NORMAL 6-VALENT CAYLEY GRAPHS OF ABELIAN GROUPS

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Abstract: We call a Cayley graph \( \Gamma = \text{Cay} (G, S) \) normal for \( G \), if the right regular representation \( R(G) \) of \( G \) is normal in the full automorphism group of \( \text{Aut(\Gamma)} \). In this paper, a classification of all non-normal Cayley graphs of finite abelian group with valency 6 was presented.

Keywords: Cayley graph, normal Cayley graph, automorphism group.

1. Introduction

Let \( G \) be a finite group, and \( S \) be a subset of \( G \) not containing the identity element 1. The Cayley digraph \( \Gamma = \text{Cay}(G, S) \) of \( G \) relative to \( S \) is defined as the graph with vertex set \( V(\Gamma) = G \) and edge set \( E(\Gamma) \) consisting of those ordered pairs \( (x, y) \) from \( G \) for which \( yx^{-1} \in S \). Immediately from the definition we find that, there are three obvious facts: (1) \( \text{Aut}(\Gamma) \) contains the right regular representation \( R(G) \) of \( G \) and so \( \Gamma \) is vertex-transitive. (2) \( \Gamma \) is connected if and only if \( G = < S > \). (3) \( \Gamma \) is undirected if and only if \( S^{-1} = S \).

A Cayley (di)graph \( \Gamma = \text{Cay}(G, S) \) is called normal if the right regular representation \( R(G) \) of \( G \) is a normal subgroup of the automorphism group of \( \Gamma \).

The concept of normality of Cayley (di)graphs is known to be important for the study of arc-transitive graphs and half-transitive graphs (see [1,2]). Given a finite group \( G \), a natural problem is to determine all normal or non-normal Cayley (di)graphs of \( G \). This problem is very difficult and is solved only for the cyclic groups of prime order by Alspach [3] and the groups of order twice a prime by Du et al. [4], while some partial answers for other groups to this problem can be found in [5-8]. Wang et al. [8] characterized all normal disconnected Cayley’s graphs of finite groups. Therefore the main work to determine the normality of Cayley graphs is to determine the normality of connected Cayley graphs. In [5, 6], all non-normal Cayley graphs of abelian groups with valency at most 5 were classified. The purpose of this paper is the following main theorem.

Theorem 1.1 Let \( \Gamma = \text{Cay} (G, S) \) be a connected undirected Cayley graph of a finite abelian group \( G \) on \( S \) with valency 6. Then \( \Gamma \) is normal except when one of the following cases happens:

(1): \( G = \mathbb{Z}_2^5 = < a > \times < b > \times < c > \times < d > \times < e >, \) \( S = \{ a, b, c, abc, d, e \} \).

(2): \( G = \mathbb{Z}_2^3 \times \mathbb{Z}_2^m = < a > \times < b > \times < c > \times < d > \times < e > ( m \geq 3 ), \) \( S = \{ a, b, c, ab, d, d^{-1} \} \).

(3): \( G = \mathbb{Z}_2^2 \times \mathbb{Z}_6 = < a > \times < b > \times < c >, \) \( S = \{ a, b, ab, c, c^{-1} \} \).

(4): \( G = \mathbb{Z}_2^4 \times \mathbb{Z}_m = < a > \times < b > \times < c > \times < d > \times < e >, \) \( S = \{ a, b, c, d, e, e^{-1} \} \).

(5): \( G = \mathbb{Z}_2^3 \times \mathbb{Z}_6 = < a > \times < b > \times < c > \times < d >, \) \( S_1 = \{ a, b, c, d^2, d, d^{-1} \}, S_2 = \{ a, b, ab, c, d, d^{-1} \}, S_3 = \{ a, b, c, ad^2, d, d^{-1} \} \).

(6): \( G = \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 = < a > \times < b > \times < c >, \) \( S = \{ a, b, ab, c, c^{-1} \} \).

(7): \( G = \mathbb{Z}_2^2 \times \mathbb{Z}_3^3 = < a > \times < b > \times < c > \times < d >, \) \( S = \{ a, b, c, d, d^{-1}, d^{-1} \} \).

(8): \( G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = < a > \times < b > ( m \geq 2 ), \) \( S = \{ a, b, c, a^2, b, b^{-1} \} \).

(9): \( G = \mathbb{Z}_2^2 \times \mathbb{Z}_2^3 = < a > \times < b > \times < c > ( m \geq 3 ), \) \( S = \{ a, b, c, a^2, b, b^2, c, c^{-1} \} \).

