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RESEARCH PAPER

Kumaraswamy-G Generalized Gompertz Distribution with Application to Lifetime Data

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ABSTRACT
Recently, generalized distributions have received much attention due to their high applicability and flexibility. This paper introduces a new five-parameter distribution called Kumaraswamy-G generalized Gompertz distribution, which is widely used in the field of survival and lifetime data. In introducing a new distribution, it is important to study the statistical properties and the estimation of its parameters. Therefore, this paper studies the statistical properties of this new distribution. In addition, the parameters of this distribution are estimated by three methods and the estimation methods are compared using simulation. Finally, using a real dataset, the performance of the introduced distribution is investigated.

KEYWORDS: *Kumaraswamy-G distribution; Generalized gompertz distribution; Inequality indices; Reliability; Monte-carlo simulation.*

1. Introduction

Although classical distributions have simpler, more comprehensible form and fewer parameters, they are not suitable for fitting the skewed data. To overcome this shortage of classical distributions, generalized distributions that are more flexible and more usable are introduced. Some of these important generators are: Marshall Olkin [1], generalized-exponential (GE) [2], beta-generated distributions [3, 4], Kumaraswamy-G distribution [5], McDonald generalized (Mc-G) distribution [6] gamma- generated type-I distribution [7,8], gamma- generated type-II distribution [9], exponentiated generalized (exp-G) distribution [10], and odd Weibull-generated distribution [11].

Gompertz distribution which was first proposed by Gompertz [12], is one of the most widely used distributions in the fields of survival, lifetime data, mortality, life tables, computer [13], biology [14], sociology [15], and marketing [16]. For more details, see references [17-23]. The cumulative distribution function (cdf) and probability density function (pdf) of the Gompertz distribution respectively are

$$F_G(x) = 1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)} \quad , \ \beta > 0\ , \gamma \geq 0\ , x \geq 0,$$

$$f_G(x) = \beta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)} \quad , \ \beta > 0\ , \gamma \geq 0\ , x \geq 0.$$

Based on the increasing or constant hazard function of this distribution, the Gompertz distribution generalizations are used to model the bathtub-shaped hazard function, some of which are: generalized Gompertz [24], beta Gompertz [25], Kumaraswamy Gompertz [26], McDonald Gompertz [27], beta generalized Gompertz [28], Marshall Olkin extended generalized Gompertz [29], odd log-logistic generalized Gompertz [30], and Marshall-Olkin Gompertz Makeham [31]. This article deals with the generalized Gompertz (GG) distribution and introduces a new generalization of the GG distribution, called Kumaraswamy-G generalized Gompertz (KG-GG) distribution.

The rest of this article is organized as follows: In Section 2, the KG-GG distribution is introduced. In Section 3, some statistical properties of this new model are studied. In Section 4, the parameters of this model are estimated in three methods: maximum likelihood estimation (MLE),

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leat square (LS) and Bayes. In Section 5, simulation studies are performed to compare these estimators. Finally, in Section 6, an application of this model is presented.

2. New Model

Before introducing the KG-GG distribution, two required definitions are given.

Definition 1. The non-negative random variable X has a GG distribution, if its cdf and pdf are as follows

$$F_{GG}(x)=\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{\theta} \quad , \ \beta,\theta>0\,,\gamma\geq 0\ \,,x\geq 0, \tag{1}$$

$$f_{GG}(x)=\theta\beta e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{\theta-1} \quad , \ \beta,\theta>0\,,\gamma\geq 0\ \,,x\geq 0, \tag{2}$$

where the parameter θ is a shape parameter.

Definition 2. Let X be a random variable with the baseline cdf F and the baseline pdf f, then the cdf and the pdf of the Kumaraswamy-G distribution (KG), that was first proposed by Corderio and de Ccastro [5] based on Kumaraswamy [32], are respectively

$$F_{KG}(x)=1-[1-(F(x))^{a}]^{b} \quad , \ a,b>0, \tag{3}$$

$$f_{KG}(x)=abf(x)(F(x))^{a-1}[1-(F(x))^{a}]^{b-1} \quad , \ a,b>0, \tag{4}$$

where a and b are the two additional shape parameters of F distribution.

By substituting (1) and (2) into (3) and (4), the cdf and the pdf of the KG-GG distribution are obtained

$$F_{KG-GG}(x)=1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b}, \tag{5}$$

$$f_{KG-GG}(x)=ab\theta\beta e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta-1}\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b-1}. \tag{6}$$

Figure 1 shows the pdf of the KG-GG distribution for some parameter values. It can be observed from the Figure 1 that increasing the value of the shape parameters the peakedness of the density function tends to increase. Similarly, the increase in the value of the scale parameter shifts the density function away from the origin.

2.1. Sub-models

The KG-GG distribution includes some distributions as submodels.

- If $a=b=1$, the KG-GG distribution reduces to the generalized Gompertz distribution $GG(\beta,\gamma,\theta)$.
- If $b=1$, the KG-GG distribution reduces to the generalized Gompertz distribution $GG(\beta,\gamma,a\theta)$.
- If $\theta=b=a=1$, the KG-GG distribution reduces to the Gompertz distribution $G(\beta,\gamma)$.
- If $\theta=1$, the KG-GG distribution reduces to the Kumaraswamy-G Gompertz distribution $KG-G(\beta,\gamma,a,b)$.
- If γ tends to zero, the KG-GG distribution reduces to the generalized exponential distribution $GE(\beta,\theta)$.
- If $\theta=1$ and γ tends to zero, the KG-GG distribution reduces to the exponential distribution $E(\beta)$.

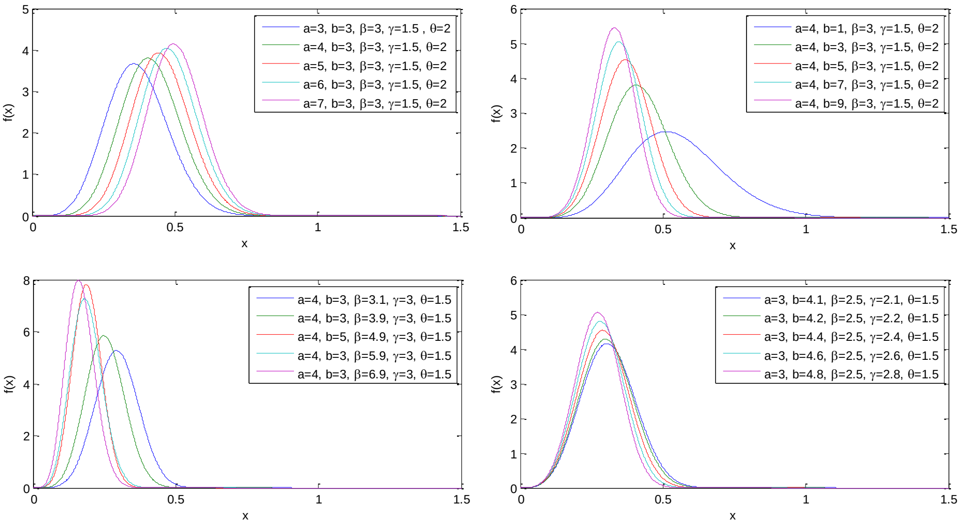

Fig. 1. The pdf of the KG-GG distribution for some parameter values.

2.2. A representation of the KG-GG pdf

Theorem 1. The pdf of the KG-GG distribution can be written as the Gompertz pdf multiplied by an infinite power series of W_{ij}.

$$f_{KG-GG}(x) = ab\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W_{ij} \times f_G(x;\beta(j+1),\gamma), \tag{7}$$

where

$$W_{ij} = \binom{b-1}{i}\binom{a\theta(i+1)-1}{j}\frac{(-1)^{i+j}}{j+1} \tag{8}$$

Proof: For a real non-integer $\alpha > 0$ and $|u| < 1$

$$(1-u)^{\alpha-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}u^i. \tag{9}$$

Thus

$$\begin{aligned}
f_{KG-GG}(x) &= ab\theta\beta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta-1}\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b-1} \\
&= ab\theta\beta e^{\gamma x} e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\sum_{i=0}^{\infty}\binom{b-1}{i}(-1)^i\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta(i+1)-1} \\
&= ab\theta\beta e^{\gamma x}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{b-1}{i}\binom{a\theta(i+1)-1}{i}(-1)^{i+j}e^{-\frac{\beta(j+1)}{\gamma}(e^{\gamma x}-1)} \\
&= ab\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{b-1}{i}\binom{a\theta(i+1)-1}{i}\frac{(-1)^{i+j}}{j+1}\beta(j+1)e^{\gamma x}e^{-\frac{\beta(j+1)}{\gamma}(e^{\gamma x}-1)} \\
&= ab\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W_{ij}\times f_G(x;\beta(j+1),\gamma).
\end{aligned}$$

□

3. Statistical Properties

Some important statistical properties of this new distribution are studied in this section.

