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The Lie Algebra of Smooth Sections of a T-bundle

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Abstract: In this article, we generalize the concept of the Lie algebra of vector fields to the set of smooth sections of a T-bundle which is by definition a canonical generalization of the concept of a tangent bundle. We define a Lie bracket multiplication on this set so that it becomes a Lie algebra. In the particular case of tangent bundles this Lie algebra coincides with the Lie algebra of vector fields.

Keywords: Vector bundle, Lie theory

1. Introduction

We know that if M is a smooth manifold, then χ(M), the set of all smooth vector fields on M (or, the set of all smooth sections of χ(M) forms a Lie algebra. (See for example [2]). In this paper first we define the concept of a T-bundle, which is a generalization of tangent bundle. Then, we define a Lie algebra structure on the set of all smooth sections of a T-bundle such that in the particular case of tangent bundle, it coincides with the Lie algebra of vector field. We are able to generalize many of definitions and theorems about vector fields to the set of smooth sections of T-bundles.

2. T-bundle

2.1. Definition

Let E and M be smooth manifolds with dimensions n +k and n respectively, such that $k = b.n$ for some $b \in .N$. Also suppose that p:E→M is a smooth map. By a T-chart on (E, p, M) we mean an ordered pair (U, y) where U is the domain of a chart (U, u) in M and y is a fiber respecting diffeomorphism as in the following diagram,

$$\begin{tikzcd} E|U := p^{-1}(U) \arrow[rr, "y"] \arrow[dr, "p"'] & & U \times R^k \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

where pr_1 is the projection on the first component. Two T-chart (U_a, y_a) and (U_b, y_b) are called TM-compatible

$$\forall x \in U_{ab} := U_a \cap U_b, \forall v \in R^k; \quad (y_a o y_b^{-1})(x, v) = (x, y_{ab}(x).v) \tag{1}$$

For some mapping $y_{ab} : U_{ab} \to GL(k, R)$ by

$$x \;\mathbf{a} \begin{bmatrix} J_{ab}(x) & 0 & \mathbf{L} & 0 \\ 0 & J_{ab}(x) & \mathbf{L} & 0 \\ \mathrm{M} & \mathrm{M} & & \mathrm{M} \\ 0 & 0 & \mathbf{L} & J_{ab}(x) \end{bmatrix} \tag{2}$$

Where $J_{ab}(x) = [\frac{\partial u_a^i}{\partial u_b^j}(x)]$. The mapping y_{ab} is called the transition function between the two T-charts.

2.2. Definition

A T-atlas $A = (U_a, y_a)$ for (E, p, M) is a set of pair-wise TM-compatible T-charts (U_a, y_a) such that $(U_a)_{a \in I}$ is an open cover of M . Two T-atlases are called equivalent, if their union is again a T-atlas.

2.3.Definition

A T-bundle (E, p, M) consists of two manifolds E (the total space) and M (the base) and a smooth mapping $p : E \to M$ (the projection) together with an equivalence class of T-atlases.

2.4. Lemma

Let (E, p, M) be a T-bundle. Then for any $x \in M$ there is a unique vector space structure on fiber $E_x := p^{-1}(x)$ which is isomorphic with R^k .

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Proof: Let (U_a, y_a) and (U_b, y_b) be two T-charts with $x \in U_{ab}$, then we have

$$\begin{aligned} y_a|_{E_x} &: E_x \longrightarrow \{x\} \times R^k \\ x_x &\ \mathbf{a}\ (x, h_a(x_x)) \\ y_b|_{E_x} &: E_x \longrightarrow \{x\} \times R^k \\ x_x &\ \mathbf{a}\ (x, h_b(x_x)). \end{aligned} \tag{3}$$