(10): \( G = \mathbb{Z}_2^4 \times \mathbb{Z}_m = < a > \times < b > ( m \geq 2 ), \) \( S = \{ a, a, a^2, b, b^2, c, c^{-1} \} \).

(11): \( G = \mathbb{Z}_2^3 \times \mathbb{Z}_2^m = < a > \times < b > \times < c > ( m \geq 3 ), \) \( S_1 = \{ a, b, b^3, c, c^{-1} \}, S_2 = \{ a, b, b^3, ab^2, c, c^{-1} \} \).

(12): \( G = \mathbb{Z}_2^4 \times \mathbb{Z}_m = < a > \times < b > \times < c > ( m \geq 2 ), \) \( S = \{ a, b, c, c^2, d, d^{-1} \} \).

(13): \( G = \mathbb{Z}_2^3 \times \mathbb{Z}_m = < a > \times < b > \times < c > \times < d > ( m \geq 3 ), \) \( S = \{ a, b, c, c^2, d, d^{-1} \} \).
S = \{ab, a b^{-1}, b, b^{-1}, c, c^{-1}\}.

(15): G = Z_2^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3),
S = \{a, b, c, d, d^{-1}, d, d^{-1}\}.

(16): G = Z_2^4 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, c, c^{-1}, c^2, c^{-3}, d, d^{-1}\}.

(17): G = Z_2^4 \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m \geq 3, m \text{ is odd}),
S = \{a, a^3, b, b^{-1}, b^{2m-1}, b^{2m-1}\}.

(18): G = Z_4^2 \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3),
S = \{a, b, ab, c, c^{-1}\}.

(19): G = Z_4m \times Z_m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2, n \geq 3),
S = \{a, a^3, a^{-2}, a^{-2}, b, b^{-1}\}.

(20): G = Z_4^2 \times Z_m \times Z_n = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3, n \geq 3),
S = \{ab, b^{2m}, b^{2m-1}, c, c^{-1}\}.

(21): G = Z_4 \times Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \quad (m = 5, 10, n \geq 3),
S = \{a, a^3, a^{-2}, a^{-3}, b, b^{-1}\}.

(22): G = Z_4^2 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,
S = \{a, b, ab, c, c^{-1}\}.

(23): G = Z_2^2 \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle,
S = \{a, b, ac, ac^{-1}\}.

(24): G = Z_2^4 \times Z_m^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle,
S = \{a, b, ab, b^{-1}, c, c^{-1}\}.

(25): G = Z_4^2 \times Z_3m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 1),
S = \{a, b, ab, abc, abc^{-1}\}.

(26): G = Z_4 \times Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, abc, abc^{-1}, ac, ac^{-1}\}.

(27): G = Z_2^2 \times Z_2m \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, ac, ac^{-1}, a^{-1}, a^{-1}\}.

(28): G = Z_2^4 \times Z_2m \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, ac, ac^{-1}, a^{-1}, a^{-1}\}.

(29): G = Z_4 \times Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3),
S = \{a, b, ab, abc, abc^{-1}\}.

(30): G = Z_2^2 \times Z_2m \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, ac, c, c^{-1}\}.

(31): G = Z_2^4 \times Z_2m \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 3, m \text{ is odd}),
S = \{a, b, c, c^2, c^{-3}, d, d^{-1}\}.

(32): G = Z_4^2 \times Z_4m \times Z_2m = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \quad (m \geq 2),
S = \{a, b, c, c^2, b^m, b^{-1}, c, c^{-1}\}.
\[ S_1 = \{ab, ab^{-1}, b, b^{-1}, ab^{-1}, ab\}, (2 \leq j < \frac{m}{2}) \], (m, j) \neq p > 2; m = (t + 1)p. \]

(48): \( G = Z_{2m} \times Z_2 = \langle a \rangle \times \langle b \rangle \), \( S_1 = \{a, ab^{-1}, b, b^{-1}, a, a^{-1}\} \), \( S_2 = \{ab, ab^{-1}, b, b^{-1}, ab^{-1}, ab\} \).

(49): \( G = Z_{2m} \times Z_2 = \langle a \rangle \times \langle b \rangle \), \( m \geq 2, n \geq 3 \), \( S = \{a, a^{-1}, b, b^{-1}, a, a^{-1}, ab, ab^{-1}\} \).

(50): \( G = Z_{2m} \times Z_2 = \langle a \rangle \times \langle b \rangle \), \( m \geq 3, n \geq 2 \), \( S = \{a, a^{-1}, b, b^{-1}, a, a^{-1}, ab, ab^{-1}\} \).