3.1. Quantile function

By solving $F(x_p) = p$, the p^{th} quantile of the KG-GG distribution are as follows

$$F(x_p) = p \iff 1 - \left[1 - \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_p}-1)}\right)^{a\theta}\right]^b = p \iff 1 - (1-p)^{\frac{1}{b}} = \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_p}-1)}\right)^{a\theta}$$
$$\iff 1 - \left[1 - (1-p)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}} = e^{-\frac{\beta}{\gamma}(e^{\gamma x_p}-1)} \iff \log\left\{1 - \left[1 - (1-p)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}$$
$$= -\frac{\beta}{\gamma}(e^{\gamma x_p} - 1) \iff 1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-p)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\} = e^{\gamma x_p}$$
$$\iff x_p = \frac{1}{\gamma}\log\left\{1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-p)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}\right\}. \quad (10)$$

By replacing $p = 0.25, 0.5, 0.75$, the first quartile (Q_1), median (Q_2), and third quartile (Q_3) are obtained, respectively

$$Q_1 = \frac{1}{\gamma}\log\left\{1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-0.25)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}\right\}, \quad (11)$$
$$Q_2 = \frac{1}{\gamma}\log\left\{1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-0.5)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}\right\}, \quad (12)$$
$$Q_3 = \frac{1}{\gamma}\log\left\{1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-0.75)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}\right\}. \quad (13)$$

3.2. Sample generating

Using the inversion method, a sample of this distribution can be generated as follows

1. $U \sim U(0,1)$.
2. $1 - \left[1 - \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^b = U \Rightarrow X$
$$= \frac{1}{\gamma}\log\left\{1 - \frac{\gamma}{\beta}\log\left\{1 - \left[1 - (1-p)^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}\right\}\right\} \sim KG - GG(\beta, \gamma, \theta, a, b).$$

3.3. Moments

Theorem 2. Let $X \sim KG - GG(\beta, \gamma, \theta, a, b)$be a random variable, then the r^{th} moment of X is given by

$$\mu^r = ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^{r+1}\Gamma(r+1), \quad (14)$$

where $\Gamma(a) = \int_0^\infty y^{a-1}e^{-y}\,dy$is gamma function and

$$W_{ijk} = \binom{b-1}{i}\binom{a\theta(i+1)-1}{j}\frac{(-1)^{i+j+k}\beta^k}{k!\,\gamma^k}e^{\frac{\beta(j+1)}{\gamma}}. \quad (15)$$

Proof. By expansion (9) and the following expansion

$$e^t = \sum_{i=0}^{\infty}\frac{t^i}{i!}, \quad (16)$$

we can write

$$\mu^r = E(X^r) = \int_0^\infty x^r f_{KG-GG}(x;\beta,\gamma,\theta,a,b)dx$$
$$= ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{b-1}{i}\binom{a\theta(i+1)-1}{j}(-1)^{i+j} e^{\frac{\beta(j+1)}{\gamma}} \int_0^\infty x^r e^{\gamma x} e^{-\frac{\beta}{\gamma}e^{\gamma x}} dx$$
$$= ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{b-1}{i}\binom{a\theta(i+1)-1}{j}(-1)^{i+j} e^{\frac{\beta(j+1)}{\gamma}} \int_0^\infty x^r e^{\gamma x} \sum_{k=0}^{\infty}\frac{\left(-\frac{\beta}{\gamma}e^{\gamma x}\right)^k}{k!} dx$$
$$= ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{b-1}{i}\binom{a\theta(i+1)-1}{j}\frac{(-1)^{i+j+k}\beta^k}{k!\,\gamma^k} e^{\frac{\beta(j+1)}{\gamma}} \int_0^\infty x^r e^{\gamma(k+1)x} dx$$
$$= ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^{r+1} \int_0^\infty z^r e^{-z} dz$$
$$= ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^{r+1} \Gamma(r+1),$$

where transform $x = -\frac{-z}{\gamma(k+1)}$ is used to solve integral.

□

Based on Theorem 2, we can calculate the mean and the variance of the KG-GG distribution

$$Mean(X) = ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{W_{ijk}}{[\gamma(k+1)]^2}, \quad (17)$$

$$Var(X) = 2ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^3 - \left(ab\theta\beta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{W_{ijk}}{[\gamma(k+1)]^2}\right)^2. \quad (18)$$

3.4. Skewness and kurtosis

Skewness and kurtosis can be calculated by $S = \frac{\mu^3}{\sigma^3}$ and $S = \frac{\mu^4}{\sigma^4}$. When the third and fourth moment does not exist, by approximating μ^3 and μ^4, the skewness and kurtosis are approximated. [33] and [34] defined the skewness and kurtosis as follows

$$Skewness = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

$$Kurtosis = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) + Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

Since the third and fourth moment do not exist for the KG-GG distribution, this approximate method can be used to calculate the skewness and kurtosis of this distribution. Table 1 shows the quartiles, kurtosis and skewness for different values of parameters and x.

Tab. 1. The quartiles, kurtosis and skewness values for $\beta = 2, \gamma = 1.5, \theta = 2$ and different values of $a, b,$ and x.

Statistics	$a = 1.5$ $b = 1.1$ $x = 1$	$a = 2.5$ $b = 1.2$ $x = 1.2$	$a = 3$ $b = 1.4$ $x = 2.1$	$a = 3.5$ $b = 1.6$ $x = 3$	$a = 4$ $b = 1.8$ $x = 3.2$
Median	0.5017	0.5874	0.5968	0.6059	0.6145
Q_1	0.3583	0.4599	0.4805	0.4982	0.5138
Q_3	0.6594	0.7244	0.7207	0.7196	0.7201
$Kurtosis$	2.6558	2.2077	2.5429	2.8890	3.2494
$Skewness$	2.9196	2.7314	3.2236	3.7394	4.2825

3.5. Mode

The KG-GG distribution mode can be calculated as follows

$$\frac{\partial f(x)}{\partial x}=e^{2\gamma x}y(1-y)^{a\theta-1}\left[1-(1-y)^{a\theta}\right]^{b-1}\left\{\gamma e^{-\gamma x}-\beta+\beta(a\theta-1)\frac{y}{1-y}-\beta a\theta(b-1)e^{\gamma x}\frac{y(1-y)^{a\theta-1}}{1-(1-y)^{a\theta}}\right\}=0,$$

where $y=e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}$.

3.6. Moment generating function and characteristic function

The moment generating function (MGF) and characteristic function (CF) of the KG-GG distribution are

$$MGF=E(e^{tx})=\sum_{m=0}^{\infty}\frac{t^m}{m!}E(X^m)=ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m}{m!}W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^{m+1}\Gamma(m+1),$$

$$CF=E(e^{Itx})=\sum_{m=0}^{\infty}\frac{(It)^m}{m!}E(X^m)=ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(It)^m}{m!}W_{ijk}\left[\frac{-1}{\gamma(k+1)}\right]^{m+1}\Gamma(m+1),$$

where W_{ijk} given in (15).

3.7. Reliability properties

The reliability properties play an important role in introducing a distribution. Some of these important features are given in this section.