Where h_a and h_b are two invertible mappings (These two mappings are exist because y_a and y_b are diffeomorphisms.). Therefore, for $x \in U_{ab}$ and $v \in R^k$, we have

$$y_a(y_b^{-1}(x,v)) = (x, h_a(y_b^{-1}(x,v)) \tag{4}$$

or $y_{ab}(x).v = h_a(y_b^{-1}(x,v))$; also, we have

$$(x, h_b(y_b^{-1}(x,v)) = y_b(y_b^{-1}(x,v)) = (x,v) \tag{5}$$

And therefore $h_b(y_b^{-1}(x,v)) = v$. Therefore,

$$y_{ab}.h_b(y_b^{-1}(x,v)) = h_a(y_b^{-1}(x,v)).. \tag{6}$$

In other words, $y_{ab}(x).h_b(x_x) = h_a(x_x)$. We know also that $y_{ab} \in GL(k,R)$ and therefore E_x has a unique vector space structure.

2.5. Corollary

Let $\{e_1, \mathbf{K}, e_k\}$ be the standard bases for the vector space R^k, then $\{y_a^{-1}(x,e_1), \mathbf{L}, y_a^{-1}(x,e_k)\}$ and $\{y_b^{-1}(x,e_1), \mathbf{L}, y_b^{-1}(x,e_k)\}$ are two ordered bases for E_x . Let

$$x_x = \sum_{i=1}^{k} y_i y_a^{-1}(x,e_i) = \sum_{i=1}^{k} z_i y_b^{-1}(x,e_i) \tag{7}$$

that $y_i, z_i \in R$ then

$$\begin{bmatrix} y_1 \\ \mathrm{M} \\ y_k \end{bmatrix} = y_{ab}(x). \begin{bmatrix} z_1 \\ \mathrm{M} \\ z_k \end{bmatrix} \tag{8}$$

2.6. Theorem

Any T-bundle admits a vector bundle structure but the inverse is not correct.

Proof: The first part of theorem is trivial, for this it is sufficient to notice that any T-atlas is a vector bundle atlas. Therefore, if A is a maximal T-atlas on E then there is a maximal vector bundle atlas A' such that $A \subset A'$.

The following example proves the second part of theorem. Let $E = J_0^2(R,R)$ be the vector bundle of jet's of second order from R to R at 0, that $p: E \to R$, $p(j_0^2 f) = f(0)$ is the natural projection. Consider the following two compatible charts for R :

$$\begin{aligned} u_1 : U_1 := R \longrightarrow R, \qquad & t\ \mathbf{a}\ t(0.1) \\ u_2 : U_2 := R \longrightarrow (0, +\infty), \qquad & t\ \mathbf{a}\ e^{2t} \end{aligned} \tag{9}$$

Then, $A = \{u_1, u_2\}$ is an atlas for standard structure of R . Consider the two mappings

$$\begin{aligned} y_1 &: p^{-1}(U_1) \longrightarrow U_1 \times R^2 \\ j_0^2(x + yt + zt^2) &\ \mathbf{a}\ (x; y, z) \end{aligned} \tag{10}$$

and

$$\begin{aligned} y_2 &: p^{-1}(U_2) \longrightarrow U_2 \times R^2 \\ j_0^2(x + yt + zt^2) &\ \mathbf{a}\ (x; 2e^{2x}y, e^{2x}y + 2e^{2x}z) \end{aligned} \tag{11}$$

It is clear that $\{U_1, y_1\}$ and $\{U_2, y_2\}$ are two vector bundle charts such that $y_2 \mathbf{o} y_1^{-1}(x; y, z) = (x; y_{21}(x)(y,z))$, where

$$\begin{bmatrix} y \\ z \end{bmatrix} \mathbf{a} \begin{bmatrix} 2e^{2x} & 0 \\ e^{2x} & 2e^{2x} \end{bmatrix} . \begin{bmatrix} y \\ z \end{bmatrix} \tag{12}$$

But $J_{21}(x) = [2e^{2x}]$. Therefore, E is a vector bundle but is not a T-bundle.