(51): \( G = Z_{2m} = \langle a \rangle \), \( m \geq 2 \), \( S = \{a, a^{-1}, a, a^{-1}, a^2, a^2^{-1}\} \), \( S_2 = \{a, a^{-1}, a^2, a^2^{-1}, a, a^{-1}\} \).

(52): \( G = Z_m = \langle a \rangle \), \( m = 7, 14 \), \( S = \{a, a^{-1}, a^3, a^3, a^3, a^3\} \).

(53): \( G = Z_{2m} = \langle a \rangle \), \( m \geq 3 \), \( S = \{a, a^{-1}, a^2, a^2, a^4, a^4, a^4, a^4, a^4, a^4\} \).

(54): \( G = Z_{2m} = \langle a \rangle \), \( m \geq 2 \), \( S = \{a, a^{-1}, a^2, a^2, a^4, a^4, a^4, a^4, a^4, a^4\} \).

(55): \( G = Z_{2m} = \langle a \rangle \), \( m \geq 1 \), \( S = \{a, a^{-1}, a^2, a^2, a^4, a^4, a^4, a^4, a^4, a^4\} \).

(56): \( G = Z_3 \times Z_3 = \langle a \rangle \times \langle b \rangle \), \( S = \{a, b, b^{-1}, a, a^{-1}\} \).

(57): \( G = Z_3 \times Z_3 \times Z_3 = \langle a \rangle \times \langle b \rangle \), \( S = \{a, b, b^{-1}, c, c^{-1}, ab, ab^{-1}\} \).

2. Primary Analysis

Proposition 2.1 [9, Proposition 1.5] Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley graph of \( G \) over \( S \), and \( A = \text{Aut}(\Gamma) \). Let \( A_1 \) be the stabilizer of the identity element \( 1 \) in \( A \). Then \( \Gamma \) is normal if and only if every element of \( A_1 \) is an automorphism of \( G \).

Proposition 2.2 [6, Theorem 1.1] Let \( G \) be a finite abelian group and \( S \) be a generating subset of \( G - \{1\} \). Assume \( S \) satisfies the condition that, if \( s, t, u, v \in S \) with \( 1 \neq st = uv \), implies \( [s, t] = \{u, v\} \). Then the Cayley graph \( \text{Cay}(G, S) \) is normal.

Let \( X \) and \( Y \) be two graphs. The direct product \( X \times Y \) is defined as the graph with vertex set \( V(X \times Y) = V(X) \times V(Y) \) such that for any two vertices \( u = [x_1, y_1] \) and \( v = [x_2, y_2] \) in \( V(X \times Y) \), \( u, v \) is an edge in \( X \times Y \) whenever \( x_1 = x_2 \) and \( y_1 = y_2 \) in \( E(X) \). A graph \( T \) is a Cayley graph of \( G \) on \( S \) with valency \( 6 \).

Let \( G = \langle a \rangle \times \langle b \rangle \) be a Cayley graph of \( G \) on \( S \) with valency \( 6 \).

Let \( G = \langle a \rangle \times \langle b \rangle \) be a Cayley graph of \( G \) on \( S \) with valency \( 6 \). Then there is a natural embedding \( nX \) in \( X[Y] \), where for \( 1 \leq i \leq n \), the ith copy of \( X \) is the subgraph induced on the vertex subset \( \{x, y_i\} \in V(X) \) in \( X[Y] \).

The deleted lexicographic product \( X[Y] - nX \) is the graph obtained by deleting all the edges of \( (\text{this natural embedding of}) \) \( nX \) from \( X[Y] \). Let \( \Gamma \) be a graph and \( \alpha \) a permutation \( V(\Gamma) \) and \( C_\alpha \) a circuit of length \( n \). The twisted product \( \Gamma \times_\alpha C_\alpha \) of \( \Gamma \) by \( C_\alpha \) with respect to \( \alpha \) is defined by:

\[ V(\Gamma \times_\alpha C_\alpha) = V(\Gamma \times V(C_\alpha)) = \{(x, i) \mid x \in V(\Gamma), i = 0, 1, \ldots, n - 1\}. \]

E(\Gamma \times_\alpha C_\alpha) = \{(x, i), (x, i+1) \mid x \in V(\Gamma), i = 0, 1, \ldots, n - 2\} \bigcup \{((x, i), (y, i+1)) \mid (x, y) \in E(\Gamma), i = 0, 1, \ldots, n - 1\}. \]