3.7.1. Survival function

$$S(t)=1-F(t)=\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right]^{b} \tag{19}$$

3.7.2. Hazard rate function

$$h(t)=\frac{f(t)}{S(t)}=\frac{ab\theta\beta e^{\gamma t}e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta-1}}{1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}}.$$

3.7.3. Cumulative hazard rate function

$$H(t)=\int_0^t h(t)dt=-ln\,S(t)=-b\,ln\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right].$$

3.7.4. Reversed hazard rate function

$$r(t)=\frac{f(t)}{F(t)}=\frac{ab\theta\beta e^{\gamma t}e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta-1}\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right]^{b-1}}{1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right]^{b}}.$$

3.7.5. Mean residual life

The mean residual life (MRL) at a given time t is defined as

$$m(t)=\frac{1}{S(t)}\int_t^{\infty}tf(t)dt=\frac{1}{S(t)}\left\{E(t)-\int_0^t tf(t)dt\right\}-t.$$

Consider

$$\begin{aligned}\int_0^t xf(x)dx &= ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W_{ijk}\int_0^t xe^{\gamma(k+1)x}dx\\ &= ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\int_0^{\frac{t}{\gamma(k+1)}} ze^{-z}dz \qquad (20)\\ &= ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\Gamma_{\frac{t}{\gamma(k+1)}}(2),\end{aligned}$$

where $\Gamma_t(a)=\int_0^t y^{a-1}e^{-y}\,dy$ is incomplete gamma, W_{ijk} is given in (15) and transform $x=-\frac{-z}{\gamma(k+1)}$ is used to solve integral. Hence, the MRL for KG-GG distribution is given as

$$m(t)=\frac{ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\left[1-\Gamma_{\frac{t}{\gamma(k+1)}}(2)\right]}{\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right]^b}-t.$$

3.7.6. Mean waiting time

Using the result of (20), we get

$$\overline{\mu}(t)=t-\left\{\frac{1}{F(t)}\int_0^t tf(t)dt\right\}=t-\frac{ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\Gamma_{\frac{t}{\gamma(k+1)}}(2)}{1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma t}-1)}\right)^{a\theta}\right]^b}.$$

3.7.7. Mean deviations

Let X be a random variable with pdf $f(x)$ and cdf $F(x)$, the mean deviation about the mean and about the median are given by

$$\delta_1(x)=\int_0^{\infty}|X-\mu|\,f_{KG-GG}(x)dx=2\mu F_{KG-GG}(\mu)-2\mu+2T(\mu), \qquad (21)$$

$$\delta_2(x)=\int_0^{\infty}|X-M|\,f_{KG-GG}(x)dx=-\mu+2T(\mu), \qquad (22)$$

where $\mu=E(X), M=Median(X)$ and

$$\begin{aligned}T(\mu)&=\int_{\mu}^{\infty}x\,f_{KG-GG}(x)dx=E(\mu)-\int_0^{\mu}x\,f_{KG-GG}(x)dx \qquad (23)\\ &= ab\theta\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\left[1-\Gamma_{\frac{\mu}{\gamma(k+1)}}(2)\right].\end{aligned}$$

Inserting the results of (5), (12), (17), and (23) in (21) and (22), the mean deviations for KG-GG distribution is obtained.

3.8. Inequality measures

In this section, some inequality measures are examined for KG-GG distribution.

3.8.1. Gini index

[35] proposed the Gini index as follows

$$G=\frac{1}{E(X)}\int_0^\infty F(x)[1-F(x)]\,dx. \tag{24}$$

Theorem 3. The Gini index for KG-GG distribution is

$$G=\frac{1}{E(X)}\left\{\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\frac{W_{hlq}}{\gamma q}-\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}\frac{W_{mup}}{\gamma p}\right\}, \tag{25}$$

where $E(X)$ given in (17) and

$$W_{mnp}=\binom{b}{m}\binom{a\theta m}{u}\frac{(-1)^{m+u+p}}{p!}\left(\frac{\beta u}{\gamma}\right)^p e^{\frac{\beta u}{\gamma}},$$
$$W_{hlq}=\binom{2b}{h}\binom{a\theta h}{l}\frac{(-1)^{h+l+q}}{q!}\left(\frac{\beta l}{\gamma}\right)^q e^{\frac{\beta l}{\gamma}}.$$

Proof. To calculate the Gini index, consider

$$F(x)[1-F(x)]=\left(1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^b\right)\left(\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^b\right)$$
$$=\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^b-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{2b}=B-C$$

Using (9) and (16), B and C can be written as follows

$$\begin{aligned}B&=\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\binom{b}{m}\binom{a\theta m}{u}(-1)^{m+u}e^{\frac{\beta u}{\gamma}}\times e^{-\frac{\beta u}{\gamma}e^{\gamma x}}\\&=\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\binom{b}{m}\binom{a\theta m}{u}(-1)^{m+u}e^{\frac{\beta u}{\gamma}}\times\sum_{p=0}^{\infty}\frac{\left(-\frac{\beta u}{\gamma}e^{\gamma x}\right)^p}{p!}\\&=\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}\binom{b}{m}\binom{a\theta m}{u}\frac{(-1)^{m+u+p}e^{\frac{\beta u}{\gamma}}(\beta u)^p}{\gamma^p p!}e^{\gamma p x}=\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}W_{mup}e^{\gamma p x},\end{aligned} \tag{26}$$

$$\begin{aligned}C&=\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\binom{2b}{h}\binom{a\theta h}{l}(-1)^{h+l}e^{\frac{\beta l}{\gamma}}\times e^{-\frac{\beta l}{\gamma}e^{\gamma x}}\\&=\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\binom{2b}{h}\binom{a\theta h}{l}(-1)^{h+l}e^{\frac{\beta l}{\gamma}}\times\sum_{q=0}^{\infty}\frac{\left(-\frac{\beta l}{\gamma}e^{\gamma x}\right)^q}{q!}\\&=\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\binom{2b}{h}\binom{a\theta h}{l}\frac{(-1)^{h+l+q}e^{\frac{\beta l}{\gamma}}(\beta l)^q}{\gamma^q q!}e^{\gamma q x}=\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}W_{hlq}e^{\gamma q x}.\end{aligned} \tag{27}$$

Now substituting (26) and (27) in (24), we get

$$G=\frac{1}{E(X)}\left\{\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}W_{mup}\int_0^{\infty}e^{\gamma px}\,dx-\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}W_{hlq}\int_0^{\infty}e^{\gamma qx}\,dx\right\}$$
$$=\frac{1}{E(X)}\left\{\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}\frac{-W_{mup}}{\gamma p}\int_0^{\infty}e^{-z}\,dz-\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\frac{-W_{hlq}}{\gamma q}\int_0^{\infty}e^{-\omega}\,d\omega\right\}$$
$$=\frac{1}{E(X)}\left\{\sum_{h=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\frac{W_{hlq}}{\gamma q}-\sum_{m=0}^{\infty}\sum_{u=0}^{\infty}\sum_{p=0}^{\infty}\frac{W_{mup}}{\gamma p}\right\},$$

where transforms $x=-\frac{-z}{\gamma p}$ and $x=-\frac{-\omega}{\gamma q}$ are used to solve integrals.

□

3.8.2. Lorenz curve

Using (17) and (20), we have

$$L(p)=\frac{1}{\mu}\int_0^x xf(x)\,dx=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\Gamma_{\frac{x}{\gamma(k+1)}}(2)}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}}.$$

3.8.3. Bonferroni curve

$$BC(p)=\frac{L(p)}{F(x)}=\frac{L(p)}{1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b}}.$$

3.8.4. Zenga index

The Zenga index defined by Zanga [36] as follows

$$Z=1-\frac{\mu^-(x)}{\mu^+(x)},$$

where

$$\mu^-(x)=\frac{1}{F(x)}\int_0^x xf(x)\,dx,$$
$$\mu^+(x)=\frac{1}{1-F(x)}\int_x^{\infty}xf(x)\,dx=\frac{1}{1-F(x)}\left\{\mu-\int_0^x xf(x)\,dx\right\}.$$

Using the results of (5), (17), and (20), the Zenga index for KG-GG distribution is

$$Z=1-\frac{1-F(x)}{F(x)}\times\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\Gamma_{\frac{x}{\gamma(k+1)}}(2)}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{W_{ijk}}{[\gamma(k+1)]^2}\left[1-\Gamma_{\frac{x}{\gamma(k+1)}}(2)\right]}.$$

3.9. Renyi entropy

The formula for this entropy was defined by Renyi [37] is given by

$$I_R(\lambda)=\frac{1}{1-\lambda}\log\left\{\int f^{\lambda}(x)\,dx\right\},\ \ \lambda>0,\ \ \lambda\neq 1. \tag{28}$$

Theorem 4. For the KG-GG distribution, the Renyi entropy is

$$I_R(\lambda)=\frac{1}{\lambda-1}\log\left\{(ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\frac{W_{cdg}}{\gamma(\lambda+g)}\right\}, \tag{29}$$

where

$$W_{cdg} = \binom{\lambda(b-1)}{c}\binom{a\theta(\lambda+c)-\lambda}{d}\frac{(-1)^{d+c+g}}{\gamma^g g!}[\beta(\lambda+d)]^g.$$

