2 ≤ j ≤ m-2 (mod 2).
When m ≠ 3k for every j and when m = 3k, for odd j; j ≠ k - 1
and for k even; j ≠ k - 1, 3k - 2.

(43): G = Z₁₄ = <a>,
S = {a, a⁻¹, a⁻², a⁻³, a⁻³, a⁻³} for j = 2, 4.

(44): G = Z₂ × Z₄ × Z₂ₘ = <a> × <b> × <c> (m ≥ 3),
S = {a, ab⁻¹b, b⁻¹, b⁻¹, c, c⁻¹}.

Now we are in a position to prove Theorem 1.1. Immediately from Lemma 2.3, [5, Theorem 1.1] and
[6, Theorem 1.2], we have the Cases (1)-(32) of Theorem 1.1. Assume that Γ is not normal. In view of
Proposition 2.2, we have the following assumption: ∃ s, t, u, v ∈ S such that st = ub ≠ 1 but {s, t} ≠ {u, v}. (*).
We divide S into four cases:

**Case 1**: S = {a, b, c, d, e, f}, where a, b, c, d, e, f are involutions. In this case G is an elementary abelian 2-
group and a, b, c, d, e, f are not independent by the assumption (*). Consequently G = Z₄₁ or G = Z₄₂ or G
13 = Z₄₃. If G = Z₄₁ = <a> × <b> × <c> by the assumption (*), we can let S = {a, b, c, ab, ac, abc}.
We have σ = (a, abc) ∈ A₄, but σ ∈ Aut(G, S); and by Proposition 2.1, G = Cay(G, S) is not normal, the Case (33) of
Theorem 1.1. If G = Z₄₂ = <a> × <b> × <c> by the assumption (*) we see that S is one of the following cases:
(i) S₁² = {a, b, c, d, abc}, (ii) S₂² = {a, b, c, d, abc},
(iii) S₃² = {a, b, c, d, abc}.

When S = S₁, σ = (a, b) ∈ A₄, but σ ∈ Aut(G, S); by Proposition 2.1, G = Cay(G, S) is not normal, the Case
(34) of Theorem 1.1. When S = S₂, we have the Case (22) of the main theorem. Also when S = S₃, G is
normal by Lemma 3.3. If G = Z₄₃ = <a> × <b> × <c> × <d> × <e> we can let S = {a, b, c, d, e, abc}
and hence G = Cay(G, S) is non-normal, the Case (1) of Theorem 1.1.

**Case 2**: S = {a, b, c, d, e, e⁻¹}, where a, b, c, d are involutions but e is not. In this case, S² = {ab, ac, ad,
ae⁻¹, bc, bd, be, be⁻¹, cd, ce, ce⁻¹, de, de⁻¹, e², e⁻²}.

By the assumption (**) d = abc, o(e) = 4 or d = e⁻³.
Suppose d = abc. Then G = Z₂ × Z₂m (m ≥ 2) or
G = Zₘₕ × Zₘₖ (m ≥ 3).

If G = Z₄₂ × Z₂m = <a> × <b> × <c> (m ≥ 2), we can let
S = {a, b, c, eⁿ, bcᵐň, c⁻ⁿ} or
S = {a, b, c⁻ⁿ, bcᵐň, c⁻ⁿ}.

When S = {a, b, c⁻ⁿ, bcᵐň, c⁻ⁿ}, σ = (ab, abcⁿ)(abc⁻ⁿ⁻¹)(abc⁻ⁿ⁻₁)(abc⁻ⁿ⁻₁)(abc⁻ⁿ⁻₁) ∈ A₄,
but σ ∈ Aut(G, S); by Proposition 2.1, G = Cay(G, S) is not normal, the Case (35) of the main theorem.
When S = {a, b, c⁻ⁿ, bcᵐň, c⁻ⁿ}, G = Cay(G, S) is normal by Lemma 3.3. If G = Z₄₃ × Z₂m = <a> × <b> × <c> × <d> × <e> (m ≥ 3), S = {a, b, c, abc, d, d⁻¹}, the
Case (2) of Theorem 1.1. Suppose o(e) = 4. Then G =
When $S = S_1$, $\sigma = (a, c)(ac, c^2)(bc, ab^2)(a, b, b, c) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (36) of Theorem 1.1.