The graph \( Q^d_4 \) denotes the graph obtained by connecting all long diagonals of 4-cube \( Q_4 \), that is, connecting all vertices \( u \) and \( v \) in \( Q_4 \) such that \( d(u, v) = 4 \). The graph \( K_{m,m} \times C_\alpha \) is the twisted product of \( K_{m,m} \) by \( C_\alpha \) such that \( \alpha \) is a cycle permutation on each part of the complete bipartite graph \( K_{m,m} \). The graph \( Q^d_3 \times \alpha C_\alpha \) is the twisted product of \( Q_3 \) by \( C_\alpha \) such that \( \alpha \) transposes each pair of elements on long diagonals of \( Q_3 \). The graph \( C_{2m}^d[2K_1] \) is defined by:

\[ V(C_{2m}^d[2K_1]) = V(C_{2m}^d[2K_1]), \]

E(C_{2m}^d[2K_1]) = E(C_{2m}^d[2K_1]) \bigcup \{(x_i, y_j), (x_{i+1}, y_j) \mid i = 0, 1, \ldots, m = 1, j = 1, 2\}, \]

where \( V(C_{2m}^d[2K_1]) = \{x_i, x_1, \ldots, x_{2m-1}\} \) and \( V(2K_1) = \{y_1, y_2\} \).

Let \( G = \times G_2 \) be the direct product of two finite groups \( G_1 \) and \( G_2 \), let \( S_1 \) and \( S_2 \) be subsets of \( G_1 \) and \( G_2 \), respectively, and let \( S = S_1 \cup S_2 \) be the disjoint union of two subsets \( S_1 \) and \( S_2 \). Then we have,

Lemma 2.3

(1) Cay(G, S) = Cay(G, S) \times Cay(G, S).

(2) If Cay(G, S) is normal, then Cay(G, S) is also normal.

(3) If both of Cay(G, S) and Cay(G, S) are normal and relatively prime, then Cay(G, S) is normal.

3. Proof of the Main Theorem

In this section, \( \Gamma \) always denotes the Cayley graph Cay(G, S) of an abelian group G on S with valency 6. Let \( A = \text{Aut}(\Gamma) \). Then \( A_1 \) and \( A_1^* \) denote the stabilizer of \( 1 \) in \( A \) and the subgroup of \( A \) which fixes \( \{1\} \bigcup S \), pointwise, respectively. In order to prove Theorem 1.1 we need several lemmas.

Lemma 3.1 Let \( G = Z_{2m} = \langle a \rangle \), \( m \geq 5 \), and \( S = \{a_i, \ a_i^{-1}, a_i, a_i^m, a_i^{-1}, a_i^{-1}\} \leq i < \frac{m}{2} \). Then \( \Gamma = \text{Cay}(G, S) \) is normal.
Proof Let $Γ_1(1)$ be the subgraph of $Γ$ with vertex set \( \{a, b, c, d\} \) and edge set \( \{(a, b), (b, c), (c, d), (d, a)\} \). By observing the subgraph $Γ_1(1)$, it is easy to prove that $A_1$ fixes $S'$ pointwise, which implies that $A_1 = 1$. Thus $A_1$ acts faithfully on $S$. Observing the subgraph $Γ_1(1)$ again, $A_1$, as a permutation group on $S$, is generated by \( (a, b, c, d) \). So $|A_1| = 2$ and $Γ = Cay(G, S)$ is normal.

Lemma 3.2: Let $G = Z_{2^m} × Z_{n^m} = \langle a \rangle × \langle b \rangle$, $m = 4k, k ≥ 2$ and $S = \{a, b, c^k, c^{k+1}\}$. Then $Γ = Cay(G, S)$ is normal.

Proof Set $G_1 = \langle a, b \rangle, G_2 = \langle c \rangle, S_1 = \{a, b\}, S_2 = \{c^k, c^{k+1}\}$. Then $Γ_1 = Cay(G_1, S_1)$ is $K_2 × K_2$. Note that $Γ_1$ and $Γ_2 = Cay(G_2, S_2)$ are relatively prime. By [5, Theorem 1.1] and [6, Theorem 1.2], $Γ_1$ and $Γ_2$ are normal and by Lemma 2.3, $Γ = Cay(G, S)$ is normal.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.

Lemma 3.3 Let $G$ and $S$ be as the following. Then the Cayley graphs $Γ = Cay(G, S)$ are normal.

1: $G = Z_2 × Z_4 = \langle a \rangle × \langle b \rangle, S = \{a, b, a^2, b^2\}$.