Proof. Using expansions (9) and (16), the λ^{th}-order pdf of the KG-GG distribution can be written as

$$\begin{aligned}
f^{\lambda}(x) &= (ab\theta\beta)^{\lambda} e^{\gamma\lambda x} e^{-\frac{\beta\lambda}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{\lambda(a\theta-1)}\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{\lambda(b-1)} \\
&= (ab\theta\beta)^{\lambda} e^{\gamma\lambda x} e^{-\frac{\beta\lambda}{\gamma}(e^{\gamma x}-1)}\sum_{c=0}^{\infty}\binom{\lambda(b-1)}{c}(-1)^c\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta(\lambda+c)-\lambda} \\
&= (ab\theta\beta)^{\lambda} e^{\gamma\lambda x}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{\lambda(b-1)}{c}\binom{a\theta(\lambda+c)-\lambda}{d}(-1)^{c+d} e^{\frac{\beta(\lambda+d)}{\gamma}} e^{-\frac{\beta(\lambda+d)}{\gamma}e^{\gamma x}} \\
&= (ab\theta\beta)^{\lambda} e^{\gamma\lambda x}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{\lambda(b-1)}{c}\binom{a\theta(\lambda+c)-\lambda}{d}(-1)^{c+d} e^{\frac{\beta(\lambda+d)}{\gamma}}\sum_{g=0}^{\infty}\frac{\left(-\frac{\beta(\lambda+d)}{\gamma}e^{\gamma x}\right)^g}{g!} \\
&= (ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\binom{\lambda(b-1)}{c}\binom{a\theta(\lambda+c)-\lambda}{d}\frac{(-1)^{c+d+g}}{\gamma^g g!} e^{\gamma(\lambda+g)x} \\
&= (ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty} W_{cdg}\, e^{\gamma(\lambda+g)x}.
\end{aligned}$$

Using transformation $x = -\frac{-z}{\gamma(\lambda+g)}$, we calculate

$$\begin{aligned}
I_R(\lambda) &= \frac{1}{1-\lambda}\log\left\{(ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty} W_{cdg}\int_0^{\infty} e^{\gamma(\lambda+g)x}dx\right\} \\
&= \frac{1}{1-\lambda}\log\left\{(ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty} W_{cdg}\left[\frac{-1}{\gamma(\lambda+g)}\right]\int_0^{\infty} e^{-z}dz\right\} \\
&= \frac{1}{\lambda-1}\log\left\{(ab\theta\beta)^{\lambda}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty}\frac{W_{cdg}}{\gamma(\lambda+g)}\right\}.
\end{aligned}$$

□

3.10. β'-Entropy

Havrda and Charvat [38] suggested β'-Entropy, which is defined as

$$I_{\beta'}(x) = \frac{1}{\beta'-1}\left\{1-\int f^{\beta'}(x)dx\right\},\ \ \beta'>0,\ \ \beta'\neq 1.$$

Similar to Theorem 4, this type of entropy for the KG-GG distribution is

$$I_{\beta'}(x) = \frac{1}{\beta'-1}\left\{1-(ab\theta\beta)^{\beta'}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{W_{rst}}{\gamma(\beta'+t)}\right\},$$

where

$$W_{rst=}\binom{\beta'(b-1)}{r}\binom{a\theta(\beta'+r)-\beta'}{s}\frac{(-1)^{r+s+t}}{\gamma^t t!}[\beta(\beta'+s)]^t. \tag{30}$$

3.11. Order statistics

The pdf of the p^{th} order statistic is

$$f_{p:n}(x)=\frac{1}{Beta(p,n-p+1)}[F(x)]^{p-1}[1-F(x)]^{n-p}f(x), \tag{31}$$

where $Beta(a_1,b_1)=\frac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)}$ is beta function and F and f are cdf and pdf, respectively.

Theorem 5. The pdf of the p^{th} order statistic for KG-GG distribution can be written as follows

$$f_{p:n}(x)=\frac{ab\theta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}W_{ijkl}\times\ f_G(x;\beta(l+1),\gamma),$$

where

$$W_{ijkl}=\binom{n-p}{i}\binom{p+i-1}{j}\binom{b(j+1)-1}{k}\binom{a\theta(k+1)-1}{l}\frac{(-1)^{i+j+k+l}}{l+1}. \tag{32}$$

Proof. For $0<F(x)<1, x>0$,

$$[1-F(x)]^{n-p}=\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i[F(x)]^i. \tag{33}$$

Substituting (33) into (31) yieldes

$$f_{p:n}(x)=\frac{1}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i\,[F(x)]^{p+i-1}f(x). \tag{34}$$

Thus,

$$f_{p:n}(x)=\frac{ab\theta\beta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i\,e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta-1}\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b-1}\left\{1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b}\right\}^{p+i-1}$$

$$=\frac{ab\theta\beta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i\,e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta-1}\sum_{j=0}^{\infty}\binom{p+i-1}{j}(-1)^j\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b(j+1)-1}$$

$$=\frac{ab\theta\beta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{n-p}{i}\binom{p+i-1}{j}(-1)^{i+j}\,e^{\gamma x}e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\sum_{k=0}^{\infty}\binom{b(j+1)-1}{k}(-1)^k\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta(k+1)-1}$$

$$=\frac{ab\theta\beta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-p}{i}\binom{p+i-1}{j}\binom{b(j+1)-1}{k}(-1)^{i+j+k}\sum_{l=0}^{\infty}\binom{a\theta(k+1)-1}{l}(-1)^l e^{\gamma x}e^{-\frac{\beta(l+1)}{\gamma}(e^{\gamma x}-1)}$$

$$=\frac{ab\theta}{Beta(p,n-p+1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{n-p}{i}\binom{p+i-1}{j}\binom{b(j+1)-1}{k}\binom{a\theta(k+1)-1}{l}\frac{(-1)^{i+j+k+l}}{l+1}\times\ \beta(l+1)e^{\gamma x}e^{-\frac{\beta(l+1)}{\gamma}(e^{\gamma x}-1)}$$

$$= \frac{ab\theta}{Beta(p, n-p+1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} W_{ijkl} \times f_G(x; \beta(l+1), \gamma).$$

4. Estimation of Parameters

In this section, the KG-GG distribution parameters are estimated using three methods, MLE, LS and Bayes.

4.1. MLE

Let $X_1, X_2, \ldots, X_n$ be a iid random sample form $KG - GG(\beta, \gamma, \theta, a, b)$ and $x_1, x_2, \ldots, x_n$ be observed values, then the likelihood function is given by

$$L(\boldsymbol{\Theta}|\boldsymbol{X}) = \prod_{i=1}^{n} f(x_i; \beta, \gamma, \theta, a, b)$$

$$= (ab\theta\beta)^n e^{\gamma \sum_{i=1}^{n} x_i} e^{\frac{\beta n}{\gamma}} e^{-\frac{\beta}{\gamma} \sum_{i=1}^{n} e^{\gamma x_i}} \prod_{i=1}^{n} \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_i} - 1)}\right)^{a\theta - 1} \left[1 - \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_i} - 1)}\right)^{a\theta}\right]^{b-1}, \tag{35}$$

where $\Theta = (\beta, \gamma, \theta, a, b)$. The log-likelihood function is

$$l(\boldsymbol{\Theta}|\boldsymbol{X}) = n \log a + n \log b + n \log \theta + n \log \beta + \frac{\beta n}{\gamma} + \gamma \sum_{i=1}^{n} x_i - -\frac{\beta}{\gamma} \sum_{i=1}^{n} e^{\gamma x_i} + (a\theta - 1) \sum_{i=1}^{n} \log\left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_i} - 1)}\right) + (b-1) \sum_{i=1}^{n} \log\left[1 - \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_i} - 1)}\right)^{a\theta}\right] \tag{36}$$

By maximizing the log-likelihood function respect to β, γ, θ, a and b parameters, the MLE of the parameters are calculated. For this purpose, it is sufficient to obtain the first partial derivatives of the log-likelihood function (36) with respect to the β, γ, θ, a and b parameters. Partial derivatives are