For $S = S_2$, when $m = 2$, $\sigma = (a, ab)(ab, ab^2)(a, b, b, c) \in A_1$, but $\sigma \not\in \text{Aut}(G, S)$; by Proposition 2.1, $\Gamma = \text{Cay}(G, S)$ is not normal, the Case (35) of Theorem 1.1. When $S = S_3$, we have the Case (29, $m=3$) of Theorem 1.1. When $S = S_4$, we have the Case (3) of Theorem 1.1.

If $G = \langle a, b, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (2) of Theorem 1.1. When $u = abd$, we have the Case (25) of Theorem 1.1. If $G = \langle a, b, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (24) of Theorem 1.1.

When $u = abd$, we have the Case (9) of Theorem 1.1. Suppose $a = c^3$, then $G$ is isomorphic to $\langle a, b, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (8) of Theorem 1.1.

If $G = \langle a, b, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (7) of Theorem 1.1.

When $G = \langle a, b, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (6) of Theorem 1.1.

For $S = S_1$, we have the Case (6) of Theorem 1.1. For $S_2$ and $S_3$, we have the Cases (2) and (3, $m=3$) of Lemma 3.3 respectively. If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (5) of Theorem 1.1.

If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (4) of Theorem 1.1. Now suppose $d = e^i$. Then $G = Z_2^i \times Z_4$ or $G = Z_2^i \times Z_6$. If $G = Z_2^i \times Z_4$, we see that $S$ is one of the following cases: $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$.

When $S = S_1$, we have the Case (6) of Theorem 1.1. For $S_2$ and $S_3$, we have the Cases (2) and (3, $m=3$) of Lemma 3.3 respectively. If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (5) of Theorem 1.1.

If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (4) of Theorem 1.1. Now suppose $d = e^i$. Then $G = Z_2^i \times Z_4$ or $G = Z_2^i \times Z_6$. If $G = Z_2^i \times Z_4$, we see that $S$ is one of the following cases: $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$.

If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (6) of Theorem 1.1. For $S_2$ and $S_3$, we have the Cases (2) and (3, $m=3$) of Lemma 3.3 respectively. If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (5) of Theorem 1.1.

If $G = \langle a, c, d, e \rangle$, $S = \{a, b, c, d, e\}$, we have the Case (4) of Theorem 1.1. Now suppose $d = e^i$. Then $G = Z_2^i \times Z_4$ or $G = Z_2^i \times Z_6$. If $G = Z_2^i \times Z_4$, we see that $S$ is one of the following cases: $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$, $S = \{a, b, c, d, e\}$.
normal, the Case (43−S₁, m = 3) of Theorem 1.1. When m ≥ 4, Γ is normal by Lemma 3.3(7−S₃). In the Case (2), S = S₁ when m = 3, σ = (b², ab)(b³, ab²) ∈ A₁, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43−S₃) of Theorem 1.1.

When m = 4, σ = (b, b')(b², b')(b³, b²) ∈ A₁, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case 39 (m = 2) of Theorem 1.1. When m ≥ 5, Γ = Cay(G, S) is normal by Lemma 3.3 (7−S₂). In the Case (2), S = S₂ when m = 5, we have the Case (26) of Theorem 1.1. When m ≥ 6, Γ is normal by Lemma 3.3 (7−S₃).

In the Case (3), S = S₁ when m = 1, we have the Case (43−S₁) of Theorem 1.1. When m ≥ 2, Γ is normal by Lemma 3.3 (8−S₁). In the Case (3), S = S₂, Γ is normal by Lemma 3.3 (8−S₂). In the Case (3), S = S₁, when m = 1, 2, we have the Cases (29,m = 3) of Theorem 1.1 respectively. When m ≥ 3, Γ is normal by Lemma 3.3(8−S₂). In the Case (3), S = S₃ when m = 1, σ = (ab, ab²) ∈ A₁, but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (40, m = 2) of Theorem 1.1. When S = S₃, m ≥ 2, Γ = Cay(G, S) is normal by Lemma 3.3(12). When S = S₄, σ = (a²b², b)(a³b², b)(ab)(ab²m−1, ab²)(a, abm−1, abm+1) ∈ A₁, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (40) of Theorem 1.1.