2: $G = Z_2 × Z_3 = \langle a \rangle × \langle b \rangle, S = \{a, b, a^2, b^2\}$.

3: $G = Z_2 × Z_2 = \{a, b, a^2, b^2\}$.

With similar arguments as in Lemmas 3.1 and 3.2, we have the following lemma.
(30): \( G = Z_2 \times Z_{m+1} \times Z_{2m-1} = \langle a, b, c \rangle \) (\( m, n \geq 1 \)), 
\( S = \{a^{2m}, a^{2m+1}, a^{2m+2}, b, c \} \).

(31): \( G = Z_{4m} = \langle a, b \rangle \) (\( m \geq 2 \)), 
\( S = \{a, a^2, a^3, a^{m-1}, a^{m}, a^{m+1}, a^{m+2}, a^{m+3} \} \) (for \( 1 < k < 2m, k \neq m, 2m-1 \)).

(32): \( G = Z_4 \times Z_m = \langle a, b \rangle \times \langle b \rangle \) (\( m \geq 3 \)), 
\( S = \{a, a^2, b, b^2, c, c^2 \} \) (for \( 1 \leq j \leq \frac{m}{2} \)).

(When \( m \neq 2k \) for every \( j \) or \( m = 2k, j \neq k \)).

(33): \( G = Z_4 \times Z_{2m} = \langle a, b \rangle \times \langle b \rangle \) (\( m \geq 2 \)), 
\( S = \{a, a^2, b, b^2, a^3, b^3, a^4, b^4 \} \) (for \( 1 \leq j \leq m \)).

(34): \( G = Z_4 \times Z_{2m-1} = \langle a, b \rangle \times \langle b \rangle \) (\( m \geq 3 \)), 
\( S = \{a, a^2, b, b^2, a^3, b^3, a^4, b^4 \} \) (for \( 1 \leq j < \frac{2m-1}{2} \)).

(35): \( G = Z_4 \times Z_m = \langle a, b \rangle \times \langle b \rangle \) (\( m \geq 5 \)), 
\( S = \{a, a^2, b, b^2, a^3, b^3 \} \) (for \( 1 \leq j < \frac{m}{2} \)).

(When \( m \neq 2k \), \( 5 \leq m \leq 2k (k \geq 3, k \neq 5) \)), \( j \neq k-1 \) or \( m = 10, j \neq 3 \). [Note: Corrected to ensure logical flow.]
If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, we have $S$ is one of the following cases:

$S_1 = \{a, b, ab, c, c^2, c^4\}$, $S_2 = \{a, b, ac^2, bc, c^2, c^4\}$, $S_3 = \{a, b, ab, c, c^2, c^4\}$, $S_4 = \{a, b, ab^2, c, c^2, c^4\}$, $S_5 = \{a, b, ab, c^2, c^3, c^4\}$.

When $S = S_1$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (36 - $S_1$) of Theorem 1.1. For $S = S_2$, we have the Case (33) of Theorem 1.1. When $S = S_3$, we have the Case (34) of Theorem 1.1. When $S = S_4$, we have the Case (35) of Theorem 1.1. When $S = S_5$, we have the Case (36) of Theorem 1.1. When $S = S_1, S_2$, $\Gamma = Cay(G, S)$ is normal, the Case (37) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (38) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (39) of Theorem 1.1.

When $S = S_2$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (40 - $S_2$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (41 - $S_2$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (42 - $S_2$) of Theorem 1.1.

When $S = S_3$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (43 - $S_3$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (44 - $S_3$) of Theorem 1.1.

When $S = S_4$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (45 - $S_4$) of Theorem 1.1.

When $S = S_5$, $\sigma = (a, c)(ab, bc)(c^2, ac^3)(bc^3, abc^3) \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (46 - $S_5$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (47 - $S_5$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (48 - $S_5$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (49 - $S_5$) of Theorem 1.1.

When $S = S_1, S_2$, $\Gamma = Cay(G, S)$ is normal, the Case (50 - $S_1$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (51 - $S_1$) of Theorem 1.1.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $S = \{a, b, ab, c, c^2, c^3, c^4\}$, then $\sigma \notin Aut(G, S)$; by Proposition 2.1, $\Gamma = Cay(G, S)$ is normal, the Case (52 - $S_1$) of Theorem 1.1.
normal, the Case \((43 - S_3, m = 3)\) of Theorem 1.1. When \(m \geq 4\), \(\Gamma\) is normal by Lemma 3.3(7 - \(S_3\)). In the Case \((2), S = S_1\) when \(m = 3\), \(\sigma = (b^2, ab^b)(b^b, b^{b'}) \in A_1\), but \(\sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is not normal, the Case \((41)\) of Theorem 1.1. In the Case \((2), S = S_2\) when \(m = 5\), we have the Case (26) of Theorem 1.1. When \(m \geq 6\), \(\Gamma\) is normal by Lemma 3.3(7 - \(S_3\)).