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \theta \sum_{i=1}^{n} \log(1 - y_i) - \theta(b-1) \sum_{i=1}^{n} \frac{(1-y_i)^{a\theta} \log(1-y_i)}{1 - (1-y_i)^{a\theta}} = 0, \tag{37}$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log\left[1 - (1-y_i)^{a\theta}\right] = 0, \tag{38}$$

$$\frac{\partial l}{\partial \gamma} = -\frac{\beta n}{\gamma^2} + \sum_{i=1}^{n} x_i + \frac{\beta}{\gamma^2} \sum_{i=1}^{n} e^{\gamma x_i} - \frac{\beta}{\gamma} \sum_{i=1}^{n} x_i e^{\gamma x_i} - (a\theta - 1) \sum_{i=1}^{n} \frac{A_i y_i}{1 - y_i} + a\theta(b-1) \sum_{i=1}^{n} \frac{A_i y_i (1-y_i)^{a\theta - 1}}{1 - (1-y_i)^{a\theta}} = 0, \tag{39}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \frac{n}{\gamma} - \frac{1}{\gamma} \sum_{i=1}^{n} e^{\gamma x_i} + \frac{a\theta - 1}{\gamma} \sum_{i=1}^{n} \frac{(e^{\gamma x_i} - 1) y_i}{1 - y_i} - \frac{a\theta(b-1)}{\gamma} \sum_{i=1}^{n} \frac{(e^{\gamma x_i} - 1) y_i (1-y_i)^{a\theta - 1}}{1 - (1-y_i)^{a\theta}} = 0, \tag{40}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + a \sum_{i=1}^{n} \log(1 - y_i) - a(b-1) \sum_{i=1}^{n} \frac{(1-y_i)^{a\theta} \log(1-y_i)}{1 - (1-y_i)^{a\theta}} = 0, \tag{41}$$

where

$$y_i = e^{-\frac{\beta}{\gamma}(e^{\gamma x_i} - 1)},$$

(42)

$$A_i = \frac{\beta}{\gamma^2}(e^{\gamma x_i} - 1) - \frac{\beta}{\gamma} x_i e^{\gamma x_i}. \tag{43}$$

From (38), we can estimate the MLE estimator of the b parameter when the other parameters are known

$$\hat{b}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \log\left[1 - (1 - \hat{y}_i)^{\hat{a}\hat{\theta}}\right]},$$

where $\hat{y}_i = e^{-\frac{\hat{\beta}}{\hat{\gamma}}(e^{\hat{\gamma} x_i} - 1)}$ and $\hat{a}, \hat{b}, \hat{\gamma},$ and $\hat{\theta}$ are the MLE of $a, b, \gamma,$ and θ parameters.

Nonlinear equations (37), (39), (40) and (41) cannot be solved analytically. Therefore, numerical methods are used to calculate the MLE estimators of parameters $a, \beta, \gamma,$ and θ.

4.1.1. Asymptotic confidence intervals

Due to the fact that the KG-GG distribution parameter estimators do not have a closed form, asymptotic confidence intervals for these parameters are calculated. Let $\widehat{\Theta} = (\hat{\beta}, \hat{\gamma}, \hat{\theta}, \hat{a}, \hat{b})$ be the MLE estimator of $\Theta = (\beta, \gamma, \theta, a, b)$. Multivariate normal $N_5\left(0, I^{-1}(\widehat{\Theta})\right)$ distribution is used to construct approximate confidence intervals for $\beta, \gamma, \theta, a,$ and b under standard regularity conditions. Therefore, we have

$$\sqrt{n}\left(\widehat{\Theta} - \Theta\right) \xrightarrow{d} N_5\left(0, I^{-1}(\widehat{\Theta})\right),$$

where I^{-1} is the variance covariance matrix of β, γ, θ, a and b parameters, which can be obtained as follows

$$I^{-1}(\widehat{\Theta}) = \begin{pmatrix} \frac{\partial^2 l}{\partial a^2} & \frac{\partial^2 l}{\partial a \partial b} & \frac{\partial^2 l}{\partial a \partial \gamma} & \frac{\partial^2 l}{\partial a \partial \beta} & \frac{\partial^2 l}{\partial a \partial \theta} \\ \frac{\partial^2 l}{\partial b \partial a} & \frac{\partial^2 l}{\partial b^2} & \frac{\partial^2 l}{\partial b \partial \gamma} & \frac{\partial^2 l}{\partial b \partial \beta} & \frac{\partial^2 l}{\partial b \partial \theta} \\ \frac{\partial^2 l}{\partial \gamma \partial a} & \frac{\partial^2 l}{\partial \gamma \partial b} & \frac{\partial^2 l}{\partial \gamma^2} & \frac{\partial^2 l}{\partial \gamma \partial \beta} & \frac{\partial^2 l}{\partial \gamma \partial \theta} \\ \frac{\partial^2 l}{\partial \beta \partial a} & \frac{\partial^2 l}{\partial \beta \partial b} & \frac{\partial^2 l}{\partial \beta \partial \gamma} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \theta} \\ \frac{\partial^2 l}{\partial \theta \partial a} & \frac{\partial^2 l}{\partial \theta \partial b} & \frac{\partial^2 l}{\partial \theta \partial \gamma} & \frac{\partial^2 l}{\partial \theta \partial \beta} & \frac{\partial^2 l}{\partial \theta^2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} var(\hat{a}) & cov(\hat{a}, \hat{b}) & cov(\hat{a}, \hat{\gamma}) & cov(\hat{a}, \hat{\beta}) & cov(\hat{a}, \hat{\theta}) \\ cov(\hat{b}, \hat{a}) & var(\hat{b}) & cov(\hat{b}, \hat{\gamma}) & cov(\hat{b}, \hat{\beta}) & cov(\hat{b}, \hat{\theta}) \\ cov(\hat{\gamma}, \hat{a}) & cov(\hat{\gamma}, \hat{b}) & var(\hat{\gamma}) & cov(\hat{\gamma}, \hat{\beta}) & cov(\hat{\gamma}, \hat{\theta}) \\ cov(\hat{\beta}, \hat{a}) & cov(\hat{\beta}, \hat{b}) & cov(\hat{\beta}, \hat{\gamma}) & var(\hat{\beta}) & cov(\hat{\beta}, \hat{\theta}) \\ cov(\hat{\theta}, \hat{a}) & cov(\hat{\theta}, \hat{b}) & cov(\hat{\theta}, \hat{\gamma}) & cov(\hat{\theta}, \hat{\beta}) & var(\hat{\theta}) \end{pmatrix}.$$

The elements of I^{-1} matrix are

$$\frac{\partial^2 l}{\partial a^2} = -\frac{n}{a^2} - \theta^2(b-1) \sum_{i=1}^{n} \frac{(1-y_i)^{a\theta} [\log(1-y_i)]^2}{[1-(1-y_i)^{a\theta}]^2},$$

$$\frac{\partial^2 l}{\partial b^2} = -\frac{n}{b^2},$$

$$\frac{\partial^2 l}{\partial \gamma^2} = \sum_{i=1}^{n} C_i - (a\theta - 1)\sum_{i=1}^{n} \frac{y_i\left(C_i + {A_i}^2 + y_i\right)}{(1-y_i)^2} + a\theta(b-1)\sum_{i=1}^{n} \frac{y_i(1-y_i)^{a\theta-1}}{[1-(1-y_i)^{a\theta}]^2}\left\{\left(C_i + {A_i}^2\right)\left[1-(1-y_i)^{a\theta}\right] - (a\theta-1)\frac{{A_i}^2 y_i}{1-y_i} - {A_i}^2(1-y_i)^{a\theta-1}\right\},$$