2.7. Definition

Let (E, p, M) and (F, q, N) be two T-bundles. By a T-bundle homomorphism we mean a pair $(j, \underline{j})$ of a smooth maps $(j: E \to F, \underline{j}: M \to N)$ such that $j: E \to F$ is fiberwise linear and the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "P"'] & F \arrow[d, "q"] \\ M \arrow[r, "\underline{j}"] & N \end{tikzcd}$$

Therefore, for any $x \in M$ the map $j_x : E_x \to F_{\underline{j}(x)}$ is linear. In this case we say that j covers $\underline{j}$. If j is invertible then we say j is an isomorophism.T-bundles together with their homomorphisms form a category TVB .

3. The Lie Algebra of Smooth Sections of a T-bundle

3.1. Definition

Let x be a point in M . We define

$$\bigoplus_{i=1}^{b} T_x^i M := T_x^1 M \oplus \ldots \oplus T_x^b M \tag{13}$$

where for any $i = 1 \ldots b, T_b^i M$ is the tangent space at x , for distinction, the index i is attributed. Also we define

$$\bigoplus_{i=1}^{b} T^i M := \bigcup_{x \in M} \bigoplus_{i=1}^{b} T_x^i M \tag{14}$$

as a manifold is diffeomorphic to Whitney-sum of b copies of TM (see [1]). In other words,

$$\begin{aligned} \bigoplus_{i=1}^{b} T^i M &= \{(v_1, \ldots, v_b) \mid \\ v_i &\in TM, \forall i, j : p(v_i) = p(v_j)\} \end{aligned} \tag{15}$$

The projection is defined naturally.

3.2.Theorem

$(\oplus_{i=1}^{b} T^i M, p, M)$ is a T-bundle.

Proof: Let (U, u) be a chart on M , it is sufficient to define the map $y : p^{-1}(U) \to U \times R^k$ by

$$\begin{aligned} &\left(\sum_{i=1}^{n} a_i^1 \frac{\partial}{\partial u^i}(x), \ldots, \sum_{i=1}^{n} a_i^b \frac{\partial}{\partial u^i}(x) \right) \mapsto \\ &(x; a_1^1, \ldots, a_n^b) \end{aligned} \tag{16}$$

3.3. Definition

Let (E, p, M) be a T-bundle with T-atlas $(U_a, y_a)_{a \in I}$. Assume x be in M and (U_a, u_a) be a chart such that $x \in U_a$, and (U_a, y_a) be the T-chart corresponding to it. In this case we define the mapping $j_x : E_x \to \oplus_{i=1}^{b} T_x^i M$ by

$$\begin{aligned} j_x(x_x) &:= \sum_{i=1}^{n} y_i \left(\frac{\partial_1}{\partial u_a^i} \right)_x \oplus \ldots \\ &\oplus \sum_{i=1}^{n} y_{mn+i} \left(\frac{\partial_{m+1}}{\partial u_a^i} \right)_x \oplus \ldots \\ &\ldots \oplus \sum_{i=1}^{n} y_{(b-1)n+i} \left(\frac{\partial_b}{\partial u_a^i} \right)_x \end{aligned} \tag{17}$$

where $x_x \in E_x$ and $x_x = \sum_{i=1}^{k} y_i y_a^{-1}(x, e_i)$, that we assume $\{e_1, \ldots, e_k\}$ is the standard bases for R^k , also for any $f \in C^\infty(M, R)$ we define

$$\left(\frac{\partial_j}{\partial u_a^i} \right)_x f := \left(\frac{\partial f}{\partial u_a^i} \right)_x . \tag{18}$$

3.4. Lemma

$j_x(x_x)$ is well define.