In the Case (IV), when S = S₁, σ = (a², ac²)(b²c, abc) ∈ A₁, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S₁) of Theorem 1.1. When S = S₂, σ = (a²c, bc²) ∈ A₁, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S₂) of Theorem 1.1. When S = S₃, σ = (a²b, ab²)(a³b², ab²)(a, abm−1, abm+1−1) ∈ A₁, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S₃) of Theorem 1.1. When S = S₄, σ = (a, ac²)(b²c, abc)(ab²m−1, ab²m) ∈ A₁, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S₄) of Theorem 1.1.
(IX), we have the Case (13) of Theorem 1.1. In the Case (X), \(m = 1\), we have the Case (14) of Theorem 1.1, and for \(m \geq 2\), \(\Gamma = \text{Cay}(G, S)\) is normal by Lemma 3.3(14). Suppose \(c^3 = d\), then \(G = Z_2 \times Z_{2m} (m \geq 4)\) or \(G = Z_2^2 \times Z_m (m \geq 5, m \neq 6)\). If \(G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle (m \geq 4)\), we can let \(S_1 = \{a, b^m, b, b^1, b, b^3, b^5\}\) or \(S_2 = \{a, ab^m, b, b^{-1}, b^3, b^5\}\). Let \(S = S_1\) for \(m = 4\), we have the Cases (29), (26) of Theorem 1.1 respectively, and for \(m \geq 6\), \(\Gamma\) is normal by Lemma 3.3(19 – S). Let \(S = S_2\) when \(m = 4\), \(\sigma = (ab^3, ab^5) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S7) of the main theorem.

\[
\sigma_{\neq} \in A_1, \quad \text{but } \sigma \not\in \text{Aut}(G, S), \quad \Gamma = \text{Cay}(G, S) \not\text{ normal, the Case } (43 – S_9) \text{ of Theorem } 1.1.
\]

In the Case (7), \(m = 2i\) and \(m = 2i + 1\), \(\sigma = (b^{i+1}, ab^{2i+1}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (43 – S9) of Theorem 1.1. In the Case (8), for \(m = 2i\) and \(m = 2i – 1\), \(\sigma = (b^{i+1}, ab^{2i+1})(b^{2i+1}, ab^i) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (43 – S9) of Theorem 1.1. If \(G = Z_2^2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle\), we can let \(S = \{a, b, c, c^{-1}\}\), the Case one of the following cases:

\[(1): S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, m \geq 2,\]
\[(2): S_2 = \{a, b, c, c^{-1}, abc, abc^{-1}\}, m \geq 2,\]
\[(3): S_3 = \{a, b, c, c^{-1}, c^{m+1}, c^{m-1}\}, m \geq 3,\]
\[(4): S_4 = \{a, b, c, c^{-1}, ac^{m+1}, ac^{m-1}\}, m \geq 2,\]
\[(5): S_5 = \{a, b, c, c^{-1}, abc^{m+1}, abc^{m-1}\}, m \geq 2,\]
\[(6): S_6 = \{a, cm, c, c^{-1}, bc, b^{-1}\}, m \geq 2,\]
\[(7): S_7 = \{a, c, c, c^{-1}, bc, b^{-1}\}, m \geq 2,\]
\[(8): S_8 = \{a, c, c, c^{-1}, bc, b^{-1}\}, m \geq 2,\]
\[(9): S_9 = \{a, c, c, c^{-1}, bc, b^{-1}\}, m \geq 2,\]

In the Case (1), \(\Gamma\) is normal, the Case (30) of Theorem 1.1. In the Case (2), \(\sigma = (ac^{i+1}, bc^{i+1}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S1) of Theorem 1.1. In the Case (3), when \(m = 2i\), \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (16) of Theorem 1.1.