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2} - \frac{(a\theta-1)}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 y_i}{(1-y_i)^2} - \frac{a\theta(b-1)}{\gamma^2}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)^2 y_i(1-y_i)^{a\theta-1}}{[1-(1-y_i)^{a\theta}]^2}\left[(1-y_i)^{a\theta} - M_i\right],$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} - a^2(b-1)\sum_{i=1}^{n}\frac{(1-y_i)^{a\theta}[\log(1-y_i)]^2}{[1-(1-y_i)^{a\theta}]^2},$$

$$\frac{\partial^2 l}{\partial a \partial b} = -\theta\sum_{i=1}^{n}\frac{(1-y_i)^{a\theta}\log(1-y_i)}{1-(1-y_i)^{a\theta}},$$

$$\frac{\partial^2 l}{\partial \gamma \partial b} = a\theta\sum_{i=1}^{n}\frac{A_i y_i(1-y_i)^{a\theta-1}}{1-(1-y_i)^{a\theta}},$$

$$\frac{\partial^2 l}{\partial \beta \partial b} = -\frac{a\theta}{\gamma}\sum_{i=1}^{n}\frac{(e^{\gamma x_i}-1)y_i(1-y_i)^{a\theta-1}}{1-(1-y_i)^{a\theta}},$$

$$\frac{\partial^2 l}{\partial \theta \partial b} = -a\sum_{i=1}^{n}\frac{(1-y_i)^{a\theta}\log(1-y_i)}{1-(1-y_i)^{a\theta}},$$

$$\frac{\partial^2 l}{\partial \gamma \partial a} = -\theta\sum_{i=1}^{n}\frac{A_i D_i y_i}{1-y_i},$$

$$\frac{\partial^2 l}{\partial \beta \partial a} = \frac{\theta}{\gamma}\sum_{i=1}^{n}\frac{D_i(e^{\gamma x_i}-1)y_i}{1-y_i},$$

$$\frac{\partial^2 l}{\partial \theta \partial a} = \sum_{i=1}^{n}\log(1-y_i) - (b-1)\sum_{i=1}^{n}\frac{(1-y_i)^{a\theta}\log(1-y_i)}{1-(1-y_i)^{a\theta}} - a\theta(b-1)\sum_{i=1}^{n}\frac{(1-y_i)^{a\theta}[\log(1-y_i)]^2}{[1-(1-y_i)^{a\theta}]^2},$$

$$\frac{\partial^2 l}{\partial \gamma \partial \theta} = -a\sum_{i=1}^{n}\frac{A_i D_i y_i}{1-y_i},$$

$$\frac{\partial^2 l}{\partial \beta \partial \theta} = \frac{a}{\gamma}\sum_{i=1}^{n}\frac{D_i(e^{\gamma x_i}-1)y_i}{1-y_i},$$

$$\frac{\partial^2 l}{\partial \gamma \partial \beta} = \frac{1}{\gamma^2}\sum_{i=1}^{n}(e^{\gamma x_i}-1) - \frac{1}{\gamma}\sum_{i=1}^{n}x_i e^{\gamma x_i} - \frac{a\theta(b-1)}{\beta}\sum_{i=1}^{n}\frac{A_i y_i(1-y_i)^{a\theta-1}}{[1-(1-y_i)^{a\theta}]^2} + \frac{a\theta(b-1)}{\gamma}\sum_{i=1}^{n}\frac{A_i M_i(e^{\gamma x_i}-1)y_i(1-y_i)^{a\theta-1}}{[1-(1-y_i)^{a\theta}]^2} - (a\theta-1)\sum_{i=1}^{n}\frac{\frac{A_i}{\beta}y_i(1-y_i) - \frac{A_i}{\gamma}(e^{\gamma x_i}-1)y_i}{(1-y_i)^2},$$

where y_i and A_i are given in (42) and (43) and

$$C_i = -\frac{2\beta}{\gamma^3}(e^{\gamma x_i}-1) + \frac{2\beta}{\gamma^2}x_i e^{\gamma x_i} - \frac{\beta}{\gamma}{x_i}^2 e^{\gamma x_i},$$

$$D_i = 1 - \frac{(b-1)(1-y_i)^{a\theta}}{1-(1-y_i)^{a\theta}} - \frac{a\theta(b-1)(1-y_i)^{a\theta}\log(1-y_i)}{[1-(1-y_i)^{a\theta}]^2},$$

$$M_i = 1 - \frac{(a\theta - 1)y_i}{1 - y_i} - y_i(1 - y_i)^{a\theta - 1}.$$

Therefore, the $100(1-\alpha)\%$ approximate confidence intervals for the unknown parameters $\beta, \gamma, \theta, a,$ and b can be expressed as follows

$$\hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\beta})},\ \ \hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\gamma})},\ \ \hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{\theta})},\ \ \hat{a} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{a})},\ \ \hat{b} \pm Z_{\frac{\alpha}{2}}\sqrt{var(\hat{b})},$$

where, $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ the percentile of a standard normal distribution.

4.2. LS method

To obtain the LS estimator, the following function must be minimized

$$LS = \sum_{i=1}^{n} \left[F(x_{(i)}) - u_i\right]^2, \tag{44}$$

where $u_i = \frac{i}{n+1}$. For the KG-GG distribution, we have

$$LS = \sum_{i=1}^{n} \left\{1 - \left[1 - \left(1 - e^{-\frac{\beta}{\gamma}(e^{\gamma x_{(i)}} - 1)}\right)^{a\theta}\right]^{b} - u_i\right\}^2. \tag{45}$$

We must calculate the first partial derivatives LS function with respect to the unknown parameters $\beta, \gamma, \theta, a,$ and b.

$$\frac{\partial LS}{\partial a} = 2b\theta \sum_{i=1}^{n} \left(1 - y_{(i)}\right)^{a\theta} \log\left(1 - y_{(i)}\right) \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b-1} B_i = 0,$$

$$\frac{\partial LS}{\partial b} = -2 \sum_{i=1}^{n} \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b} \log\left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right] B_i = 0,$$

$$\frac{\partial LS}{\partial \gamma} = -2ab\theta \sum_{i=1}^{n} A_{(i)} y_{(i)} \left(1 - y_{(i)}\right)^{a\theta - 1} \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b-1} B_i = 0,$$

$$\frac{\partial LS}{\partial \beta} = \frac{2ab\theta}{\gamma} \sum_{i=1}^{n} (e^{\gamma x_{(i)}} - 1) y_{(i)} \left(1 - y_{(i)}\right)^{a\theta - 1} \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b-1} B_i = 0,$$

$$\frac{\partial LS}{\partial \theta} = 2ab \sum_{i=1}^{n} \left(1 - y_{(i)}\right)^{a\theta - 1} \log\left(1 - y_{(i)}\right) \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b-1} B_i = 0,$$

Where

$$y_{(i)} = e^{-\frac{\beta}{\gamma}(e^{\gamma x_{(i)}} - 1)},$$
$$A_{(i)} = \frac{\beta}{\gamma^2}(e^{\gamma x_{(i)}} - 1) - \frac{\beta}{\gamma} x_{(i)} e^{\gamma x_{(i)}},$$
$$B_i = \left\{1 - \left[1 - \left(1 - y_{(i)}\right)^{a\theta}\right]^{b} - u_i\right\}.$$

The above equations cannot be solved by analytical methods and appropriate numerical methods must be used.

4.3. Bayesian analysis

In this method, assume that the parameters are independent of each other and the joint prior distribution of the parameters is given by

$$\pi(\beta, \gamma, \theta, a, b) \propto \pi(\beta)\pi(\gamma)\pi(\theta)\pi(a)\pi(b), \tag{46}$$

And

$$\beta \sim Gompertz(a_1, b_1),$$
$$\gamma \sim Gompertz(a_2, b_2),$$
$$\theta \sim Gompertz(a_3, b_3),$$

$a \sim Gompertz(a_4, b_4)$,
$b \sim Gompertz(a_5, b_5)$,

where the Gompertz pdf is

$$f(x; a_i, b_i) = a_i e^{b_i x} e^{-\frac{a_i}{b_i}(e^{b_i x}-1)}, \quad x, a_i, b_i > 0.$$

The joint posterior distribution for β, γ, θ, a, and b is obtained by combining the likelihood function (35) and the prior distribution (46) as follows

$$\pi(\beta,\gamma,\theta,a,b|\boldsymbol{x}) \propto (ab\theta\beta)^n e^{\gamma \sum_{i\in F} x_i} e^{-\frac{\beta}{\gamma}\sum_{i\in F}(e^{\gamma x_i}-1)} \prod_{i\in F}\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}\right)^{a\theta-1} \left[1 - \left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x_i}-1)}\right)^{a\theta}\right]^{b-1} \times \prod_{i\in C}\left\{1-\left[1-\left(1-e^{-\frac{\beta}{\gamma}(e^{\gamma x}-1)}\right)^{a\theta}\right]^{b}\right\} \times \pi(\beta,\gamma,\theta,a,b). \tag{47}$$

Since solving (47) by analytical methods is not an easy task, "important sampling method" is used [39].