Proof: It is sufficient to show the definition is independent of charts. Let $x \in U_{ab}$ and

$$x_x = \sum_{i=1}^{k} y_i y_a^{-1}(x, e_i) = \sum_{i=1}^{k} z_i y_b^{-1}(x, e_i) \tag{19}$$

Therefore, we have

$$\begin{aligned} j_x(x_x) &= j_x \left(\sum_{i=1}^{k} y_i y_a^{-1}(x, e_i) \right) \\ &= \sum_{i=1}^{n} y_i \left(\frac{\partial_1}{\partial u_a^i} \right)_x \oplus \ldots \oplus \sum_{i=1}^{n} y_{mn+i} \left(\frac{\partial_{m+1}}{\partial u_a^i} \right)_x \oplus \ldots \\ &\ldots \oplus \sum_{i=1}^{n} y_{(b-1)n+i} \left(\frac{\partial_b}{\partial u_a^i} \right)_x \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} z_j \left(\frac{\partial u_a^i}{\partial u_b^j} \right)(x) \right) \left(\frac{\partial_1}{\partial u_a^i} \right)_x \oplus \ldots \\ &\ldots \oplus \sum_{i=1}^{n} \left(\sum_{j=1}^{n} z_{mn+j} \left(\frac{\partial u_a^i}{\partial u_b^j} \right)(x) \right) \left(\frac{\partial_{m+1}}{\partial u_a^i} \right)_x \oplus \ldots \\ &\ldots \oplus \sum_{i=1}^{n} \left(\sum_{j=1}^{n} z_{(b-1)n+j} \left(\frac{\partial u_a^i}{\partial u_b^j} \right)(x) \right) \left(\frac{\partial_b}{\partial u_a^i} \right)_x \end{aligned} \tag{20}$$

By using (3.6), we have

$$= \sum_{i=1}^{n} z_i \left(\frac{\partial_1}{\partial u_b^i}\right)_x \oplus \mathbf{L} \oplus \sum_{i=1}^{n} z_{mn+i} \left(\frac{\partial_{m+1}}{\partial u_b^i}\right)_x \oplus \mathbf{L} \oplus \sum_{i=1}^{n} z_{(b-1)n+i} \left(\frac{\partial_b}{\partial u_b^i}\right)_x = j_x(x_x) \qquad (21)$$

Therefore, j_x is well define.

3.5. Theorem

Let (E, p, M) be a T-bundle such that for any $x \in M$, $\dim(p^{-1}(x)) = b$ then the mapping $j : E \to \oplus_{i=1}^{b} T^i M$ with $j(x_x) := j_x(x_x)$ is an isomorphism of T-bundles.

Proof: Let (U, u) be a chart on M and (U, y) be the correspondence T-chart on E. It is clear that j is invertible and by using the proof of theorem (3.2) the following diagram is commutative

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "j"] \arrow[d, "y"'] & p^{-1}(U) \arrow[d, "y_T"] \\ U \times R^k \arrow[r, "\Phi"] & U \times R^k \end{tikzcd}$$

Where $j(x, a) = (x, a)$ that is isomorphism for any x.

3.6. Corollary

Any two T-bundles on the same base such that their fibers have the same dimension are isomorphic.

3.7. Definition

Suppose (E, p, M) be a T-bundle. By a section of the T-bundle E we mean an smooth map $x : M \to E$ such that $p \circ x = Id_M$.

We denote the set of all sections of a T-bundle (E, p, M) by $C^\infty(E)$.

3.8. Definition

Let (E, p, M) be a T-bundle; $x, h \in C^\infty(E)$ and $f \in C^\infty(M, R)$ then we define:

$$(x + h)_x := x_x + h_x, \quad (fx)_x := f(x)x_x \qquad (22)$$

3.9. Definition

Let x and h be in $C^\infty(E)$ and (U_a, u_a) be a chart on M, in this case we can write in local coordinates

$$x \,|U_a = \sum_{i=1}^{k} f_i y_a^{-1}(.,e_i) \quad \text{and}$$

$$h \,|U_a = \sum_{i=1}^{k} g_i y_a^{-1}(.,e_i) \qquad (23)$$

where $f_i, g_i \in C^\infty(U_a, R)$. Then we define

$$x.h := j^{-1}(j(x).j(h)) \qquad (24)$$

and

$$j(x).j(h)\,|U_a := \sum_{i=1}^{n}\sum_{j=1}^{n} f_j \frac{\partial_1 g_i}{\partial u_a^j}\frac{\partial_1}{\partial u_a^i} \oplus \mathbf{L}$$