In the Case \((3), S = S_1\), when \(m = 1\), we have the Case \((43 - S_3)\) of Theorem 1.1. When \(m \geq 2\), \(\Gamma\) is normal by Lemma 3.3(8 - \(S_3\)). In the Case \((3), S = S_2\), when \(m = 1, 2\), we have the Cases \((29, m = 3, 5)\) of Theorem 1.1 respectively. When \(m \geq 3\), \(\Gamma\) is normal by Lemma 3.3(8 - \(S_3\)). In the Case \((3), S = S_4\), when \(m = 1\), \(\sigma = (ab, ab^b) \in A_1\), but \(\sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is not normal, the Case \((27, m = 1)\) of Theorem 1.1. When \(m \geq 2\), \(\Gamma = Cay(G, S)\) is normal by Lemma 3.3(9). In the Case \((4), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(10). In the Case \((5), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(11). In the Case \((6), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(12). In the Case \((7), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(13). In the Case \((8), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(14). In the Case \((9), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(15). In the Case \((10), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(16). In the Case \((11), \Gamma = Cay(G, S)\) is normal by Lemma 3.3(17). Suppose \(o(\sigma) = 4\). Then we have one of the following cases:

(I) \(G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b^2, b^{-1}, a, ab\},\)

(II) \(G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b^{2m}, a^{b}, b^{2m}, b^{-1}, a, ab\}, (m \geq 1), S_2 = \{a, b^{2m}, b^{2m}, b^{-1}, a, ab\}, (m \geq 2), S_3 = \{a, ab^m, b^m, b^{-1}, a, b\}, (m \geq 2).\)

(III) \(G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a^2, a^{b}, a^{-1}, b, b^{-1}\}, S_2 = \{a^2, a^{b}, a^{-1}, b, b^{-1}\}.

(IV) \(G = Z_2 \times Z_4 = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, S_2 = \{a, b, c, c^{-1}, ac, ac^{-1}\}.

(V) \(G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, S_2 = \{a, b, c, c^{-1}, ac, ac^{-1}\}.

(VI) \(G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, S_2 = \{a, b, c, c^{-1}, ac, ac^{-1}\}.

(VII) \(G = Z_2 \times Z_{2m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, ac, ac^{-1}\}, S_2 = \{a, b, c, c^{-1}, ac, ac^{-1}\}.

(IX) \(G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, d, d^{-1}\}.

(X) \(G = Z_2 \times Z_{4m} = \langle a \rangle \times \langle b \rangle, S_1 = \{a, b, c, c^{-1}, d, d^{-1}\}.

In the Case \((I), \sigma = (ab, b^2) \in A_1\), but \(\sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is not normal, the Case \((43 - S_3)\) of Theorem 1.1. In the Case \((II), S = S_2\), \(\Gamma = Cay(G, S)\) is normal by Lemma 3.3(11 - \(S_3\)). In the Case \((II), S = S_3, \sigma = (b, b^{-1})(b^2, b^{-2})...(b^{2m}, b^{2m-1})(a, ab^m, b^{2m-1}, b^{2m-1})(a, ab^m, b^{2m-1}, b^{2m-1}) \in A_1\), but \(\sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is not normal, the Case \((39)\) of Theorem 1.1. In the Case \((III), S = S_3, \sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is normal by Lemma 3.3(3), the Case \((11 - S_3)\). In the Case \((IV), S = S_1\), \(S = S_2, m = 2, \sigma = (a^2, b^2)(b^2, b^{-2})(ab, ab^{-1})(ab^2, ab^{-2}) \in A_1\), but \(\sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is not normal, the Case \((30 - S_1)\) of Theorem 1.1. When \(m = 1, \sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is normal, the Case \((47)\) of Theorem 1.1. When \(m = 2, \Gamma = Cay(G, S)\) is normal by Lemma 3.3(13 - \(S_3\)); In the Case \((V), S = S_2, m = 1, \sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is normal by Lemma 3.3(13 - \(S_3\)), we obtain the same result. When \(m \geq 2\), \(\Gamma = Cay(G, S)\) is normal by Lemma 3.3(13 - \(S_3\)); In the Case \((VI), S = S_3, m = 1, \sigma \not\in Aut(G, S)\); by Proposition 2.1, \(\Gamma = Cay(G, S)\) is normal, the Case \((57)\) of Theorem 1.1.
(IX), we have the Case (13) of Theorem 1.1. In the Case (X), m = 1, we have the Case (14) of Theorem 1.1, and for m ≥ 2, Γ = Cay(G, S) is normal by Lemma 3.3(14). Suppose c^3 = d, then G = Z_2 × Z_m (m ≥ 4) or G = Z_2 × Z_m (m ≥ 5, m ≠ 6). If G = Z_2 × Z_m = <a> × <b> (m ≥ 4), we can let S to be S_1 = {a, b, b^1, b, b^2} or S_2 = {a, ab, b, b^(-1), b^1, b^3}. Let S = S_1, for m = 4, 5 we have the Cases (29), (26) of Theorem 1.1 respectively, and for m ≥ 6, Γ is normal by Lemma 3.3(19−S). Let S = S_2, when m = 4, σ = (ab^3, ab^7)(ab^2, ab^8)(ab, ab^2, ab^8) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S) of the main theorem.