5. Simulation Study

In this section, Mont-Carlo simulation is used to compare estimators. The algorithm in subsection 3.2 is used to generate a sample of the KG-GG distribution with the actual values $\beta = 2, \gamma = 3, \theta = 2, a = 4$ and $b = 3$. The simulation study are performed for samples with size $n = 10, 20, 30, 40, 50, 60$. For each sample size, the number of repetitions is 1000. To compare these estimation methods, criteria such as mean, variance, bias and mean square error (MSE). These criteria are given by

$$\bar{\hat{\Theta}}_{MC} = \frac{1}{n}\sum_{i=1}^{n}\hat{\Theta}_i,$$

$$Var(\hat{\Theta}_{MC}) = \frac{1}{n-1}\sum_{i=1}^{n}\left(\hat{\Theta}_i - \bar{\hat{\Theta}}_{MC}\right)^2,$$

$$Bias(\hat{\Theta}_{MC}) = E(\hat{\Theta}_{MC}) - \Theta,$$

$$MSE(\hat{\Theta}_{MC}) = Var(\hat{\Theta}_{MC}) + Bias^2(\hat{\Theta}_{MC}),$$

where $\hat{\Theta} = (\hat{\beta}, \hat{\gamma}, \hat{\theta}, \hat{a}, \hat{b})$.

The results of the simulation studies are summarized in Figures 2, 3, 4, 5 and 6. According to these Figures, because the MLE method has estimators very close to the actual value, its bias is close to zero and has the lowest values of variance and MSE, so it is the most appropriate estimation method. Also, by increasing the sample size, the performance of the MLE method improves.

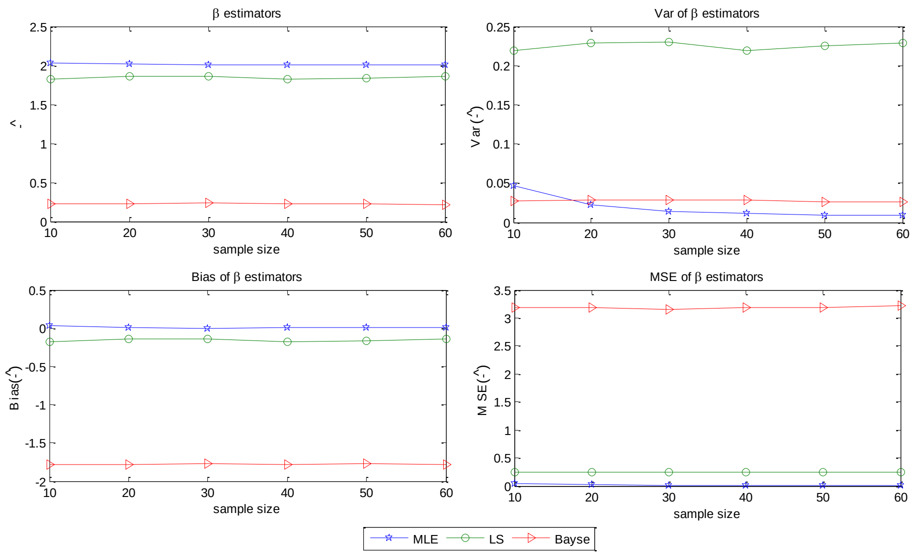

Fig. 2. Result of β estimators.

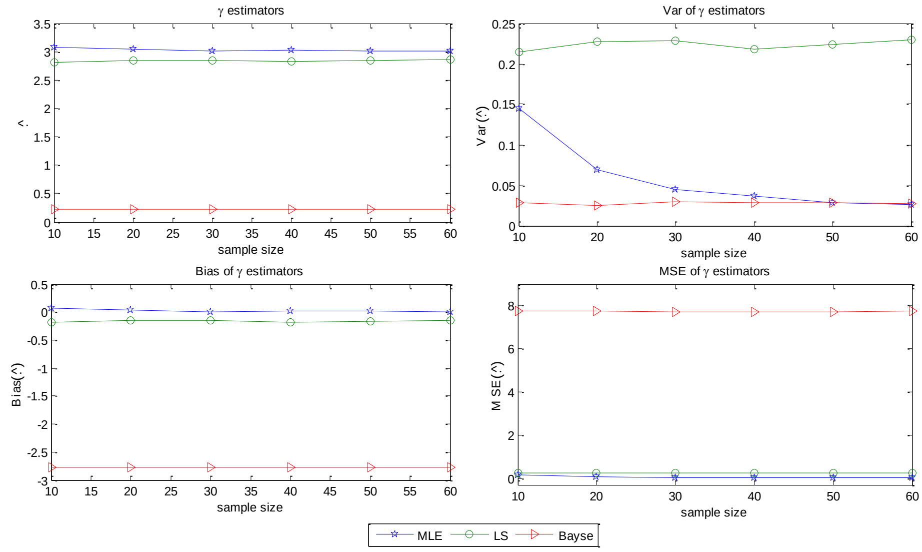

Fig. 3. Result of γ estimators.

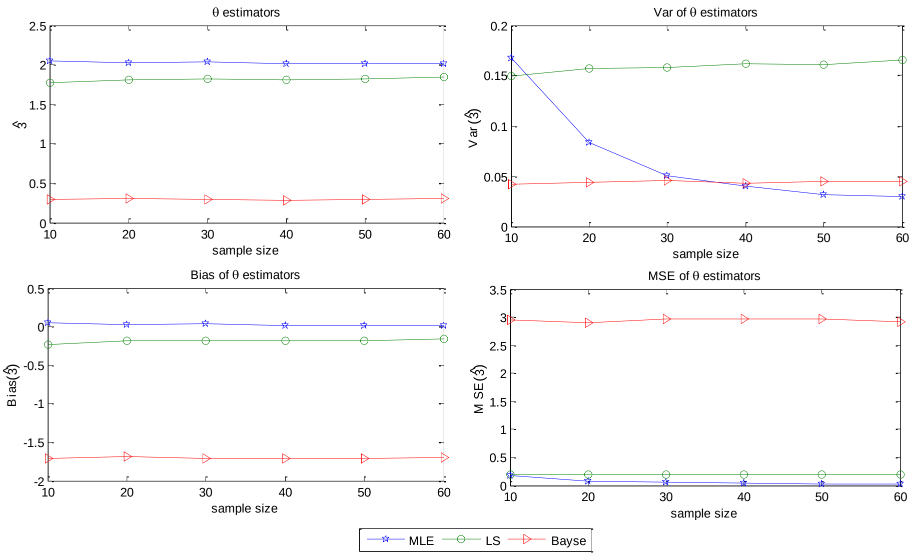

Fig. 4. Result of θ estimators.

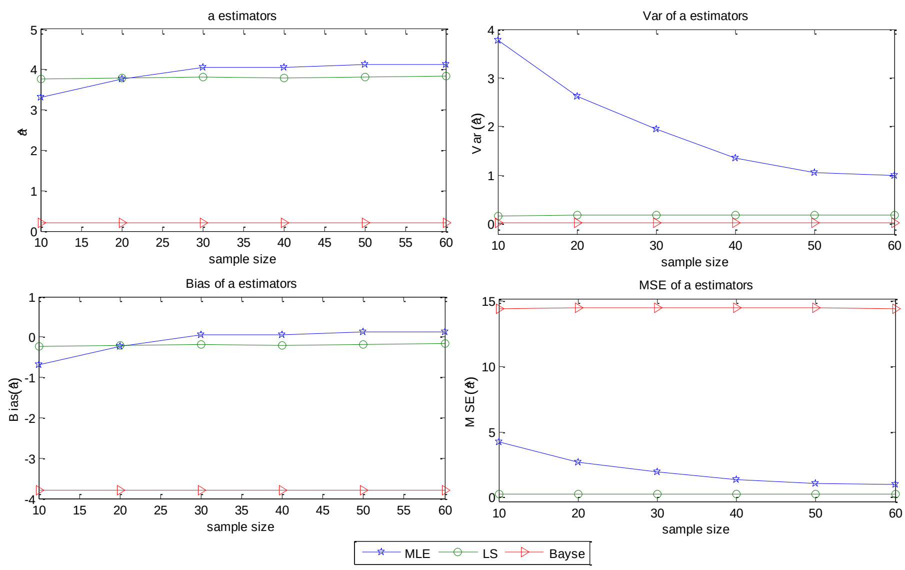

Fig. 5. Result of a estimators.