$$\mathbf{L} \oplus \sum_{i=1}^{n}\sum_{j=1}^{n} f_{mn+j} \frac{\partial_{m+1} g_{mn+i}}{\partial u_a^j}\frac{\partial_{m+1}}{\partial u_a^i} \oplus \mathbf{L} \qquad (25)$$

$$\mathbf{L} \oplus \sum_{i=1}^{n}\sum_{j=1}^{n} f_{(b-1)n+j} \frac{\partial_b g_{(b-1)n+i}}{\partial u_a^j}\frac{\partial}{\partial u_a^i}$$

Therefore, we have

$$x.h = \sum_{p=0}^{b-1}\sum_{i=1}^{n}\sum_{j=1}^{n} f_{pn+j} \frac{\partial g_{pn+i}}{\partial u_a^i} y_a^{-1}(.,e_{pn+i}). \qquad (26)$$

3.10. Definition

Let x be a section of T-bundle (E, p, M), suppose that $x \in M$ and (U_a, u_a) is a chart of M such that $x \in U_a$. Also we assume that

$$f = (f_1, \mathbf{K}, f_b) \in C^\infty(U_a, R^b) \qquad (27)$$

where $f_i \in C^\infty(U_a, R)$, in this case we define $x : C^\infty(U_a, R^b) \to C^\infty(U_a, R^b)$ by

$$x(f)(x) = (j(x)(f_1, \mathbf{L}, f_b))(x)$$

$$:= \left(\sum_{i=1}^{n} g_i(x)\frac{\partial f_1}{\partial u_a^i}(x), \mathbf{L}, \sum_{i=1}^{n} g_{n(b-1)+i}(x)\frac{\partial f_b}{\partial u_a^i}(x)\right) \qquad (28)$$

Where $x\,|_{U_a} = \sum_{i=1}^{k} g_i y_a^{-1}(.,e_i)$.

Now we can define the Lie bracket of to sections of a T-bundle.

3.11. Definition

Let $x, h \in C^\infty(E)$, then we define the Lie bracket of x and h by $[x, h] = xh - hx$, if we assume that $x\,|U_a = \sum_{i=1}^{k} f_i y_a^{-1}(.,e_i)$ and $h\,|U_a = \sum_{i=1}^{k} g_i y_a^{-1}(.,e_i)$ then we have

$$[x,h]|U_a = \sum_{p=0}^{b-1}\sum_{i=1}^{n}\sum_{j=1}^{n}(f_{pn+j}\frac{\partial g_{pn+i}}{\partial u_a^j} \qquad (29)$$
$$-g_{pn+j}\frac{\partial f_{pn+i}}{\partial u_a^j}).y_a^{-1}(.,e_{pn+i})$$

3.12. Theorem

Suppose that (E,p,M) is a T-bundle then $C^\infty(E)$ with Lie bracket and the addition of functions and scaler product is a Lie algebra. *Proof:* The product that we define in definition (3.9) is associative (It is clear from the definition)so $C^\infty(E)$ is an associative algebra and also we define the Lie bracket of two sections x and h by $xh-hx$, therefore the $C^\infty(E)$ is a Lie algebra.

3.13. Corollary

If (E,p,M) be a T-bundle with standard fiber R^k such that $\dim(M)=n$ then the Lie algebra $C^\infty(E)$ is isomorphic to multiplication of $b=k/n$ copies of Lie algebra $c(M)$ (the Lie algebra of all vector fields on M).

4. Conclusion

T-bundle which defined in this article is a vector bundle such that it is a natural generalization of tangent bundle. Also we defined a Lie algebra structure on the set of smooth sections of a T-bundle such that in particular case of tangent bundle it coincides with the Lie algebra of vector fields.

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