In the Case (1), m = 2i, when m = 2i + 1, σ = (b, b^(-1), a, ab, ab^(-1)) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43−S) of Theorem 1.1. In the Case (9), for m = 2i and m = 2i − 1, σ = (b, ab^(-1))(bc, ab) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (43−S) of Theorem 1.1. If G = Z_2 × Z_m = <a> × <b> × <c>, we can let S to be one of the following cases:

(1): S_1 = {a, b, c, c^(-1), ac, ac^(-1)} (m ≥ 2),
(2): S_2 = {a, b, c, c^(-1), abc, abc^(-1)} (m ≥ 2),
(3): S_3 = {a, b, c, c^(-1), c^m, c^(-1)} (m ≥ 3),
(4): S_4 = {a, b, c, c^(-1), ac^m, ac^(-1)} (m ≥ 2),
(5): S_5 = {a, b, c, c^(-1), abc^m, abc^(-1)} (m ≥ 2),
(6): S_6 = {a, cm, c, c^(-1), bc, bc^(-1)} (m ≥ 2),
(7): S_7 = {a, ac^2, ac^(-2), bc, bc^(-1)} (m ≥ 2),
(8): S_8 = {a, c^2, c^(-1), bc, bc^(-1)} (m ≥ 2),
(9): S_9 = {a, ac^2, ac^(-2), abc, abc^(-1)} (m ≥ 2).

In the Case (1), Γ is not normal, the Case (30) of Theorem 1.1. In the Case (2), σ = (ac^2, bc^2) ∈ A_1, but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S) of Theorem 1.1. In the Case (3), when m = 2i, Γ = Cay(G, S) is not normal, the Case (16) of Theorem 1.1.

In the Case (4), when m = 2i, i ≥ 2, σ = (c^2, ac^2)(ac^(-2), bc^2)(abc, abc^(-1)) ∈ A_1, but σ ∉ Aut(G, S), and when m = 2i + 1, σ = (c^2, ac^2)(ac^(-2), bc^2)(abc, abc^(-1)), but σ ∉ Aut(G, S), by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S) of Theorem 1.1. In the Case (5), when m = 2i, i ≥ 2, σ = (c^m, ac^m, bc^m)(abc^m, abc^(-1)) ∈ A_1, but σ ∉ Aut(G, S), and when m = 2i + 1, σ = (c^m, ac^m, bc^m), but σ ∉ Aut(G, S); by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S) of Theorem 1.1.

In the Case (6), m ≥ 2, Γ is not normal, we have the Case (27) of Theorem 1.1. In the Case (7), if m ≥ 3, for m = 2i and m = 2i − 1, σ = (ci, bci)(aci)(ci+m, bci+m)(aci+m, abci+m) ∈ A_1, but σ ∉ Aut(G, S), and when m = 2i + 1, σ = (ci, bci)(aci)(ci+m, bci+m), but σ ∉ Aut(G, S). Then by Proposition 2.1, Γ = Cay(G, S) is not normal, the Case (44−S) of Theorem 1.1.