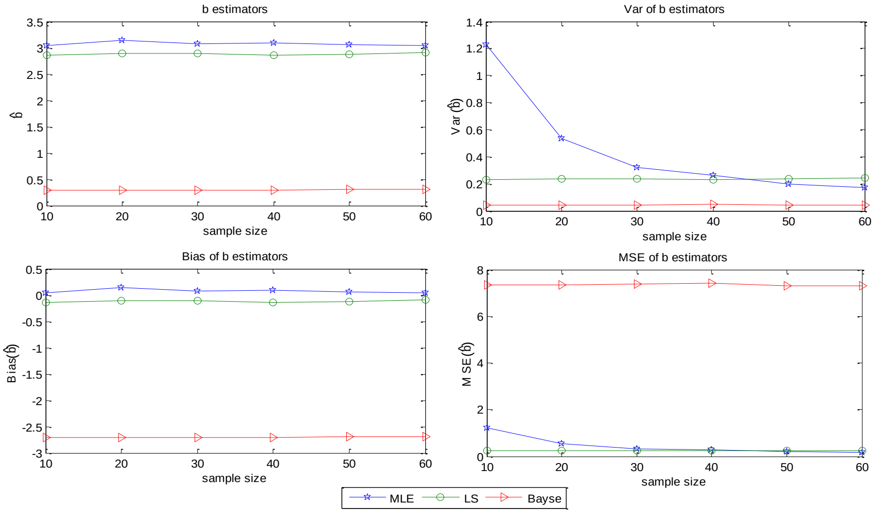

Fig. 6. Result of *b* estimators

6. Application

In this section, the lifetime data of 50 devices, taken from source [24], is used. These data are given in Table 2. The MLE of the parameters has been calculated for this data set. To compare the proposed model with other models, criteria such log-likelihood function (l), Kolmogorov-Smirnov (KS), $Pvalue$, root mean square error ($RMSE$) and coefficient of determination (R^2) are calculated in Table 3. We compare the performance of the KG-GG distribution with distributions GG and G. Figures 7, 8, and 9 show the empirical density and cumulative distribution, histogram and theoretical pdfs, and empirical and theoretical cdfs, respectively. Based on Table 3 and Figures 8 and 9, the following results are obtained:

- The KG-GG distribution has the highest l, $Pvalue$ and R^2 values among other distributions.
- The KG-GG distribution has the lowest KS and $RMSE$ values among other distributions.
- The KG-GG model has the best performance in fitting the lifetime data of 50 devices.

Tab. 2. The lifetimes data of 50 devices.

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Tab. 3. The MLE, l, KS, $Pvalue$, $RMSE$ and R^2values for the lifetime data of 50 devices.

Distribution	MLE	l	KS	$Pvalue$	$RMSE$	R^2
KG-GG	$\hat{\beta} = 0.000296$ $\hat{\gamma} = 0.076634$ $\hat{a} = 0.421159$ $\hat{b} = 0.6062$ $\hat{\theta} = 0.076634$	-221.9666	0.1367	0.3073	0.06342818	0.9598253
GG	$\hat{\beta} = 8.937e-05$ $\hat{\gamma} = 8.273e-02$ $\hat{\theta} = 2.619e-01$	-222.2444	0.1403	0.2788	0.06446936	0.9590554
G	$\hat{\beta} = 0.009715$ $\hat{\gamma} = 0.0203$	-235.3308	0.1697	0.1123	0.09907338	0.9188625

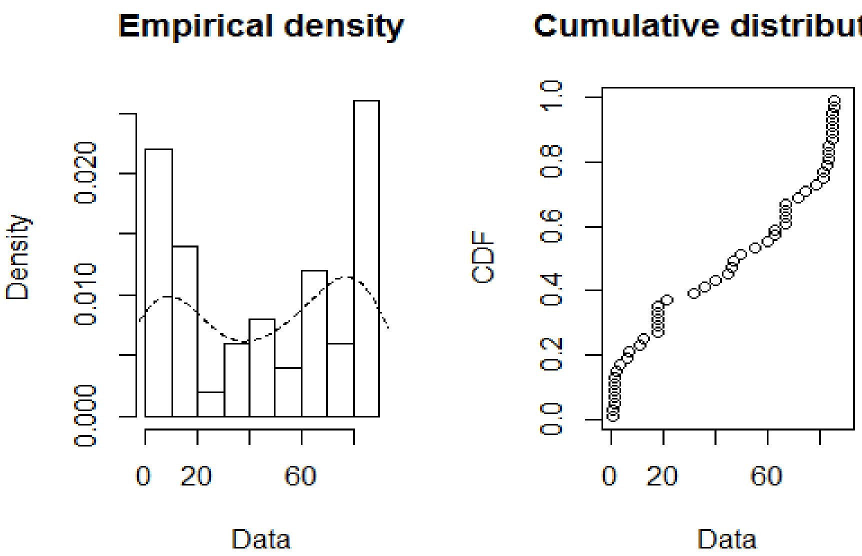

Fig. 7. Empirical density and cumulative distribution for the lifetime data of 50 devices.

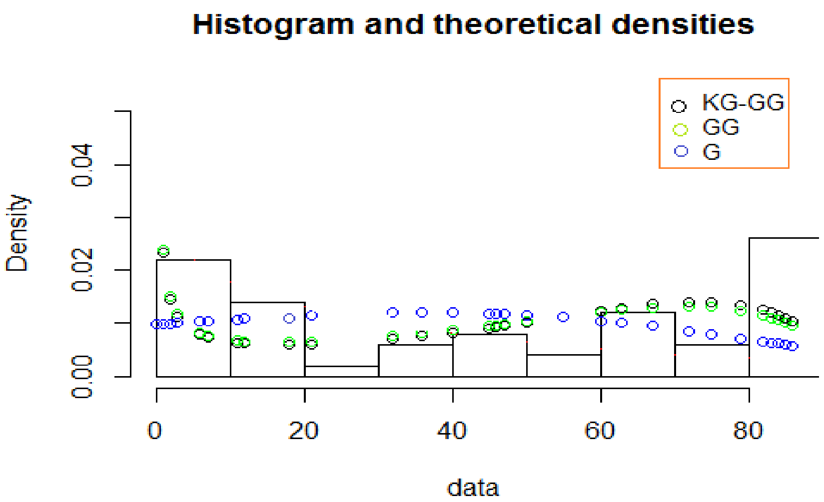

Fig. 8. Histogram and theoretical pdfs for the lifetime data of 50 devices.

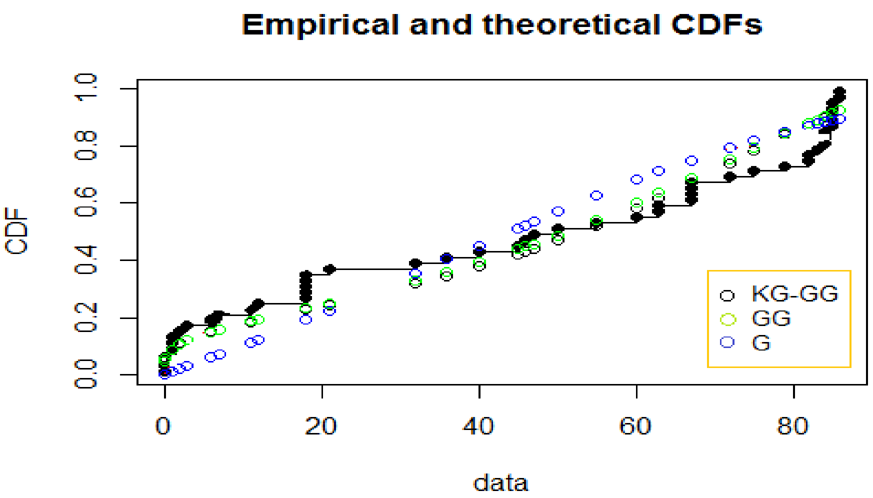

Fig. 9. Empirical and theoretical cdfs for the lifetime data of 50 devices.

7. Conclusion

In this article, a new distribution called KG-GG distribution was introduced. Some of its important statistical features were studied. The parameters of this distribution were estimated using three methods of MLE, LS and Bayse and these methods are compared using simulation. Finally, using a real data set, with the help of some criteria such as l, KS, $Pvalue$, $RMSE$, and R^2 the superiority of the performance of this model over some other distributions was proved.

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