When \(m = 2i+1\), \(\Gamma = \text{Cay}(G, S)\) is not normal, we have the Case 14 (with \(m\) odd) of Theorem 1.1. In the Case (4), when \(m = 2i\), \(i \geq 2\), \(\sigma = (c^i, ac^i)(ac^i, c^i)(bc^i, abc^i)(abc^i, b^{2i}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), and when \(m = 2i+1\), \(\sigma = (c^i, ac^i)(ac^i, c^i)(bc^i, abc^i)(abc^i, b^{2i}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S5) of Theorem 1.1. In the Case (5), when \(m = 2i\), \(i \geq 2\), \(\sigma = (c^i, ac^i)(ac^i, bc^i)(bc^i, abc^i)(abc^i, b^{2i}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), and when \(m = 2i + 1\), \(\sigma = (c^i, ac^i)(ac^i, bc^i)(bc^i, abc^i)(abc^i, b^{2i}) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S5) of Theorem 1.1. In the Case (6), \(m \geq 2\), \(\Gamma\) is normal, we have the Case (27) of Theorem 1.1. In the Case (7), when \(m = 3\), \(m \geq 2\), \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S7) of Theorem 1.1. In the Case (8), similarly Case (8), \(\Gamma = \text{Cay}(G, S)\) is not normal, the Case (44 – S7) of Theorem 1.1. In the Case (9), similarly Case (8), \(\Gamma = \text{Cay}(G, S)\) is not normal . We have the Case (44 – S4) of Theorem 1.1.

IF \(G = Z_2^2 \times Z_{2m+1} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle\), then \(S = S_1 = \{a, b, c, c^i, ac, ac^i\}\) or \(S_2 = \{a, b, c, c^i, bc, b^{-1}\}\). When \(S = S_1\), \(\sigma = (cm, acm)(bcm, abc) \in A_1\), but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S)\) is not normal , the Case (44 – S4) of the main theorem.
When \( S = S_2, \sigma = (a c^{m-1}, b c^{m-1}) \in A_1, \) but \( \sigma \not\in \text{Aut}(G, S), \) by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal, the Case (44–S) of Theorem 1.1. If \( G = Z_2^{2} \times Z_m = <a \times b \times c \times d>, \) we can consider \( m \geq 3, S = \{a, b, d, c, d, c'\}. \) In this case for \( m = 2i \) and \( m = 2i-1, \) \( (\geq 2) \sigma = (d', c')(a'd', a'c')(bd'c'd')(ab'd', ab'd'), \) if \( \sigma \not\in \text{Aut}(G, S), \) and by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal the Case (14) of Theorem 1.1.

**Case 4:** \( S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}, \) where the elements of the set \( S \) are not involutions By the assumption, \( o(a) = 4, a^2 = b^2, a^4 = b^4 = c^4 = 1. \) Suppose \( o(a) = 4, \) then \( G \) is isomorphic to one of the following: \( Z_{4m}(m \geq 2), Z_4 \times Z_m \times Z_{4m} \times Z_n (m \geq 2, n \geq 3), Z_4 \times Z_{4m} \times Z_n (m \geq 1, n \geq 1), Z_4 \times Z_{4m} \times Z_n (m = n \geq 3). \) If \( G = Z_{4m} = <a, b>, \) we can let \( S = \{a^m, a^{-m}, a, a^{-1}\}, \) where \( 1 \leq m \leq 2m, j \neq m. \) When \( j = 2m - 1, \sigma = (a, a^{-m}) \in A_1, \) but \( \sigma \not\in \text{Aut}(G, S), \) by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal, the Case (45) of Theorem 1.1. When \( j = 2m - 1, \) \( \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(31). If \( G = Z_4 \times Z_{4m} = <a \times b>, \) we can let \( S \) to be one of the following cases:

1. \( S_1 = \{a^2, b^2, ab, a^{-1}b, a, a^{-1}\}, m \geq 3, 1 \leq j \leq m/2, \)
2. \( S_2 = \{a, a^2, b^2, a^{-1}b, a^2b, a^3b^{-1}, a^{-1}b^{-1}\}, m \geq 2, 1 \leq j \leq m/2, \)
3. \( S_3 = \{a, a^2, b^2, a^{-1}b, b^2, b^3\}, m \geq 5, 1 < j < m/2. \)