In the Case (8), if G = Z_2 × Z_m = <a> × <b> × <c> (m ≥ 2), then S

S_1 = {a, b, c, c^(-1), ac, ac^(-1)}, S_2 = {a, b, c, ac, ac^(-1)}.
When \( S = S_2, \sigma = (ac^{m-1}, bc^{m-1}) \in A_1 \), but \( \sigma \not\in \text{Aut}(G, S) \), by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal, the Case (44–S1) of Theorem 1.1. If \( G = Z_2^n \times Z_m = \langle a^n, b \rangle \times \langle b, c \rangle \), we can consider \( m \geq 3, S = \{ a, b, c, d, e, f, g \} \). In this case for \( m = 2i \) and \( m = 2i-1 \), \((\geq 2) \sigma = (d^i, e^i)(a^j)(a^{j+1}) \in A_1 \), but \( \sigma \not\in \text{Aut}(G, S) \) and by Proposition 2.1, \( \Gamma = \text{Cay}(G, S) \) is not normal the Case (14) of Theorem 1.1.

**Case 4:** \( S = \{ a, a^2, b, b^2, c, c^2 \} \), where the elements of the set \( S \) are not invertible by the assumption (*), \( o(a) = 4, a^2 = b, a^3 = b = c = a^2b \). Suppose \( o(a) = 4 \), then \( G \) is isomorphic to one of the following: \( Z_4m = \langle m \rangle \leq Z_4 \times Z_m \times Z_n \), \( Z_3 \times Z_4 \times Z_n \) (\( m \geq 2, n \geq 3 \)), \( Z_3 \times Z_4 \times Z_2m \) (\( m \geq 1 \)), \( Z_3 \times Z_4 \times Z_n \) (\( m \geq 2, n \geq 3 \)). If \( G = Z_4m = \langle a \rangle \) (\( m \geq 2 \)), we can let \( S = \{ a^i, a^{i+1}, a^{i+2}, a^{i+3} \} \), where \( 1 \leq i < 2m, 2 \leq j \leq m \). When \( S = S_2, j = k \), we have the Case (49) of Theorem 1.1. If \( S = S_3, j > m/2 \) or \( S = S_4, j > m/2 \), we have the Case (49) of Theorem 1.1. When \( m = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( n = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( j = k, 2k \), \( k > 3 \), \( S = \{ a, b, a^i, a^{i+1} \} \), \( 1 < j < 2m, 2 \leq j \leq m \). When \( S = S_2, j = k \), we have the Case (49) of Theorem 1.1. When \( m = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( n = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( j = k \), \( k > 3 \), \( S = \{ a, b, a^i, a^{i+1} \} \), \( 1 < j < 2m, 2 \leq j \leq m \). When \( S = S_2, j = k \), we have the Case (49) of Theorem 1.1. When \( m = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( n = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( j = k \), \( k > 3 \), \( S = \{ a, b, a^i, a^{i+1} \} \), \( 1 < j < 2m, 2 \leq j \leq m \). When \( S = S_2, j = k \), we have the Case (49) of Theorem 1.1. When \( m = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( n = 2i \), \( i = 1, 2, 3 \), \( S = \{ a, a^i, b, b^i \} \), \( j = k \), \( k > 3 \), \( S = \{ a, b, a^i, a^{i+1} \} \), \( 1 < j < 2m, 2 \leq j \leq m \).
S = \{a^i b^j, a^{2i+1} b^{2j}, a^i, a, a^{-1}, b, b^{-1}\}.

(4): \Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(27) .

In the Case (1), when \(m = 6k, j = 3k-1, k \geq 2\), \(\sigma = (a, a^{3k-1})(a, a^{3k-4})... (a, a^{3k-2}, a^{6k-2}) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S) is normal, the Case (52) of Theorem 1.1.\)

When \(m = 14, j = 5, \sigma = (a^{2j}, a^{3j}) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(42, 43) .\)

In the Case (2), if \(m = 7, j = 4, \sigma = (a^j, a^k) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), and if \(m = 14, j = 5, \sigma = (a^j, a^k) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(42, 43) .\)

In the Case (3), if \(m = n = 3, \sigma = (a^j, a^{j+1}) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(42, 43) .\)

In the Case (4), if \(m = 2\), we have the Case (21) of Theorem 1.1. if \(m \geq 3, \Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(28) .\)

In the Case (5), if \(m = n = 1, \sigma = (a^2, a^3) \in A_1, \) but \(\sigma \not\in \text{Aut}(G, S)\), by Proposition 2.1, \(\Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(28) .\)

In the Case (6), \(\Gamma = \text{Cay}(G, S) is normal by Lemma 3.3(30) .\)

4. Conclusion

Let \(\Gamma = \text{Cay}(G, S) be a connected Cayley graph of an abelian group G on S. In this paper we have shown all non-normal Cayley graph \(\Gamma\) with valency 6.

References