When \( S = S_1, \) for \( m = 2j, \sigma = (a^2, a^2b)(a^2b, a^2b^{j-1})...\langle a^2b^{j-1}\rangle, \) \( \sigma \not\in \text{Aut}(G, S), \) by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal, the Case (49) of the main theorem. Otherwise, \( \Gamma \) is normal by Lemma 3.3(32). When \( S = S_2, j = 1 \) for \( m = 2k \) and \( m = 2k - 1, k \geq 2, \sigma \not\in \text{Aut}(G, S), \) and when \( j = k - 1, m = 2k \) \( (k \geq 3), \sigma = (b^{k-1}, a^{k-1})...\langle b^{k-1}\rangle, \) \( \sigma \not\in \text{Aut}(G, S), \) and \( \sigma \not\in \text{Aut}(G, S), \) with these graphs are non-normal and we have the Cases (49, 50) of Theorem 1.1. Otherwise, \( \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(33, 34). When \( S = S_3, j = k - 1, m = 2k, \) if \( k \) is odd we have the Case (17) of Theorem 1.1 and if \( k \) is even we have the Case 19 \( (m = 4) \) of the main theorem. For \( m = 5, j = 2 \) and \( m = 10, j = 3 \) we have the Case 21 \( (m = 4) \) of the main theorem.

Otherwise, \( \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(35). If \( G = Z_{4m} \times Z_n = <a \times b, m \geq 2, n \geq 2>, S = \{a^m, a^{-m}, a, a^{-1}, b, b^{-1}\}, \) then \( \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(20). If \( G = Z_{4m} \times Z_{4m} = <a \times b, m \geq 1, n \geq 1>, S = \{a^2b, a^2b^{-1}, a^m, a^{-m}, a, b^{-1}\}, \) then \( \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(21). If \( G = Z_{4n} \times Z_{m} = <a \times b, c \times d \times c >, m \geq 3, \) we can consider \( S = \{a, a, b, b^{-1}, c, c^{-1}\}. \) In this case, for \( m = 4, \) \( \Gamma = \text{Cay}(G, S) \) is not normal, the Case (18) of Theorem 1.1, and for \( m, n \neq 4, \Gamma = \text{Cay}(G, S) \) is normal by Lemma 3.3(22). Suppose \( a^2 = b^2. \) Then \( G \) is isomorphic to one of the following: \( Z_{m} \times Z_{2m} = <a \times c, b \times c \times d >, m \geq 3, 2 \leq j \leq m/2, \)

\( S_4 = \{a, a^{-1}, a^3, a^{3-1}, a, a^{-1}\}, \)

\( (j = 3, 2 \leq j \leq m/2), \)

\( S_5 = \{a, a^3, a^{-1}, a^{3-1}, a, a^{-1}\}, \)

\( (2 \leq j \leq m/2, 3 \neq 0, \)

\( 1 - m - 1, j, m \neq 2(\text{mod } m)), \)

\( S_6 = \{a, a^3, a, b, b^{-1}\}. \)

\( (2) \): \( G = Z_{4m} \times Z_n = <a \times b, c >, n \geq 3, m \geq 5, \) \( S = \{a, a^{-1}, a, b, b^{-1}\}. \)

\( (3) \): \( G = Z_{4m} \times Z_{4m} = <a \times b, c >, m \geq 2, n \neq 1, \)
A1, but

Proposition 2.1, If m = 6k, k ≥ 2, σ = (a, a3k+1), Cay(G, S) is not normal, the Case (51) of Theorem 1.1. When m = 7, j = 3, σ = (a, a4), Cay(G, S) is not normal, the Case (52) of Theorem 1.1. When m = 8, j = 2, σ = (a2, a6) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of the main theorem.

When m = 14; j = 5, σ = (a2, a12i)a(n, a9) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (54) of Theorem 1.1, if m = 4j, j ≥ 2, σ = (a, a2k) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (50) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(43). In the Case (2), if m = 6k, j = 3k + 1, k ≥ 3, σ = (a3k, a2k) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (53) of Theorem 1.1. If m = 4j, j = k, k ≥ 2, σ = (a, a2k) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (55) of Theorem 1.1. In the Case (3), if m = n = 3, σ = (ab, a2b2) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(27).

In the Case (4), if m = 2, we have the Case (21) of Theorem 1.1, if m ≥ 3, Γ = Cay(G, S) is normal by Lemma 3.3(28). In the Case (5), if m = n = 1, σ = (ab, a2b2) ∈ A1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (56) of Theorem 1.1. Otherwise, Γ = Cay(G, S) is normal by Lemma 3.3(29). In the Case (6), Γ = Cay(G, S) is normal by Lemma 3.3(30).

4. Conclusion

Let Γ = Cay(G, S) be a connected Cayley graph of an abelian group G on S. In this paper we have shown all non-normal Cayley graph Γ with valency 6.

References


